The Fourth Dimension

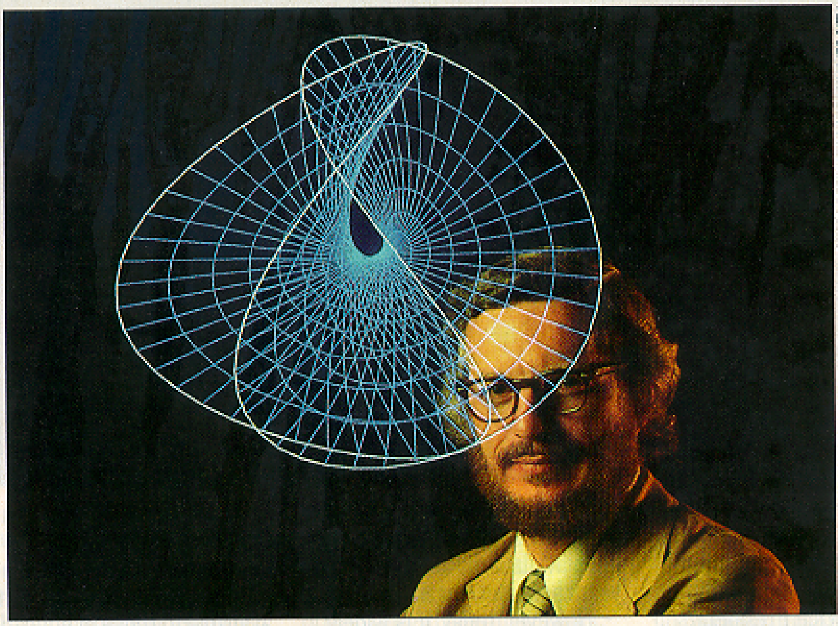
Modern computer graphics shed new light on our more familiar space

By Thomas Banchoff

Climatologists pound a hollow stake into a dry Midwestern lake bed and extract a record of 10,000 years of history. The new insights that enable them to extract information from this record come from one of the most mysterious parts of mathematics, the geometry of higher dimensions. And the new techniques that make this interaction possible come from modern computer graphics.

Each layer of soil in the long, slender core sample is like a chapter in a book, every chapter chronicling the passage of years. In place of words, the climatologists read the record by measuring the fossilized grains of pollen from ancient forests or fields of prairie grass. Knowing what pollen predominated when and where helps them understand what the weather was like 3,000, 5,000, or even 10,000 years ago. The presence of prairie grass pollen, for example, suggests a warm, dry climate. Spruce pollen indicates a moist, cooler climate. The climate history of a particular location is completely given by two numbers for each observation, the ratio of pollen concentrations for each specific time.

But what if we want to know about the climate history along an entire dry river bed? It becomes necessary to interpret not just one, but a sequence of core samples, so now each observation is also described by a third number, representing the distance along the river. To understand the behavior of climate over an entire region requires yet another coordinate. An imaginative climatologist can study the data points, and try to visualize the relentless advance of the treeline across the plains as the temperature drops, and the re-emergence of the prairie grass as more temperate times return. This process involves dealing with a data set that is four-dimensional, since each observation is determined by four coordinates: the latitude and longitude of the sample, and the pollen ratio at each time (as determined by the depth of the sample). To visualize such configurations is a challenge that stretches our imaginative powers, and fortunately we now have computer graphics to open our eyes to phenomena that truly come from "The Fourth Dimension."



JOHN FORASTE/BROWN UNIVERSITY

Fantasy becomes reality: Banchoff's research in the fourth dimension stems from his boyhood interest in science-fiction books.

Few phrases from mathematics convey as much sense of mystery and fascination as the fourth dimension. Almost a synonym for the unfathomable and incomprehensible, this elusive concept has defied attempts to visualize it for well over a hundred years. Now all that is changing, in dramatic and sometimes spectacular ways.

Modern graphics computers reveal for the first time photographic images of objects that could never be built in our three-dimensional universe, because their existence depends on a fourth dimension that is physically inaccessible to us. How is it that mathematicians are able to visualize things they have never seen? As they investigate these higher-dimensional realms, what new things are we learning that shed new light on our more familiar space? What insights can we expect in the future?

From ancient times, geometers and philosophers have sought analogies to help understand complicated ideas, and often these involved interaction among different dimensions of experi-

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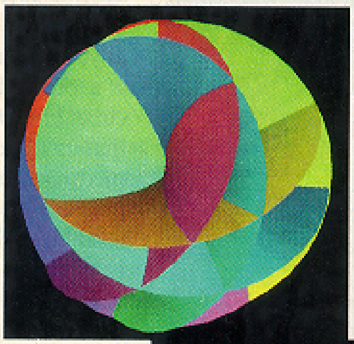
ence. Plato introduced the idea of projection to describe ways in which we have to deal with incomplete lower-dimensional information. In his "Allegory of the Cave," he told about a race of people whose gaze was fixed on a wall on which they could see the shadows of objects passing behind them. How imperfect their knowledge of reality would be if the only images they received were these two-dimensional silhouettes of solid phenomena. In a similar way, Plato argued that the solid forms we see in our ordinary experience are only poor shadows of the real ideas behind them, and that we have to train the mind to interpret these projections if we are to rise to a higher level of understanding.

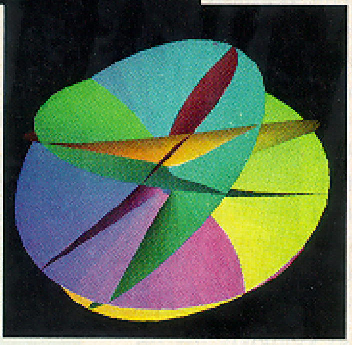
In our day, we gaze at two-dimensional images on television screens and try to discern the solid reality that produces the pictures. The team of physicians studying a CAT scan or the astronomers analyzing photographs of Neptune are facing some of the same problems that challenged Plato's cave dwellers when they tried to interpret limited data. We only see a finite number of planar projections, and we want to know the shape that causes them. The breakthrough in our ability to deal with such problems is the high-speed interactive graphics computer. The power it provides literally leads us into new dimensions.

DAVID MARGOLIS

Creating the impossible, a computer ties a trefoil knot, projected from four dimensions into three.

My own experience with computer graphics is typical of that of many researchers —it just began a bit earlier. I had become fascinated with the idea of the fourth dimension as a junior high student reading science fiction comic books. All during my education, I had felt frustrated in my attempts to visualize four-dimensional shapes. Then, 23 years ago, when I came to Brown University, I met Charles Strauss, who had constructed for his Ph.D. project a marvelous computer program that seemed like an answer to my dreams.





DAVID MARGOLIS

Origami-like structures turning in six-dimensional space

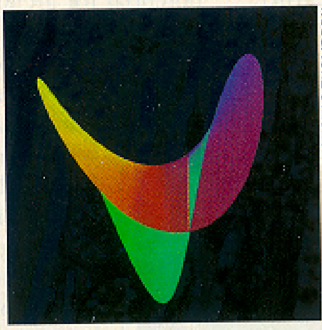
In these days of sophisticated video games, no one is surprised to see complex shapes moving around on a computer screen as a result of the twist of a dial or a joystick; but two decades ago, it was a very exciting phenomenon indeed. Strauss's program would take complicated architectural blueprints or designs for machine tools and display on the graphics screen not just the traditional front, top, and side views, but any desired view at all. It was fast enough to produce many views, each slightly different from the other, that could be shown one after another, giving an animated cartoon impression of a tour around the outside of a building that had not yet been constructed. If the view turned up something that didn't look quite right, it was possible to change the parameters and try again. This interactive design program had great potential for architects and engineers, who make their living in the three-dimensional world, but it also offered an excellent opportunity for a geometer who wanted to see objects from higher dimensions.

One of the most interesting things about computers is that they do not share the dimensional prejudices of human experience. A computer really doesn't know which way is up, unless we tell it—and we can tell it anything we choose. The way we communicate with computers about the location of points—say, the corners of a cube—is to use sets of numbers, giving each location an "address" like a house number on a street map. But the computer doesn't really know or care whether we provide the address for a point in three-dimensional space or one in a dimension that doesn't exist in our world.

To specify a point on a one-dimensional line, we need only one number. To give an address for a point on a two-dimensional plane, we need a pair of numbers—for example, the latitude and longitude. We can very simply describe a basic object like a square by giving four pairs of numbers (0, 0), (1, 0), (1, 1), and (0, 1). We can then locate these points on the grid of a blackboard or a graphics screen and draw the lines connecting pairs, which differ in just one coordinate—(0, 1) to (1, 0), for example, but not (0, 0) to (1, 1). A person or a computer following these instructions can reproduce the two-dimensional square.

Three-dimensional objects require a

little more work. For each corner of a cube, we require three coordinates, something akin to latitude, longitude, and depth. We get eight points, from (0, 0, 0) to (1, 1, 1) using all combinations of zeros and ones, and once again we can give the instructions to construct the framework of a cube by connecting corners with addresses that differ in just one coordinate. A model maker could follow these instructions and build a structure from sticks that could then be photographed to give a projection of the cube onto a two-dimensional plane. To get such a picture, however, we do not have to go through the work of the actual construction. If someone shows us the images of the three edges coming from a single point, we can draw the parallelograms that complete the figure, or we can instruct a computer to carry out the same task.



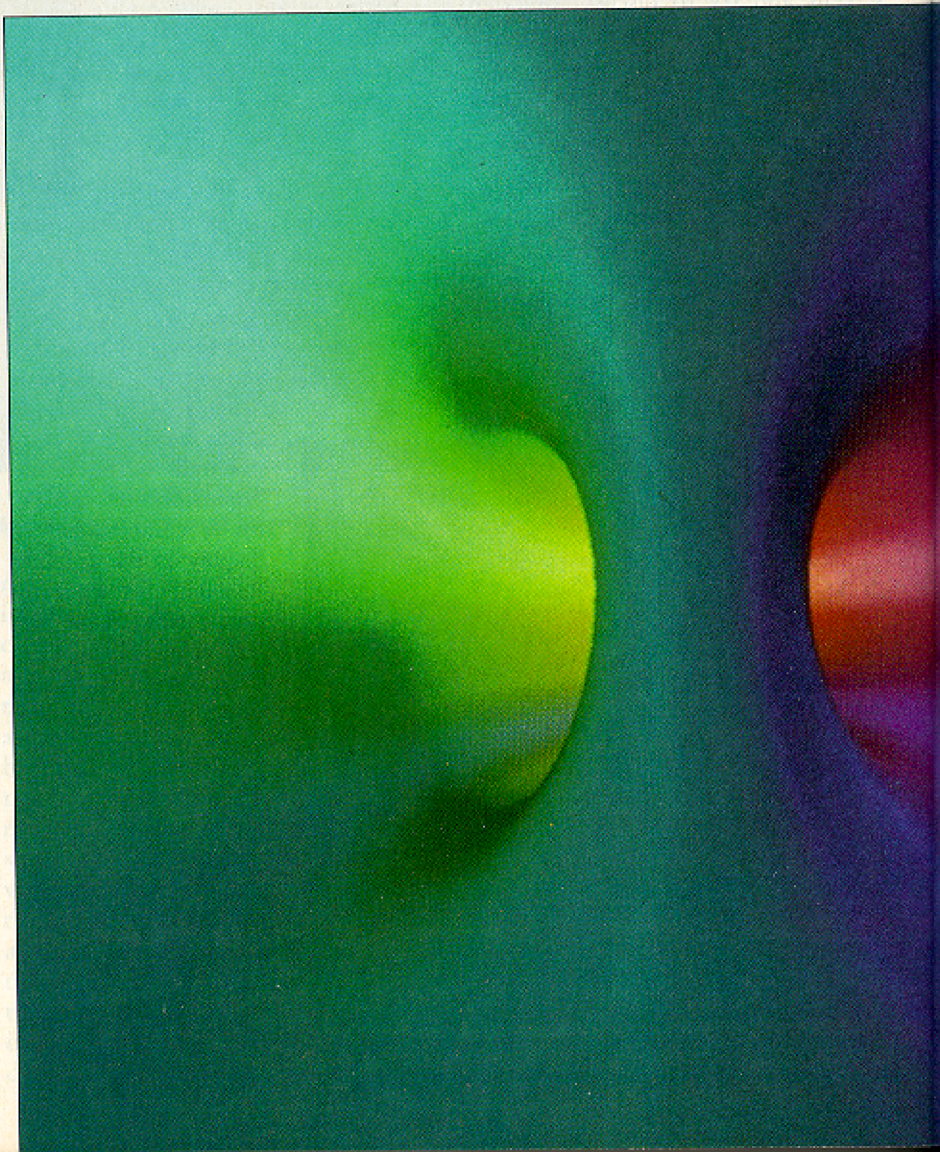
NICHOLAS THOMPSON

There are still challenges in interpreting these figures and seeing them as projections of a three-dimensional structure. For a complicated object, we might need a large number of views before feeling that we had seen enough for our purposes. Just a few views might suffice to orient us with respect to the plan of an entrance way, but many more might be required to understand the structure of a complicated molecule. The three-dimensional information is stored in the computer, and it is up to us to find ways of retrieving it on a screen so we can visualize it.

The surprising benefit of this exercise in seeing information is that we can use the same methods to deal with structures that cannot be built, for example, structures that require not three but four or more numbers to give the location of each point. In precise analogy with our description of the square and the cube, we can define a "hypercube" in four dimensions by enumerating all quadruples of zeros and ones to get 16 corners, and we can then use the same instruction as before to connect two corners if their addresses differ in exactly one coordinate. But how do we see this object if it is impossible for us to build it? Actually, we don't. What we see is a projection of the hypercube. Fortunately, the mathematics that helps us draw pictures of the cube makes it possible to draw almost as easily a collection of projections of the hypercube. Once we show the image of the four edges emanating from a single corner, we, or a computer, can complete the picture on a two-dimensional screen. By changing the positions of the edges slightly we can obtain a whole sequence of images which can then be interpreted as the projections of a hypercube rotating in four-dimensional space.

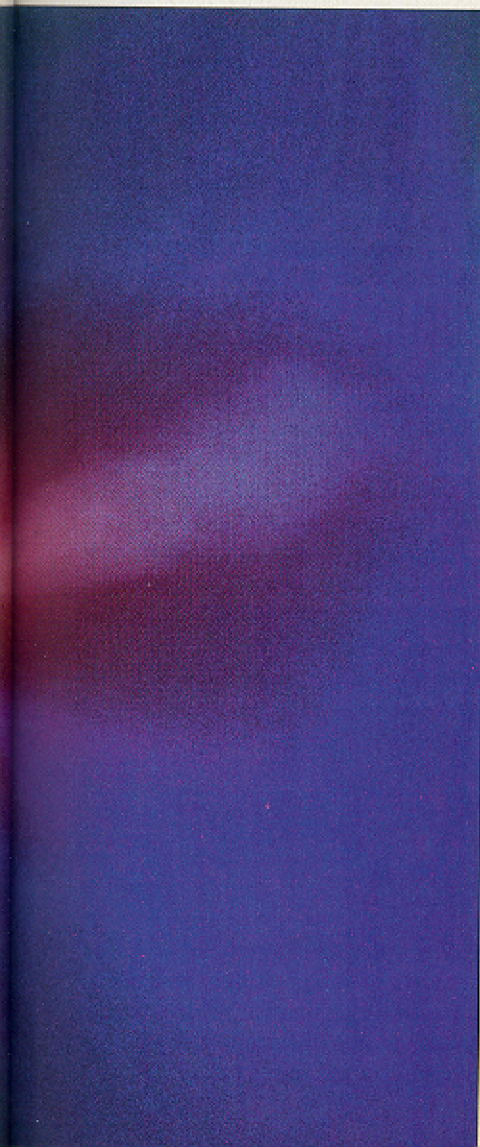
It takes a good deal of practice to see what is happening with these images, and that experience makes it possible to visualize more complicated data sets that involve four coordinates for each data point. It is this facility that makes computer graphics such a potent technique for analyzing the high-dimensional data sets of economics or biology, where observations sometimes include dozens of numbers detailing characteristics of a neighborhood or

Below: A hypertorus stretches to infinity as it turns in the fourth dimension. Left and right: Plotting the twists and turns of function graphs



an organism. The job of the researcher is to identify patterns and trends that can help in prediction and classification, and the methods developed for studying high-dimensional geometric objects are a valuable tool in carrying out this "exploratory data analysis."

A wonderful thing about interactive computer graphics was that we could not only simulate a walking tour around a building, but we could also enter into it to see its interior structure in new ways. Within a year after the start of our collaboration, Charles Strauss and I had produced our first computer-graphics film, a rotating hypertorus (a torus is a doughnut shape) that engulfed all of space as it turned in the fourth dimension. We have come back to those images again and again



DAVID LAIDLAW/HUSEYIN KOCAK

as our equipment has improved, but nothing has ever matched the thrill of seeing those wire-frame shapes moving through space for the first time. Just as the photographs of Neptune's moons will forever change our views of that planet's astronomy, the experience of seeing the projections of an object rotating in four-dimensional space totally altered the way we interact with important geometric objects.

The study of projections is just one of the important ways of dealing with high-dimensional objects. Another technique, slicing, has produced some of the most significant insights, especially when augmented by modern computer graphics.

NICHOLAS THOMPSON

The story of slicing began 105 years ago, when a busy Victorian headmaster used a dimensional analogy to challenge his students to cast off artificial limitations on their perspectives and to open their minds to new visions. Edwin Abbott Abbott wrote *Flatland*, in which he invented a plane world where all of existence was described in only two dimensions, north-south or east-west. There was no notion of up or down—a third direction perpendicular to the first two. Like amoebae floating on the surface of a still pond, the inhabitants of Flatland could sense only the edges of their neighbors. Drawing a circle around a treasure would be a sufficient way of sealing it off from the rest of the flat universe.

How mysterious it is when A Square (Abbott Abbott), the narrator of this tale, is visited by a round ball, a being from the higher and inconceivable third dimension who can never be seen all at once, but only as a changing series of circular slices as its spherical shape passes through the plane. From its higher-dimensional vantage point, the ball can see the contents of the locked two-dimensional treasure chest, and even see the inner organs of the Flatlanders in the same way we can observe the nucleus of an amoeba on a microscope slide. We gain some empathy with A Square as he struggles to visualize phenomena in a dimension that is beyond his abilities. We begin to appreciate how difficult it would be for us to come to terms with a visit from a "hyperball," plunging through our universe from a direction we cannot even imagine, appearing in our world as a growing and changing series of three-dimensional slices.

Abbott was a social and educational reformer who used satire to ridicule the shortsightedness of his contemporaries, especially with respect to the low place of women in Victorian society. His little book has become a classic. It is a social allegory, a challenging examination of ways of seeing and knowing and communicating, and a brilliant invitation to contemplate the idea of higher-dimensional space.

Slicing a cube is a more challenging exercise than slicing a ball, precisely because in the case of a cube the results depend on the direction of the slicing. If the slices are parallel to one of the faces, we just get squares. If we slice corner first, perpendicular to the long diagonal, then the slice begins as a point, then becomes a small triangle. Just before it disappears, it is a point, and just before that, another triangle. What do we get halfway through? Many people have difficulty figuring out that the answer is a hexagon, although it becomes perfectly apparent once we see the picture. Such exercises in visualizing three-dimensional shapes in terms of their two-dimensional slicing sequences are a necessary preliminary to appreciating the three-dimensional slicing sequences of a four-dimensional hypercube. Fortunately, the computer graphics programs are very good at figuring out what the slices will be and at displaying them on-screen. We can finally accept the "Flatland" invitation to explore the fourth and higher dimensions.

Abbott's masterpiece was written a

full generation before Einstein's theories brought out a different notion of a fourth dimension. So successful was this effort that many people today, when they hear the words "fourth dimension," think immediately of time. It is certainly true that, on some occasions, time is a very valuable fourth dimension. If we want to make an appointment to meet someone in New York, say on the corner of 3rd Avenue and 4th Street on the 5th floor, then (3, 4, 5) would be a convenient shorthand notation in a date book. But we would miss the appointment without a fourth number, the time—say 10 o'clock. The four coordinates (3, 4, 5, 10) would then determine a point in space-time. It is in this sense that physicists constructed the theory of relativity, where each event had four dimensions, three of space and one of time. It is significant to note that the physicists did not have to invent the mathematics for relativity—it already existed as an abstract theory, developed by geometers in the 19th century in at least six different countries.

In recent years, physicists have been studying spaces of dimension even higher than four. Some current theories need 11 or 26 dimensions. Again, it is fortunate that the abstract mathematics is in place to help in these investigations, and that the experience of visualization in lower dimensions can give some insights about the structure in situations where we must rely almost entirely on formal arguments.

It is undoubtedly true that in some cases time is *a* fourth dimension, but it is not *the* fourth dimension. There are many circumstances when we need four or more numbers to specify an observation, for example when a physician records height, weight, blood pressure, and cholesterol level, a sequence of four coordinates that tell something, not everything, to be sure, but something about a particular patient or group of patients. By visualizing pairs of numbers on a diagram, a researcher can identify trends or potentially dangerous configurations. By observing the variation of these patterns, she can diagnose an ailment and prescribe a treatment. Looking at three of these numbers at a time, she can study a richer class of relationships, using the projection techniques described above to obtain insights into the structure of this data set. But this collection of three-dimensional data points is still just a three-dimensional "slice" of the four-dimensional data set, and the true understanding of the whole complex of information depends on assimilating the entire entity. Just as we can slice the hypercube to reveal its many different aspects, we can also make use of this technique as we slice our data sets in different ways.

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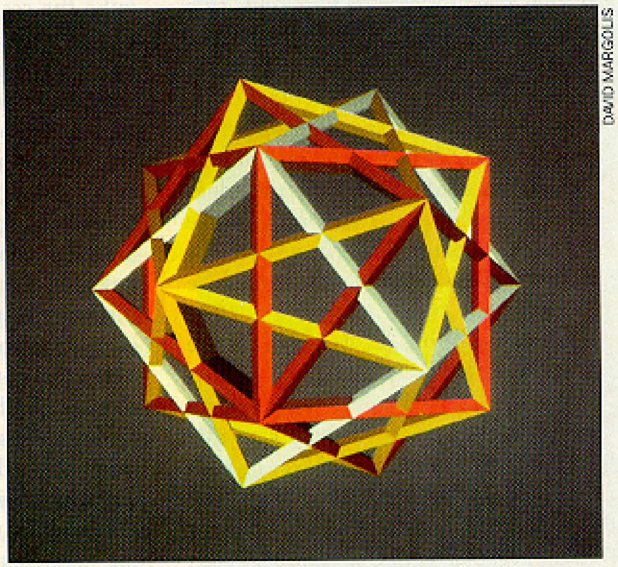

DAVID MARGOLIS

Five interlocking cubes in a dodecahedron

Take the brain, for example.

Magnetic resonance imaging, or MRI, is a relatively new and sophisticated technology that offers "snapshot" cross-section views of any section of the body, from a variety of angles. Assume, for a moment, that one two-dimensional view of a patient's brain shows a cancerous growth. Looking down on the brain from an imaginary vantage point above the crown—the so-called axial view—the physician is afforded a fairly informative, but still limited, view of the growth. But suppose the physician wants to monitor the growth, from top to bottom, front to back, and side to side—*and* to see it growing and changing in density, over time. Scientists are still at work trying to translate the surgeon's data into reference points on the computer—with time as the fourth dimension. Ultimately, we should be able to plot the course of a tumor as the patient undergoes treatment and visualize it from all angles. The MRI gives the surgeon an immense amount of data, but it only becomes information when structured in a certain way.

These techniques are already producing new results in collaborations at Brown University in such diverse areas as paleoclimatology, anthropology, modern dance, and surrealist art. We are using locally developed software in graphics laboratories that enables students to investigate geometric phenomena in two, three, and higher dimensions. We can even anticipate a time when this kind of visualization tool will be available in schools at all levels.

Our new access to the fourth and higher dimensions provides stimuli for all sorts of imaginations. It will continue to be an inexhaustible source of challenge and inspiration in all areas of visualization in the future. □