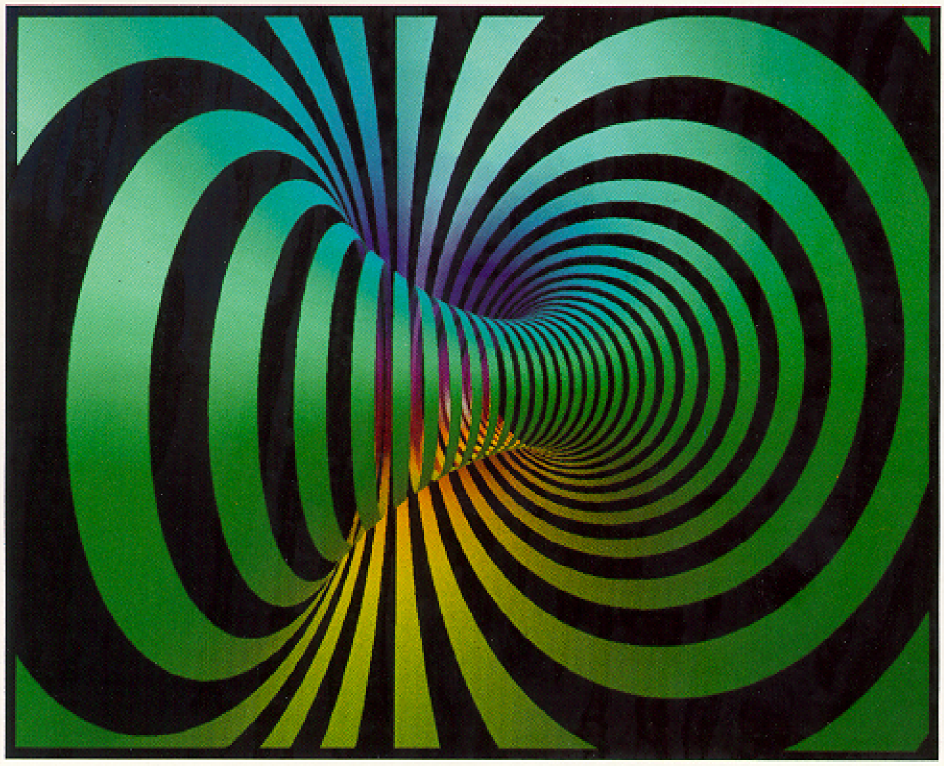
Prime™

Discovering The Fourth Dimension



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Using Computer Graphics to Explore the Generation of Surfaces in Four Dimensional Space

by

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1. Introduction

Galileo discovered the larger moons of Jupiter. They had been there orbiting their planet for eons, and it took the development of a new technology, the telescope, to make it possible for human beings to see them. This extension of human vision literally opened up new worlds for observation even though it was not possible to travel to explore them directly. We now take for granted a myriad of astronomical phenomena that have been revealed by more and more sophisticated viewing instruments. Interplanetary voyages by unmanned spacecraft provide us each day with new information which continues to transform our ideas, for example about the moons of Jupiter. We still can't visit them directly, but we see more and more.

Over a hundred years ago, mathematicians discovered the fourth dimension. It was always there, but it took an extension of human insight to be able to begin to visualize four dimensional phenomena. In our present generation, progress in computer graphics technology has guided our understanding of the fourth and higher dimension to new levels, as dramatic an advance for geometry as radio telescopes have been for astronomy. The latest developments in interactive computer graphics parallel the achievements of a space probe in leading us to see things that were inaccessible before. This brief report describes the experience of a small team working with a new high performance engineering workstation, the PXCL 5500, which makes it possible to investigate complex mathematical objects in four dimensional space very quickly.

Two decades of computer graphics projects at Brown University have identified a variety of difficult visualization problems in four dimensional geometry which challenge any computational and graphics configuration. In this report we review some of the background for the geometry of higher dimensions and indicate the way that computer graphics aids in such investigations. We then describe some of the work carried out at Prime Computer, Inc. over a six-week period in the summer of 1987 in collaboration with Dr. Robert Gordon, in charge of Advanced Workstation Technology, and Nicholas Thompson, responsible for nearly all of the programming, who in the spring of 1987 was a freshman in my course in higher dimensions at Brown University.

2. Discovering Different Dimensions

Nineteenth century workers in four dimensional geometry used many of the synthetic methods which had been used by the first mathematicians, namely the ancient Greeks, to discover another dimension. They were the ones who discovered ideal two dimensional space, the familiar plane of Euclidean geometry. No one has ever constructed a perfect circle, or a perfectly straight infinitely thin line. Those objects exist only in our minds. Experience with diagrams drawn in the sand or on the blackboard can suggest facts about plane geometry, for example, no line can intersect a perfect circle in more than two (infinitesimally small) points. But in the last analysis these facts are about ideal objects that no human being has even seen directly.

Euclid's famous textbook shows how to extend the ideas of plane geometry to the geometry of ordinary space. Once again this geometry is about ideal objects, perfectly round spheres and infinitely thin planes which would intersect, if at all, in ideal circles. The water line on a floating beach ball approximates a circle. This can suggest what occurs when an ideal plane cuts an ideal sphere, and it can even suggest new relationships that we might try to prove using Euclid's techniques. But ultimately solid geometry deals with objects we cannot construct perfectly. *There has always been an interplay between the things we can model and the things we can prove, and this interaction will continue to have an important role in the discovery and exploration process.*

It took the development of a new mathematical technology to bring home crucial facts about plane and solid geometry, and to pave the way for discovering the fourth and higher dimensions. The new technique was the analytic geometry of Descartes.

3. The Analytic Geometry Framework

On a given line, we may choose a point as the origin, labelled 0, and another point a unit distance away, labelled 1. We can then identify each point of the line with a real number, its coordinate. The line is then a one-dimensional space as shown in figure one.

In a plane, through any point on a given line we can construct exactly one perpendicular line. We can then establish coordinates on each line, with the origin of each line lying at their intersection point. For any other point in the plane, there is a unique closest point to the origin on each of the lines, and we identify the point by giving an ordered pair (x,y) of numbers which label the point "x" on the first line and the point "y" on the second. In this way the plane is coordinatized as a two dimensional space as in figure two. In this space, we can describe the four vertices of a square as (-1,-1),(1,-1),(1,1),(-1,1). *Two of these vertices are to be connected if they differ in exactly one coordinate,* so we connect (1,1) to (-1,1) and (1,-1) but not to the diagonally opposite point (-1,-1). The result is shown in figure three.

In ordinary space, we may imagine a unique coordinatized line constructed through the origin of a given coordinatized plane. For any point in space there is a unique closest point on each of the

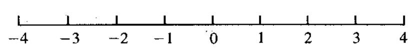


Figure 1.
A One-Dimensional Cartesian Space

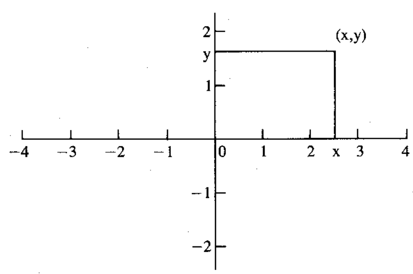


Figure 2.
A Two-Dimensional Cartesian Space

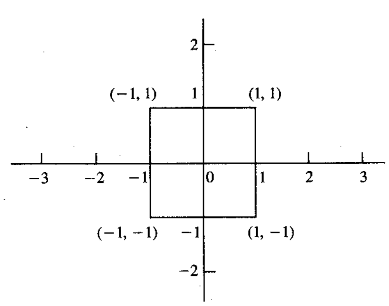


Figure 3.
The "Unit Square"

three coordinate lines, and the labels of the points on the first, second, and third coordinate axes give an ordered triple of real numbers, the coordinates of the point. *Any space in which we can find three mutually perpendicular lines but not four is a three dimensional space.*

In a three-space, the analogue of a square is a cube, with eight vertices given by the triples with all coordinates 1 or -1. The same rule determines which pairs of vertices are to be connected, namely that two triples are to be joined by an edge if and only if they differ in exactly one coordinate. It is easy to construct a representation of this abstract cube in ordinary space by building a model out of sticks and wire, and we can then take photographs from different angles to obtain various two dimensional projections of the cube. *This projection process can be described mathematically so it is not actually necessary to construct a model in order to find out what the image in the plane will look like.* For each of the eight vertices or the cube, we find the closest point in the given plane, and then connect the images of two vertices precisely when the two vertices are connected by an edge of the cube. This gives a straight-on, or orthographic, projection of the unit cube to a plane as seen in figure 4.

Any single projection of a cube to a plane always involves distortions, and so we can achieve a better visualization of the cube by studying large numbers of projections to various planes in space.

Of course the mathematics of projecting points in a three dimensional space to points in a two dimensional space (e.g. a viewing plane) have been known for some time. Typically, in computer graphics, this involves a matrix operation such that each triple of real numbers in three-space results in a pair of integers in screen space where the points are electronically painted. In the past such matrix operations have been carried out on a general purpose computer, and objects having thousands of triples (vertices) took many minutes to compute a

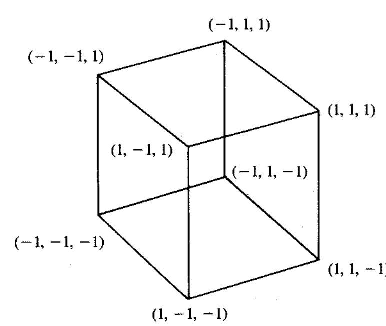


Figure 4.
An "Orthographic" Projection of the "Unit Cube"

single projection. With recent advances in VLSI it is possible to design custom circuits specifically for doing matrix operations. Such circuits allow high performance workstations, like the PXCL 5500, to transform hundreds of thousands of vertices per second. Complex objects with thousands of vertices can now be "reprojected" several times a second thus allowing one to view objects from any position in "real time".

4. The Leap to a Higher Dimension

We can manipulate ordered pairs of real numbers and describe facts about geometry. Similar algebraic manipulations of ordered triples of real numbers describe what is happening in a three dimensional space. *From this stage, it is a short step to consider the same sort of algebraic manipulations with four-tuples of real numbers, even though we have no direct experience of a physical space to which such algebraic concepts might correspond.* Thus mathematicians discovered an abstract four dimensional space which generalized the domain of solid geometry in the same way that three dimensional geometry generalized the plane.

In such a four dimensional space, what corresponds to a square in the plane or a cube in three-space? It is easy to give a formal answer: simply take all four-tuples where each coordinate is either 1 or -1. Since there are two choices for each of the four coordinates there are sixteen vertices on the four dimensional cube. As before, two vertices are connected by an edge if and only if they differ in exactly one coordinate. If we could find four mutually perpendicular lines, we could build a model of this hypercube and photograph it to obtain various two dimensional representations of it. We can't find such a set of axes in ordinary space, but nonetheless we can describe a process of projection from a hypercube to a plane in four-space. Just as we did before, we find for each of the sixteen vertices the closest point lying in the plane, and we connect these images if the vertices are endpoints of an edge in the hypercube.

We can also find the projection of a hypercube into three dimensional space lying in four-space by the same process: send each vertex of the hypercube to the closest point in the three-space and connect two images if the vertices are endpoints of an edge of the hypercube. Figure 5 shows such a projection with all sixteen vertices connected by the above rule.

5. Visualizing Hypercubes

More than one hundred years ago, mathematicians attempted to visualize the hypercube by

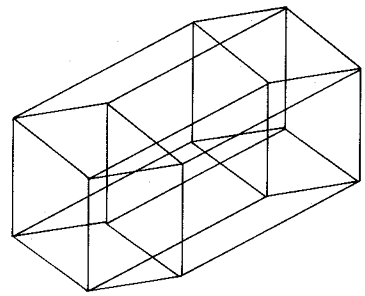


Figure 5.
An "Orthographic" Projection
of the "Unit Hypercube"

constructing pictures of projections of hypercubes into planes and models of projections of hypercubes into three-spaces. Each single projection involves distortions, *so no one view gives a totally satisfactory visualization.* To minimize ambiguities it is desirable to have many different projections which the mind can then try to integrate into a coherent mental image of the entire object. In the nineteenth century, this was difficult to achieve.

All this has changed with the advent of modern computer graphics. For the first time we can realize the dream of mathematicians of the last century and study three and four dimensional objects directly. Analytic geometry is a perfect way to communicate the structure of an object to a computer. If we give a collection of pairs of real numbers, a computer can show points on the screen which correspond to those coordinate pairs, and then connect certain pairs of points by edges to form a polygonal image. If we have a polygonal object in three-space, we can again enter into the machine the list of triples of real numbers corresponding to the vertices and the list of pairs of points which are to be connected by edges. For every plane that we select in space, the computer can show us the projection of the object into that plane. If the computer can form images quickly enough, it is possible to use an input device (i.e. a mouse, joystick or slider bar) to change a single parameter from a family of planes to show what would happen if we built the object in space and flew around it in an airplane. In this way a computer can enable us to explore three dimensional objects which have not yet been constructed physically!

In the same way it is possible to use a high performance workstation to tour an object in a mathematical four dimensional space. There are more degrees of freedom in four dimensional space so it is necessary to have more dials or slider bars to give full control over the planes into which the object is being projected. In writing the program for the four dimensional rotations we made use of the fact that there is hardware support for four-by-four matrix transformations, a consequence of the homogenous coordinate system commonly used in computer graphics to support three dimensional transformations and projections. This four-by-four hardware was used to calculate new three dimensional triples for each particular four to three-space projection. Thus "real-time" rotation of four dimensional objects is now feasible. In fact, it is now even possible to explore simple wire-frame four dimensional objects slowly on a microcomputer, as is described by the April 1986 *Computer Recreations* section of Scientific American.[1]

6. Central Projections of the Cube and Hypercube

Up to this time we have been discussing perpendicular projections and line drawings in which the images of parallel lines in space are parallel or coincident lines in a plane. Another extremely important technique is central projection, where we consider the rays emanating from a fixed light source and then gather the shadows cast on a plane. If the light source is very close to the top face of the cube, for example, then the farthest away square will be small and completely contained within the image of the closer square, producing the "long hallway" illusion of figure 6.

If we rotate the cube about a horizontal axis while keeping the plane and the source of light fixed, then the image of figure 7 appears to "swim through itself" as the inner and outer squares change places.

For a hypercube in four dimensional space the analogous central projection from a point source of light just above one of the boundary cubes produces an image in three-space which has the form of a cube within a cube, as in Figure 8.

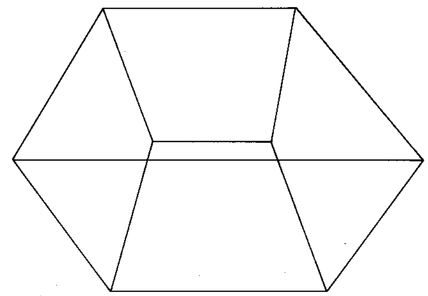


Figure 7.
A "Central Projection"
of a "Rotated" Cube

As before, when we rotate this object in four dimensional space, it is possible to get a sequence of images in which the inner and outer cubes interchange positions. The effect is considerably stronger if sixteen of the two dimensional faces of the hypercube are filled in to form a polyhedral torus. This construction suggests that we describe the polyhedral torus as a rectangular four-by-four grid folded up into four-space. Once we have written the program to be able to do this, it is a simple matter to subdivide further to obtain an eight-by-eight grid which can be folded together in four-space and then centrally projected into three-space. We added this feature to our program so that we could also have fine control over the degree of surface subdivision for other objects we wished to study.

It is worthwhile to pause here to mention a word about the lighting and shading support available on new high performance workstations that allow the exploring of these shapes with different coloring assignments of the faces and different views.

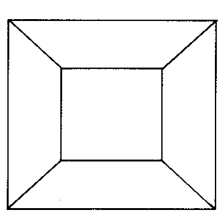


Figure 6.
A "Central Projection" of a Cube

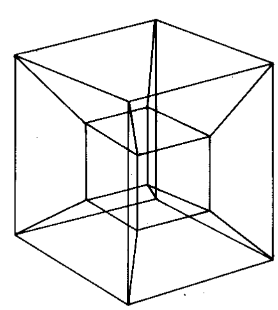


Figure 8.
A "Central Projection" of a Hypercube

[1]Computer Recreations, by Alexander K. Dewdney Scientific American, April 1986

To achieve the lighting effects shown in many of the images in the rest of this paper requires a calculation to determine the "color" of each face and the ability to rapidly "fill" the interior pixels of each face with the appropriate color. A square is, of course, really a four sided polygon, and in certain projections one face can hide another. The hidden face problem has been well researched over the past twenty years and modern workstations have excellent support for filling and sorting polygons very quickly. In modelling the effect of lights which are a source of "spectral energy", the "color" calculation for each face is really an approximation of the reflected energy of the light source that is directed towards the viewer's eye. It requires that the normal, or perpendicular of each face, be computed so that the angular difference between the light ray and the surface is available to model the reflected light energy. One really needs a high speed floating point processor and a high speed central processor, as in the PXCL 5500, to calculate the normals for each polygon and the corresponding "color" value in near real time for complex objects.

It is not always the case that each polygon should be "flat shaded", or have all pixels within it the same color. A better shading algorithm is to compute a color for each vertex and "interpolate" the color throughout the polygon between all vertices. This technique is known as "Gouraud" shading after Henri Gouraud.[2]

The projection mathematics used by the PXCL 5500 do not throw away the "z" value of each vertex in computing a two dimensional projection onto the screen space. Instead, the "z" value, computed as a result of the two dimensional projection, is used to determine which polygon or face is closer to the viewer so that "front facing" polygons are colored last. This technology, known as "z-buffer sorting", is supported by special memory (the z-buffer) in the graphics processor, and hardware comparison circuitry to allow real time sorting of polygons. Many of the well known computer graphics algorithms and techniques briefly mentioned here can be found in an excellent text by Rogers.[3]

Plates 1 through 4 at the back of this paper show a sequence of four successive polyhedral tori images, based on the successive subdivision of a rectangular grid. At each stage the normal vectors are recomputed, and once we find a view we would like to see in finer detail, we may use these normal vectors and the hardware supported Gouraud shading algorithm to remove all traces of the approximating polygons, as in Plate 5.

7. The Hypertorus

In order to study geographical features on the surface of the earth, geographers have devised all sorts of projections from the curved surface of a sphere in three-space to flat two dimensional maps. One of the mapping techniques that has especially nice geometrical properties is *stereographic projection.* To obtain this mapping, we begin with a transparent sphere resting on a plane and imagine a bright light situated at the North Pole. Each point on the sphere other than the North Pole itself will have an image point on the plane, every point of the plane corresponds to exactly one point on the sphere. The rays from the North Pole through the points of the Equator form a right circular cone, which cuts the plane in a circle. Similarly the image of any parallel of latitude on the sphere is a circle in the plane. This projection, shown in figure 9, provides a very accurate map of the Antarctic region, and it becomes more distorted near the Equator. Land masses in the Northern Hemisphere are further distorted, and Greenland for example has an image which is huge. In order to use this projection to get a reasonable image of Greenland, one can rotate the globe, keeping the plane and the light source fixed, so that Greenland is moved near to the point where the sphere touches the plane.

One of the significant properties of stereographic projection which makes it easier to keep track of geometric features is that it not only sends parallels of latitude to circles --- it sends all circles on the sphere to circles in the plane, with the exception of circles passing through the North Pole which have straight lines as their images. As we

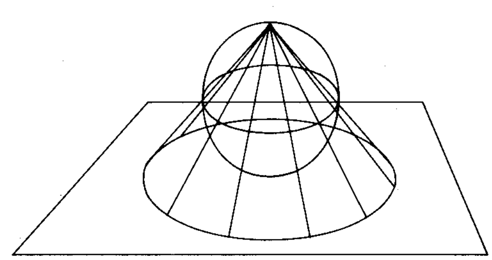


Figure 9.
A "Stereographic Projection" of a Sphere

[2] Gouraud, Henri; Continuous Shading of Curved Surfaces, IEEE Transactions on Computers, Vol. C-20, No. 6, June 1971

[3] Rogers, David F.; *Procedural Elements for Computer Graphics*, McGraw Hill, C1985, ISBN 0-07-053534-5

rotate the globe we can keep track of what happens to a region bounded by a particular circle, for example, the Equator. At the beginning, the image of the Southern Hemisphere is the interior of a disc centered at the point of tangency as in figure 10. As we begin to rotate, the image of the Equator is still a circle, and the image of the Southern Hemisphere is the interior of the off-center disc of figure 11. Ultimately the Equator rotates so far that it passes through the light source at the top point. The image of the Equator is then a straight line and the image of the Southern Hemisphere becomes the infinitely large half-plane depicted in figure 12.

If we continue the rotation, the light source is contained in the region bounded by the Equator, *so the image of this region is the exterior of the circular image of the Equator.* By the time the rotation has brought the Equator back to a horizontal position, the Southern and Northern hemispheres have been interchanged and the images in the plane have been *"turned inside out"* as in figure 13.

The same method of stereographic projection also works to produce accurate images of objects on the hypersphere in four dimensional space. In this case the hypersphere is the collection of all points which are at a fixed distance from a center in four dimensional space. This mapping from the hypersphere to ordinary three-space also preserves circles, so it is ideal to use in studying a surface which is covered by circles, namely the torus described in the previous section.

We can reveal the structure of the torus in four-space by removing half of the cylindrical bands which cover it. (This permits us to gain an appreciation of the back as well as the front of the object without resorting to to modelling transparent objects by time-consuming ray tracing algorithms.) We can then begin to rotate the hypersphere in four-space so that some portions of the torus have more precise images and other parts become distorted. If we rotate so that one of the points of the torus goes through the light source, then the image down in our three dimensional space stretches out to infinity. This infinitely large surface is remarkable in that, like the plane, it totally and symmetrically separates all of space into two congruent pieces. If we continue the rotation in four dimensional space, then the image returns to a familiar torus of revolution but with an

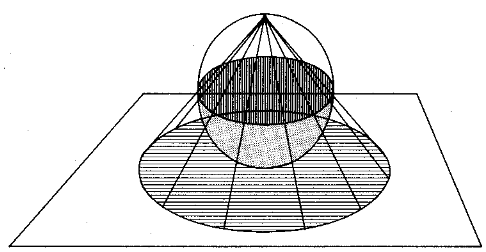


Figure 10.
A "Stereographic Projection"
of the "Southern Hemisphere"

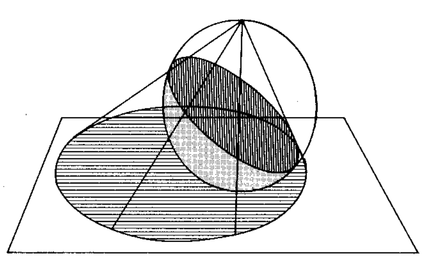


Figure 11.
A "Stereographic Projection" of the
rotated "Southern Hemisphere"

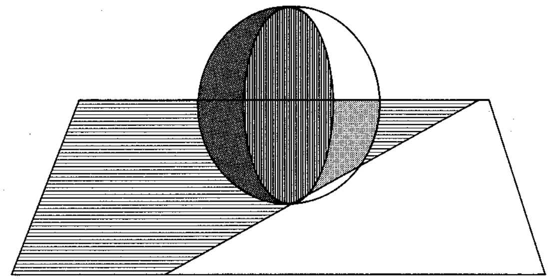


Figure 12.
A "Stereographic Projection"
of the "Southern Hemisphere" rotated
through the Light Source

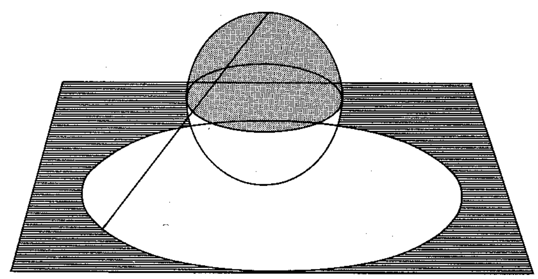


Figure 13.
A "Stereographic Projection"
of the "Southern Hemisphere"
pointing at the light source

essential difference: The cylindrical bands which formerly were parallels of latitude have now become meridians of longitude. During this deformation, the torus has turned completely inside out. Latitude and longitude have been interchanged. We illustrate several stages in this deformation in the series of photographs in Plates 6 through 10.

Several features of the Prime PXCL 5500 have increased our ability to interact with these phenomena. The speed of the machine enables us to recompute the normal directions at each face several times a second so we may apply lighting models during the deformation and still have performance approaching real time motion. The hardware clipping features make it possible to keep track of images even as they stretch out toward infinity. At any particularly interesting view it is possible to engage the Gouraud shading algorithm to display the smooth surface which the polygonal objects are approximating. The 24 bits of color available allow us to use the full range of 16 million colors in the spectrum to identify positions on the torus even as it turns inside out. Previous computer graphics investigations were limited to black and white rotating wire-frame images, or raster graphics shaded images which took a considerable amount of time to produce. The films made by single frame animation were very useful in showing properties of these objects, but their slowness inhibited interactive exploration. The development of interactive raster graphics devices such as the PXCL 5500 opens a new range of possibilities to a researcher who wishes to visualize complex phenomena, in four-space as well as three.

8. Torus Knots Projected From Four-Space

Knotted curves in three dimensional space have been the subject of mathematical investigations for over a century, and it is only recently that computer graphics has been used to study their properties. By definition, a closed curve is *knotted* if it cannot be deformed to a planar circle without passing through itself. It is no simple matter to look at a picture of a curve in space and to decide whether or not it is knotted. More generally, it is often quite difficult to determine if one knot can be deformed into another, especially if the knots are complicated in appearance.

Strangely enough, one of the best places to find knotted curves of mathematical interest is not in three dimensional space but in the three dimensional sphere in four dimensional space. We have already seen how the torus can be studied by building it on a sphere in four-space and projecting it in three-space by stereographic projection. It is not hard to find closed curves on the surface of the torus and many of these turn out to be knotted when we project them down into three dimensional space. Rotating the objects in four-space sometime produces surprising deformations of the images of knotted curves. In particular the curious property that reverses the inside and the outside of a torus will deform a familiar trefoil knot with threefold symmetry to a double loop with a different sort of symmetry, as indicated in the sequence of images in Plates 11 through 14.

In studying knots, it is useful to surround each curve with a thin tube so that the z-buffering automatically shows which portion of the curve is in front near an apparent crossing point. We can then employ color to show how the tube is twisting as we go around the curve. For rotating curves projected from four dimensional space, it is necessary to find at each point a unit vector perpendicular to the velocity vector at the point of the curve. It is then possible to construct a circle of fixed radius perpendicular to the velocity vector at a finite number of points on the curve, and then to connect these circles by cylindrical tubes. Using the acceleration vector of a point moving at unit speed is a common way of building such tubes, but this method is ineffective near points where the curve is highly twisted. The speed of the PXCL 5500 made it possible to construct another collection of circles by recomputing at each stage the position vector from the origin to the curve and taking its projection into the plane perpendicular to the velocity vector at a point. This method produced tubes of uniform thickness to which we could apply various lighting models.

9. Complex Function Graphs

One of the most important applications of visualization of objects in four dimensional space occurs in complex analysis, the mathematics used in electrical and aeronautical engineering for the study of flows. This subject generalizes ordinary analytic geometry and calculus of one variable, and it combines a rich mixture of algebra, analysis, and geometry. One of the most effective techniques in ordinary calculus and analytic geometry is graphing functions of one real variable on a two dimensional grid. A single complex number already requires two real numbers, its real and imaginary parts, so graphing a complex function presents a special challenge. The total graph requires four real dimensions, two for the domain and two for the range, giving a two dimensional surface in four dimensional space. In the last century, mathematicians created plaster models to show projections of such surfaces into three-space, but it was often difficult to imagine how these various views could be integrated into a single object in four-space.

Interactive computer graphics makes it possible to analyze these important phenomena in a direct way. The PXCL 5500 can rotate filled-in surfaces in four dimensional space almost as fast as it rotates in three so we can easily see what happens as we rotate from the graph of the real part of a complex function to its imaginary part. One of the greatest rendering challenges occurs when we rotate in a different way and exhibit the real or imaginary parts of the inverse function. During such a rotation, the image in three-space cuts through itself even though the object in four-space has no self-intersections. (This is analogous to the apparent crossings we see when we project a knotted curve in three-space down to a plane). The most interesting point for the complex square-root function is at the origin since it has only one square root whereas all other complex numbers have two. The real and imaginary parts of the square root function possess "pinch points" which are endpoints of arcs of double curves. The pieces of surface which intersect arbitrarily near a pinch point are nearly identical , so conventional rendering algorithms have a great deal of difficulty determining which part should be in front. The 24 bit z-buffering capability of the PXCL 5500 makes it possible to render a surface containing such singular points without ambiguity.

We now briefly discuss an example of the way the computer handles the equations of complex functions. In ordinary analytic geometry, the equation:

$$u=x^2$$

can be graphed in the plane to give the parabola of figure 14.

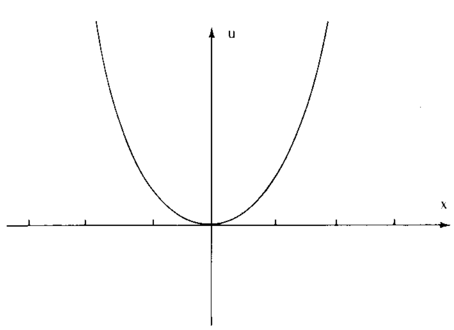


Figure. 14
The "Parabola": $u = x^2$

In complex analysis, each complex number "z" is written as:

$$z=x+iy$$

where x and y are real numbers and where the imaginary unit "i" has the property that:

$$i^2=-1$$

The usual rules of algebra then give:

$$z^2=(x+iy)^2=x^2+i2xy+(iy)^2=x^2-y^2+i2xy$$

If we express z^2 as the complex number:

$$w=u+iv$$

we then have:

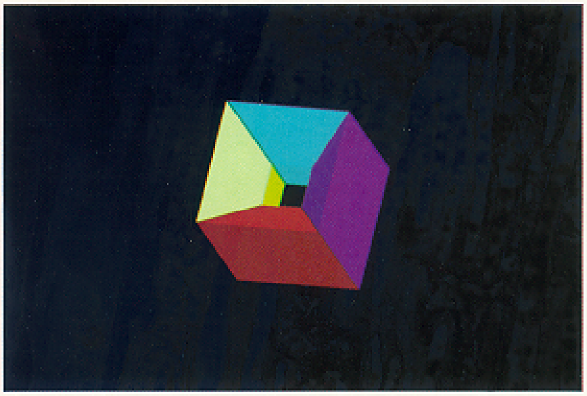
$$u=x^2-y^2 \text{ and } v=2xy$$

We may then enter these four coordinate functions: $(x, y, x^2-y^2, 2xy)$ into a computer. If we then specify a domain in the (x,y)-plane, the computer can show any desired view of this function graph in four-space. An appropriate domain is a disc of radius "1" about the origin. In Plates 15 through 18 we indicate various views of mapping of this disc into four-space as we rotate from the real part of the squaring function (x, y, x^2-y^2) to the imaginary part of the square root relation $(y, x^2-y^2, 2xy)$. During this rotation, the image acquires a segment of double points ending at a pinch point, one of the most difficult rendering challenges for a raster graphics computer, and one for which the PXCL 5500 is extremely well equipped.

In this short paper we have only explored a few of the many four dimensional objects that mathematicians have know about and want to explore. We have, in fact programmed several more that are not mentioned here. Interested readers may want to read more about such objects in Ruckers[4] book. For those interested in the difficulties of explaining three dimensions to persons only familiar with two dimensions you may want to read Abbots[5] delightful essay on a two dimensional world.

[4] Rucker, Rudy; *The Fourth Dimension: Toward a Geometry of Higher Reality*, Houghton Mifflin Company, 1984

[5] Abbot, Edwin A.; *Flatland: A Romance of Many Dimensions*, New American Library 1984



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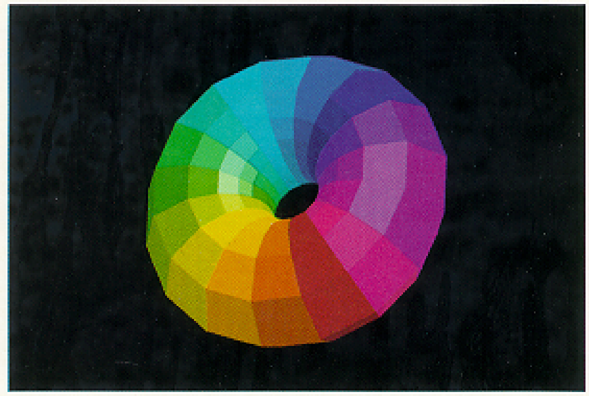

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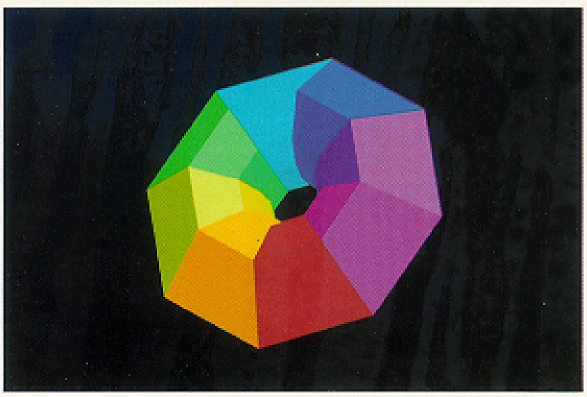

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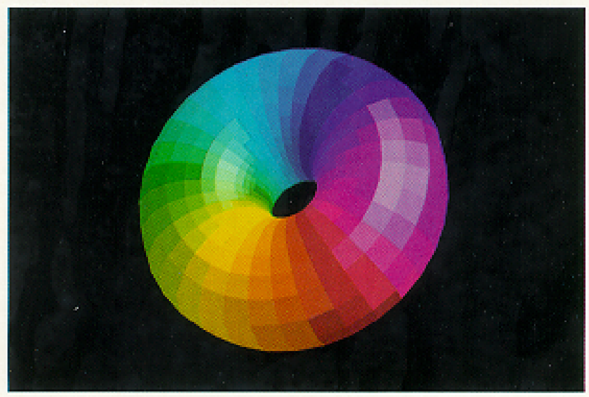


4

Plates 1-4
Polyhedral Tori based upon successive subdivision of the basic Hypercube faces.

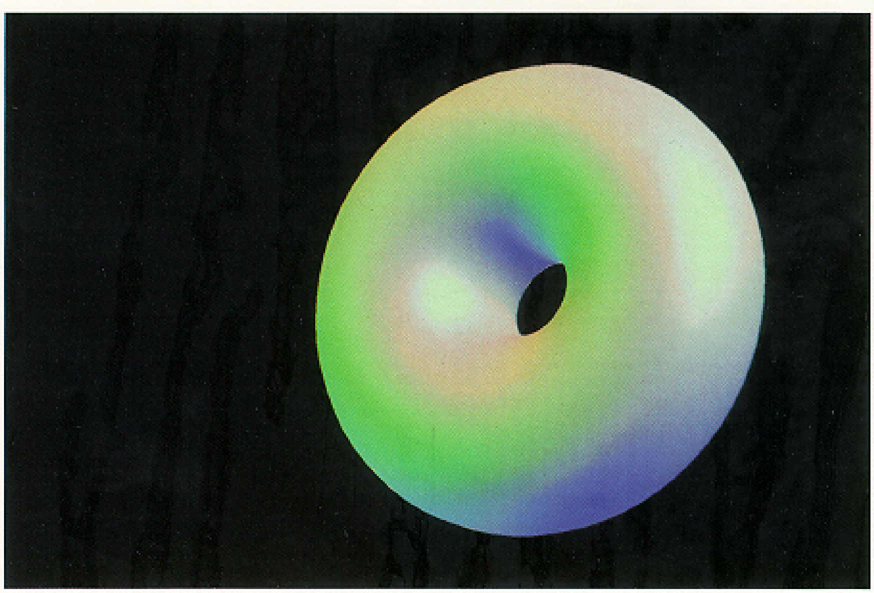
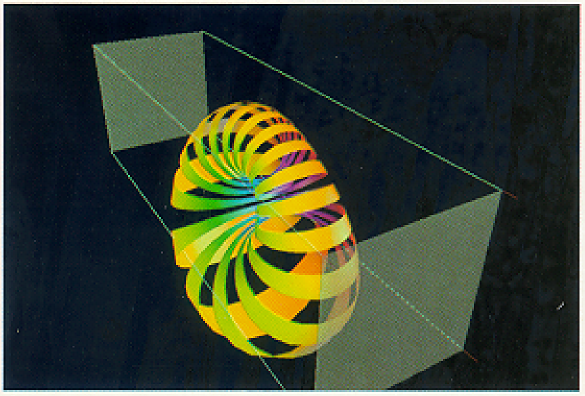


Plate 5
A ''Gouraud Shaded'' Polyhedral Torus with Multiple Colored Light Sources.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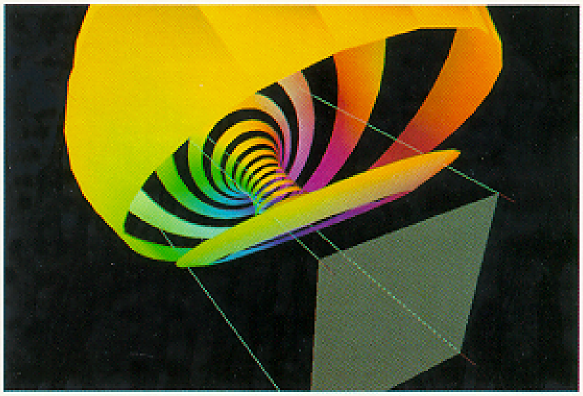
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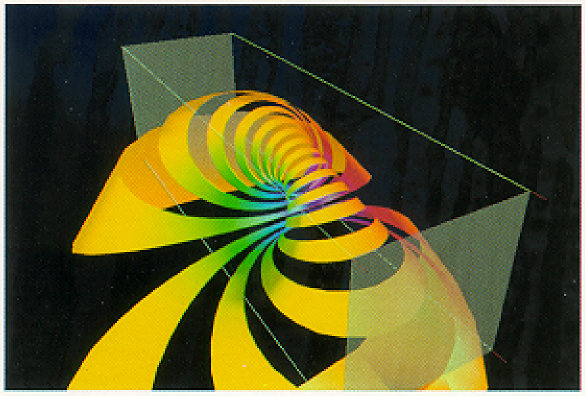
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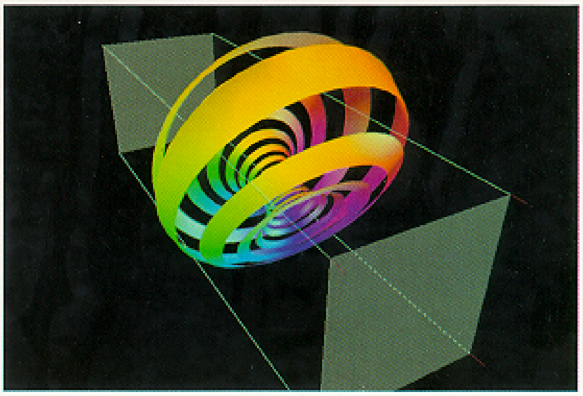
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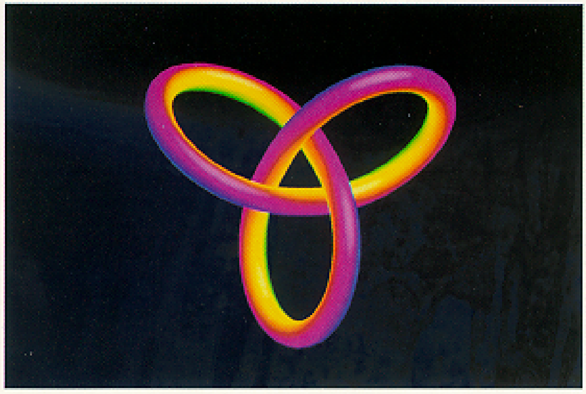
8

Plates 6-10
Successive Stereographic Projections of a Polyhedral Tori with alternate rows removed.



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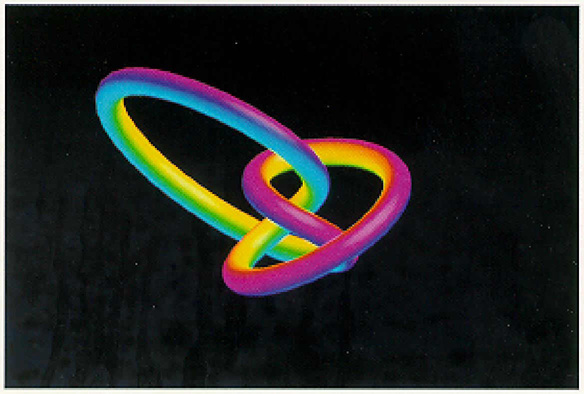

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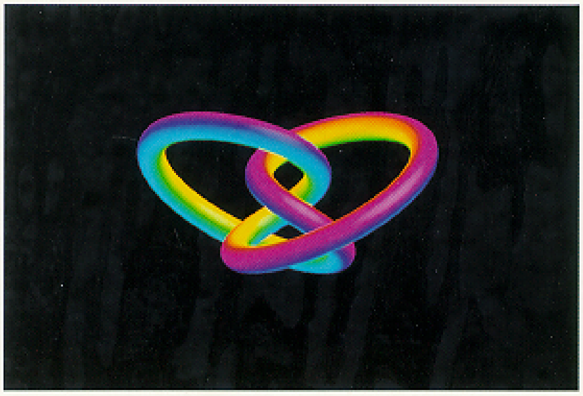
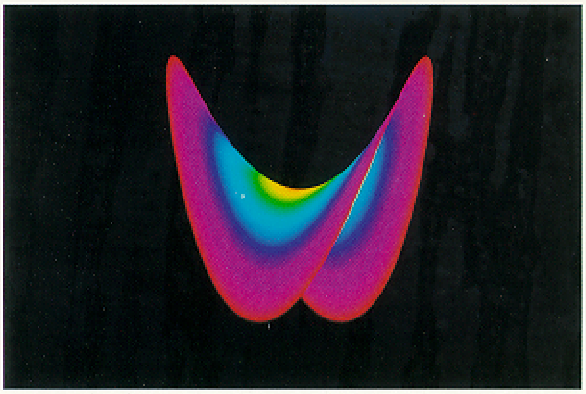


14

Plates 11-14
Deformation of the ''Trefoil Knot'' with threefold Symmetry.



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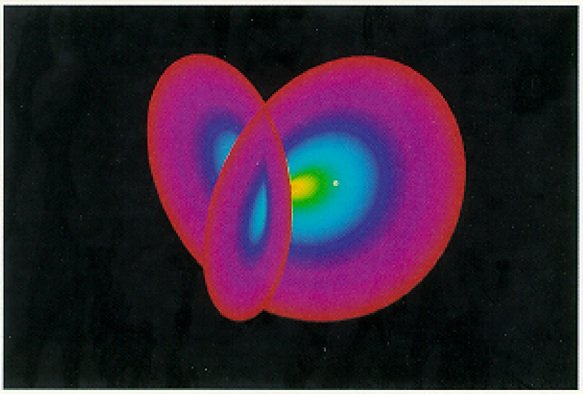

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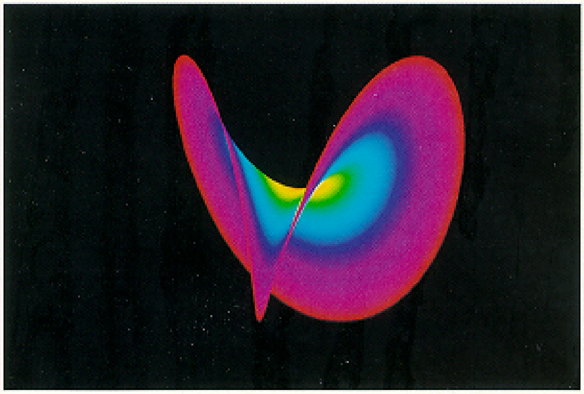

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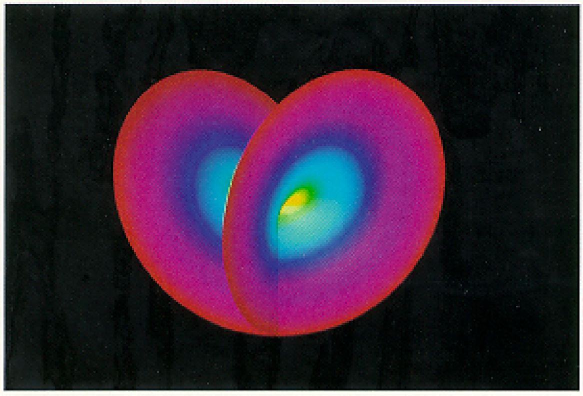

18

Plates 15-18
Rotations of the ''Complex Function'' in Four Space.