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REAL TIME COMPUTER GRAPHICS TECHNIQUES IN GEOMETRY

BY

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This paper contains three geometric examples, each of which could be given as an exercise to an undergraduate. Like most exercises, they represent illustrations of some mathematical phenomenon which had been observed by a teacher or textbook writer. What unifies the three and makes them noteworthy is that each was first observed by the authors while interacting at a graphics terminal with pictures produced by a computer. Once the observation has been made, it has not been difficult to write out the verification, so far anyway. The hope is that soon such methods will lead to more subtle observations and ultimately to some significant mathematical results. For now, these examples may serve as a demonstration of the power of computer graphics in suggesting geometric problems and in aiding in their solutions.

Example. **The evolute of the cardioid.** A graphics routine for parallel curves to curves in the plane takes in a curve in parametric form $X(t), a \leqslant t \leqslant b$, and a number n of equal subdivisions of the parameter domain and displays the parallel curve

$$Y_r(t) = X(t) + rN(t)$$

where the oriented distance r is put in by a control dial. The unit normal $N(t)$ is either given by a formula or is obtained by taking the average of the normal vectors to the edges adjacent to the point $X(t)$ in the polygonal approximation determined by the partition.

If $X(t)$ is smooth, and if $X'(t) \neq 0$ for all t, then, for sufficiently small r, the parallel curve $Y_r(t)$ is also smooth; but, for large r, the curve may develop cusps:

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$$\begin{aligned} Y_r'(t) &= X'(t) + rN'(t) \\ &= X'(t) + r(-\kappa(t)X'(t)) \\ &= (1 - r\kappa(t))X'(t). \end{aligned}$$

We get a cusp only when $Y_r'(t) = 0$, i.e., when $r = 1/\kappa(t)$. The locus of cusps of parallel curves is called the *evolute*, so the equation of $E(t)$, in the smooth case for $\kappa(t) \neq 0$, is given simply by

$$E(t) = X(t) + (1/\kappa(t))N(t).$$

We may also describe the evolute as the envelope of the normal lines, so that $E(t)$ is the limit as $h \longrightarrow 0$ of the intersection of the line $X(t) + uN(t)$ and the line $X(t+h) + uN(t+h)$. In a visual display which includes the segments from $X(t)$ to $X(t) + rN(t)$ for selected points along the parameter domain, the evolute appears as the fold curve where this strip intersects itself.

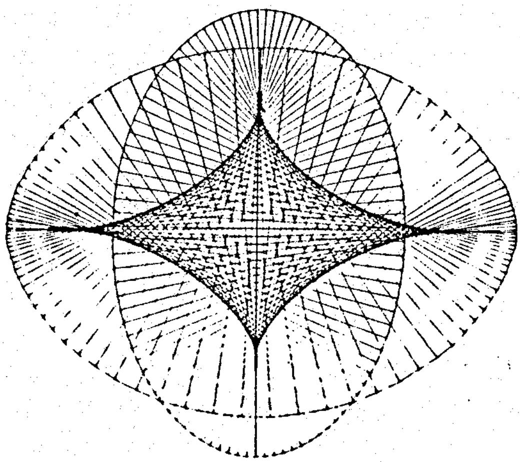

FIGURE 1

The visual display can suggest results which might not be apparent from the equations themselves. After an observation has been made, it is often straightforward to make the verification.

The cardioid is given in parametric form by

$$X(t) = (1 + \cos(t))(\cos(t), \sin(t)), \qquad 0 \leqslant t \leqslant \pi,$$

with tangent vector $X'(t) = (-\sin(t) - \sin(2t), \cos(t) + \cos(2t))$ which is nonzero except at the point $t = \pi$ where the curve has a cusp. The computer picture of the curve plus its normal lines suggested that the evolute itself was similar to the original curve, although rotated, dilated and translated. (See Figure 2.) Once this observation was made, it was not difficult to check.

By standard formulas of elementary calculus, we find

$$\|X'(t)\| = (2 + 2\cos(t))^{1/2}, \qquad \kappa(t) = \frac{3}{2}(2 + 2\cos(t))^{-1/2},$$

$$N(t) = (2 + 2\cos(t))^{-1/2}(-\cos(t) - \cos(2t), -\sin(t) - \sin(2t)).$$

Thus

$$\begin{aligned} E(t) &= X(t) + (1/\kappa(t))N(t) \\ &= \Big((1 + \cos(t))\cos(t) - \frac{2}{3}(\cos(t) + \cos(2t)), \\ &\qquad (1 + \cos(t))\sin(t) - \frac{2}{3}(\sin(t) + \sin(2t))\Big) \\ &= \left(\frac{1}{3}(\cos(t) - \cos^2(t)) + \frac{2}{3}, \frac{1}{3}(\sin(t) - \sin(t)\cos(t))\right) \\ &= \frac{1}{3}(1 - \cos(t))(\cos(t), \sin(t)) + \left(\frac{2}{3}, 0\right). \end{aligned}$$

Thus the evolute of the cardioid is another cardioid, scaled down by 1/3, rotated, and shifted by (2/3, 0).

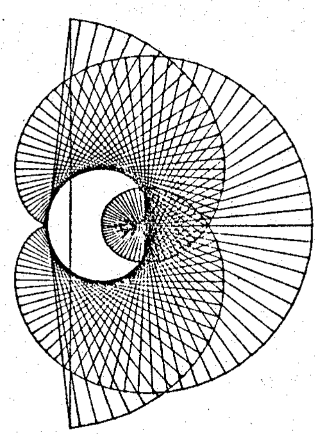

FIGURE 2

For a general discussion of such evolute curves, see for example the book of Lockwood [2].

2. Example. The graph of the complex exponential and its inverse. As part of a program to demonstrate the usefulness of the computer graphics routine in studying surfaces in 4-space, an investigation was carried out of the graph of $w = e^z$ in complex 2-space viewed as real 4-space. Letting $z = x + iy$ and $w = u + iv$, we obtain the locus

$$(x, y, e^x \cos(y), e^x \sin(y)).$$

The domain used was $-2 \leqslant x \leqslant 2, -2\pi \leqslant y \leqslant 4\pi$, cross-hatched by choosing equal subdivisions in the x- and y-domain. Projection to the (x, y, u) space gives the graph of the real part of e^z as a surface in 3-space, and projection to (x, y, v) gives the graph of the imaginary part. A smooth rotation in the u-v-plane through angle α corresponds to the graph of

$$(x, y, e^x(\cos(y)\cos(\alpha) + \sin(y)\sin(\alpha)))$$
$$= (x, y, \operatorname{Re}(e^{x+iy})e^{i\alpha}) = (x, y, e^x e^{i(y+\alpha)}).$$

A rotation of this locus into (x, u, v) space produced a surface which should have been expected but which as a matter of fact appeared as a surprise. Under a rotation in the y-v-plane, the surface wraps itself around into a surface of revolution, an exponential horn with equation

$$(x, e^x \cos(y), e^x \sin(y)).$$

(See Figure 3.)

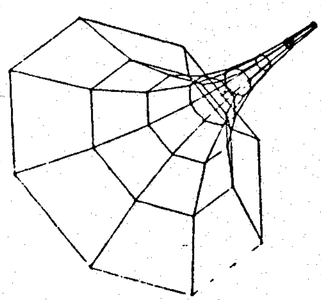

FIGURE 3

A further rotation in the x-y-plane brings the surface into a right conoid

$$(y, e^x \cos(y), e^x \sin(y)).$$

(See Figure 4.)

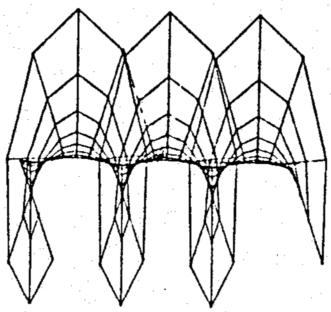

FIGURE 4

We can identify the significance of these last two projections of the surface in 4-space by reparametrizing in terms of u and v. The first becomes

$$(\ln((u^2 + v^2)^{1/2}), u, v)$$

and the second becomes

$$(\tan^{-1}(v/u), u, v).$$

But the functions $\ln((u^2 + v^2)^{1/2})$ and $\tan^{-1}(v/u)$ are precisely the real and imaginary parts of the function $\ln(w)$. This should have been anticipated since the locus (z, e^z) viewed another way gives the locus $(\ln(w), w)$. As in the case of a curve graphed in the Euclidean plane, an accurate graph of a function can be looked at from a different point of view to give the *inverse* function.

The domain of the inverse function $\ln(w)$ here is given by the image of the function e^z, where we have

$$e^{-2} \leqslant |w| \leqslant e^2 \quad \text{and} \quad -2\pi \leqslant \arg(w) \leqslant 4\pi,$$

a multiply covered domain in the w-plane.

3. Example. Deformation of a cusp of an algebraic curve. An algebraic curve in complex 2-space may be described by a relation between variables z and w, for example, $z^3 = w^2$.

Such a complex equation yields two real equations so the locus can be considered a real surface in real Euclidean 4-space. We may parametrize this locus by choosing a radial domain $\zeta = re^{i\theta}$, $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, and taking $z = \zeta^2$, $w = \zeta^3$. We then have a mapping of the disc into complex 2-dimensional space

REFERENCES

1. T. F. Banchoff and C. M. Strauss, *Real time computer graphics analysis of* $z^3 = w^2$ *in Euclidean* 4-*space*, Talk presented to the Annual Meeting of the Amer. Math. Soc., Dallas, January 1973. (With film)

2. E. H. Lockwood, *A book of curves*, Cambridge Univ. Press, Cambridge, 1971.

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