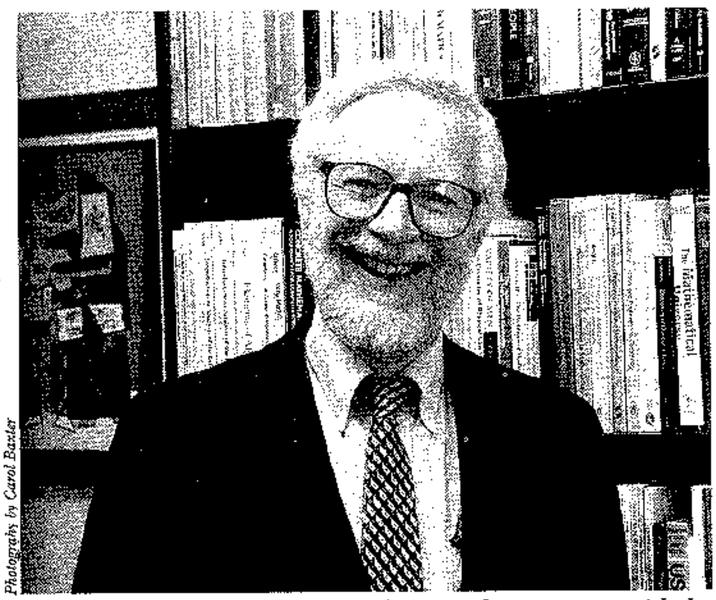
DONALD J. ALBERS

Tom Banchoff: Multidimensional Mathematician

Professor Thomas Banchoff is one of the first people on planet Earth to interact with the fourth dimension. Thanks to a blend of his powerful imagination and computer technology, he has made it possible for all of us to see beyond the three-dimensional space in which we live. The images that he displays on his computer screen are beautiful, exciting, and often surprising. To be more precise, what Banchoff enables us to "see" are three-dimensional shadows of four-dimensional objects. "We're trained from very early childhood," Banchoff explains, "to interpret the two-dimensional shadows of three-dimensional objects. We all learn to infer the shapes of three-dimensional objects from their shadows. So if we want to visualize a four-dimensional object, the best thing to do is work with three-dimensional shadows."



Photographs by Carol Baxter

Professor Tom Banchoff, just back from another encounter with the fourth dimension.

Captain Marvel

Banchoff has been fascinated with the fourth dimension since he was ten years old. He first read about it in a comic book, "Captain Marvel Visits the World of Your Tomorrow." One of the panels shows a boy reporter going into a futuristic laboratory where his guide says, "This is where our scientists are working on the seventh, eighth, and ninth dimensions." A thought balloon rises above the hero's head: "I wonder whatever happened to the fourth, fifth, and sixth dimensions." Banchoff, too, has been wondering ever since. He decided to keep trying until he understood the fourth dimension completely or until it became boring. Very soon he realized that he would never be able to figure it all out, and that it would never get boring.

He really got into thinking about the fourth dimension as a student at Trenton Catholic Boys' High School. By the time he was a sophomore, he had developed a full-fledged theory of the Trinity. He remembers one afternoon cornering Father Jeffrey, a sympathetic biology teacher, and trying as hard as he could to explain his theory—if God came from the fourth dimension into our three-dimensional world, all we would see is a 'slice' person who would look like us, but there would still be two other parts of God that we couldn't see, and that's where the Trinity comes in. Father Jeffrey was amused by his earnestness and asked why it was so important for the theory to be validated that day. He answered, "Because tomorrow I'm going to be sixteen years old."

Banchoff grew up in Trenton, New Jersey. His father, who was a payroll accountant, impressed upon him the fact that English and arithmetic were the most important subjects in the curriculum. He was also very concerned that Tom should be a "regular guy." His father was actually rather suspicious of intellectuals. He knew people who read books and discussed them, but he himself wasn't a reader. His favorite example was a middle-aged, somewhat eccentric neighbor who spent most of his time in the library and carried his laundry in a paper sack. "My father warned me," recalls Banchoff, "You don't want to grow up like him." On the other hand, his mother, a kindergarten teacher, was a great reader and was very encouraging.

So Tom proceeded to become a regular guy. He played on the tennis

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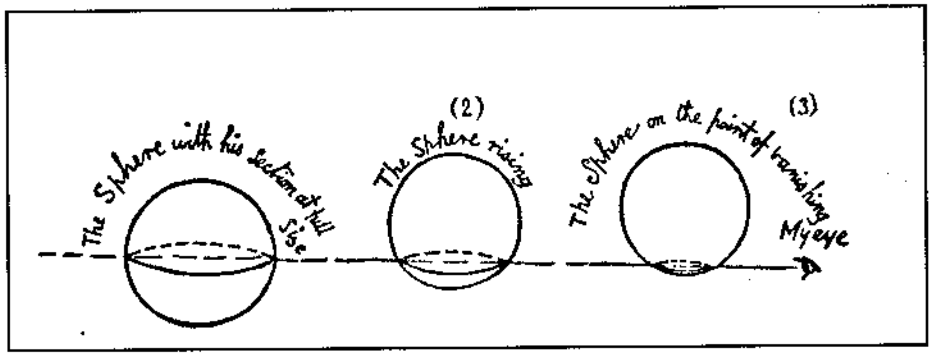
Flatland

In 1884, a British schoolmaster, Edwin Abbott Abbott, wrote the classic introduction to the dimensional analogy. His small book *Flatland* is narrated by A Square, living on a two-dimensional flat universe and as incapable of comprehending our geometry as we are when we try to conceptualize a fourth spatial dimension. We are invited to empathize with the experiences of A Square, first of all in his own two-dimensional world, a social satire on Victorian England, and then as he is confronted with a visitation by a being from a higher dimension, A Sphere, who passes through his universe, giving to the two-dimensional onlooker the impression of a growing and changing circular figure. This experience challenges A Square to rethink all that he had previously taken for granted about the nature of reality. Analogously, we are challenged to imagine the experience of being visited by beings from a fourth spatial dimension.



As A. Sphere passes through Flatland, the two-dimensional section changes, starting as a point, reaching maximal extent as a circular section, and then reducing to a point as it leaves flatland.

and soccer teams. In addition, he was in the school play, orchestra, band, and debating team as well as editor of the school paper and the yearbook. At his high school, it was considered all right to get high grades as long as it was clear that you weren't spending all the time studying, so he maintained a 99+ average over four years. He won the regional science fair, represented his district in the National Student Congress, and was in the first class of National Merit Scholars. And he was an altar boy, too!

In January 1996, Banchoff won one of the MAA's Distinguished Teaching Awards. He cares deeply about teaching and is proud of his award. He always knew that he would be a teacher, but he wasn't sure at first what it was he wanted to teach. For several years it was a toss-up between mathematics and English. At Notre Dame, he remembers, "In mathematics when I made a mistake and the teacher pointed out a counterexample or a flaw in the argument, I could accept that. I didn't have the tolerance for ambiguity that was necessary to become a scholar of literature. But in mathematics, I knew by that time that I could come up with original ideas, and the courses definitely were challenging. So I decided to become a mathematics teacher."

Many teachers influenced Banchoff. In his freshman year of high school, Father Ronald Schultz stood out. "Although I never took a class from him, he was the first one who really listened to my mathematical ideas and encouraged me, especially in geometry."

Shadows

Banchoff discovered his first geometry theorem as a freshman. "Every Friday morning, the whole school would file into church for Mass, and our home room was the first to enter. While waiting for the rest to come in, there was plenty of time to contemplate the shadows advancing across the tiles at the base of the altar rail. When we first arrived, the narrow of the altar rail covered only a small portion of the triangular tiles, and by the end of Mass, almost the entire triangle was in shadow. When, I asked myself, did the shadow cover half the area? I hadn't studied any formal geometry yet, but I figured that if you cut an isosceles right triangle in half by a line perpendicular to the hypotenuse, then one of those halves could be rotated to give the triangle that remains when the shadow was covering half of the original triangle. It surprised me that the line did not pass through the centroid of the triangle! To this day, I still use that example when I teach calculus students about centroids."

Another early influence was Herbie Lavine, who was three years older than Banchoff and "real smart." Herbie worked in his father's grocery store, and when Tom was in grammar school, he would teach him the mathematics he was learning. His father would remind him that he was supposed to be unloading packing crates and not doing algebra on them! When Tom was in seventh grade, Herbie told him about a classic problem involving twelve billiard balls, one of which was either heavier or lighter than all the rest. How could you find the 'odd ball' in three weighings using just a balance scale? Tom couldn't solve it right away and Herbie was about to show him the answer. Tom said "no," he wanted to work it out by himself. "We both forgot about it," Banchoff recalls, "and soon afterward Herbie went off to college, while I started high school. Once again it was at one of the Friday morning masses that I received an inspiration: a pattern on one of the stained glass windows reminded me of the billiard ball problem from three years earlier, and the pattern gave me the idea for solving it. I sent my solution off to Herbie at the University of Michigan, and got a letter back saying it was right. It made me feel good to know that I could solve a problem that took a long time, and not just the usual problems that you can either do immediately or not at all." (Herbie went on to become an actuary and a professional bridge player. He and his wife and son visited Brown a couple of years ago, and Tom took them out to lunch. He astounded them by telling them that Herbie had been his mathematical hero when he was young.)

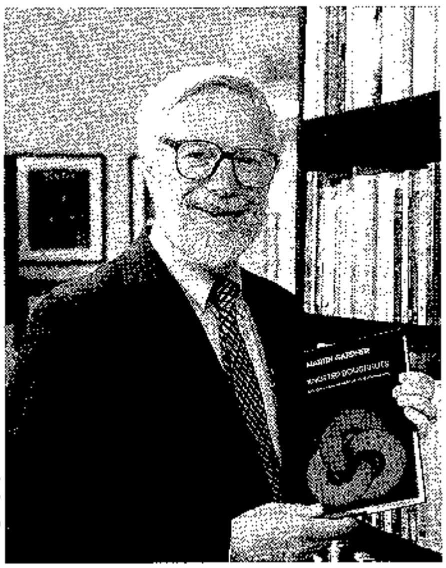
In high school, one of the things Tom liked most about mathematics was that he was asking questions that were different from the ones that his classmates and teachers were asking. "When I got to college, I realized that was still true. I knew that most of the things I observed had been seen before, but I thought even then that maybe I might have some insights that nobody else would have, that I would prove something that nobody ever would have thought of if I hadn't done it. And that was very appealing to me. I loved the creative aspect of mathematics. I was lucky enough to realize something about the creative aspect of mathematics when I was young Individually, the theorems I proved are almost trivial things, but I remember them very clearly. Curiously enough some of them keep showing up—I'm still watching shadows and cutting things in half!"

During his senior year of high school, Banchoff's math project on three-dimensional graphs of complex valued functions of a real variable won the regional science fair and earned him a trip to the National Science Fair in Oklahoma. On his return trip, he rerouted himself so he could make his first visit to the University of Notre Dame, where he had just been admitted. He met the Dean of the Arts and Letters College who introduced him to the well-known research mathematician, Professor Ky Fan. When he started explaining his science project, Fan interrupted him to say that he should spend his time learning mathematics, not trying to do original projects.

The next person he met was the mathematics department chairman, Dr. Arnold Ross. "I hear you are interested in becoming a mathematician," said Ross. "I was until five minutes ago," responded Banchoff. "Oh? Tell me about it," he said in his very fatherly way. He listened as Banchoff explained his project, and then said, "There is a mistake in this expression for the fourth root of -1. Go up to the blackboard right now and find the correct form." I said, "You mean, go to the blackboard right now and just do it?" He nodded, so I went up and figured it out. As I turned around, as surprised as I was proud, he smiled. Just like that I wanted to be a mathematician again."

You will never be a mathematician

As a senior at Notre Dame, Banchoff received from Ky Fan the only C of his life in a second year graduate course in general topology. He didn't realize at the time that he didn't really have enough background in advanced analysis to appreciate the very formal generalizations and proofs. "It was in that class that I began to appreciate how it feels to be lost most of the time. I was also taking courses in literature and philosophy at the time, and Dr. Fan had formed the impression I was not a serious mathematics student. I was so intimidated by the lectures that I would study the notes from the previous year before each class just in case he would ask a question. One day my fears were realized. He turned and said, 'Banchoff, what is a set of the second category?' I started my answer, "Well, a set of the..." and that was as far as he let me get. He shouted back, "'Well, you say 'Well'? This is not an English class, this is a mathematics class. When I ask a mathematical question I want a mathematical answer, not 'well'." I then responded: "A set of the second category is a set that cannot be expressed as a countable union of nowhere dense sets." "Why didn't you say that the first time? 'Well'!" It was pretty clear that the chemistry between us wasn't very good."

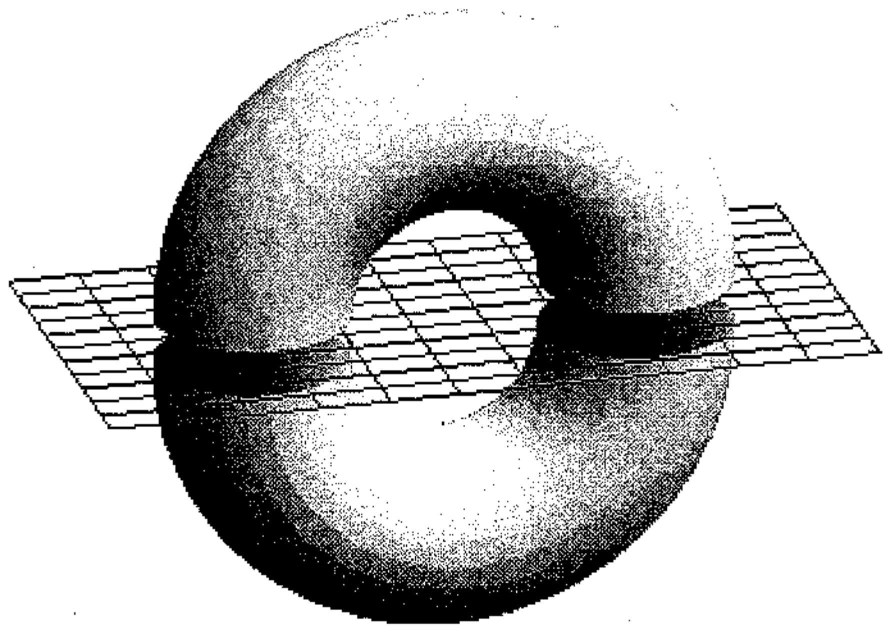

Photography by Carol Baxter

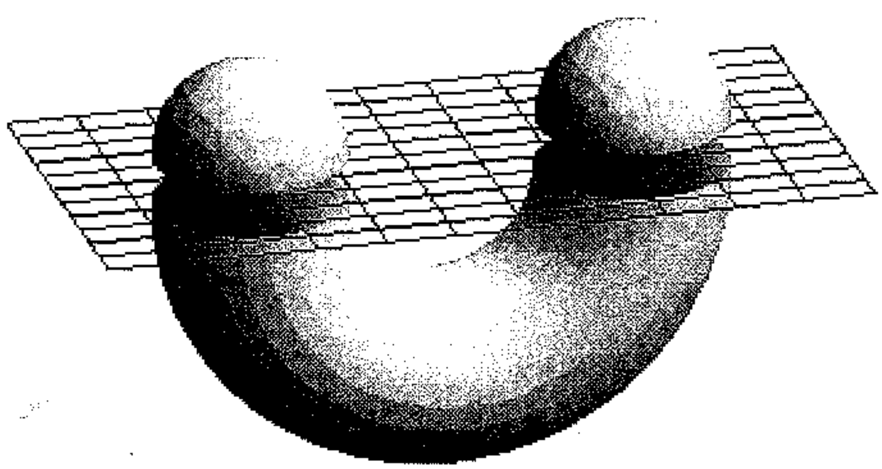
Banchoff is particularly fond of shadows, especially those of four-dimensional objects.

He stayed in that course for the second semester and gradually began to catch on so that he was able to raise his grade to a B. After missing class one day, he managed to infuriate Dr. Fan by a question he asked him in the hallway after class. At the top of his lungs he delivered his estimation of Banchoff, loud enough for the whole department to hear: "Mr. Banchoff, you will never be a mathematician, never! never!" At that moment he seriously wondered if he was in the right field.

After receiving his bachelor's degree in 1960, Banchoff began graduate studies at the University of California, Berkeley. Algebraic and combinatorial topology with topologist Edwin Spanier and differential geometry with visiting professor Marcel Berger introduced him to the areas that became his specialty. He became a research assistant to the differential geometer Shiing-Shen Chern who suggested a thesis project in total absolute curvature, the study of surfaces like a torus of revolution, that are "as convex as they can be." After making a great many drawings and models, he found an elementary way of interpreting that condition called the Two-Piece Property (TPP), and he found himself in the unusual position of being able to explain what his thesis topic was about, even to non-mathematicians. Using the TPP, he could consider polyhedra as well as smooth surfaces, and he came up with some models to show the difference between the smooth and the polyhedral cases. He was making some progress, but he still didn't have a big breakthrough.

One day Professor Chern told Banchoff that he wanted to introduce him to Nicolaas Kuiper, a special visitor from Holland. "You two think alike" he said. Kuiper showed him some of his

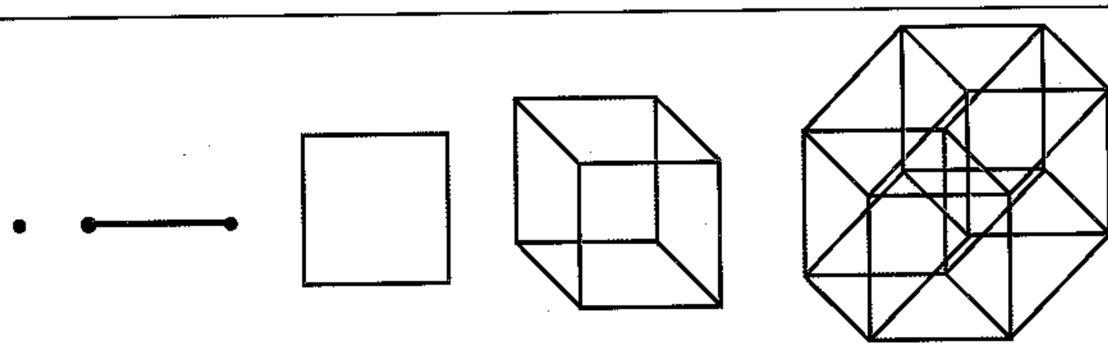




Two-Piece Property

Certain items on a breakfast table have the property that they fall into at most two pieces when they are sliced with a long straight knife, for example an orange, or a hard-boiled egg. Other objects, like a fork or a sufficiently curved banana will fall into three or more pieces when cut in a certain direction, so they do not have the two-piece property (TPP). Any convex object has the TPP, but there are also non-convex objects with this property, for example a doughnut or bagel;, or a half-cantaloupe, or a stemless apple (but not a pear or a peach). The study of smooth surfaces that have the TPP involves "total absolute curvature," a notion from differential geometry, the subject that applies calculus techniques to problems in the geometry of curves and surfaces in the plane, in three-space, and in higher-dimensional spaces. A closed curve with the TPP must lie in a plane, whether it is smooth or polygonal. By a theorem of Nicolaas Kuiper, a smooth surface with the TPP has to lie in a five-dimensional space, but, surprisingly, there are polyhedral surfaces with the TPP in six-space that do not lie in any five-dimensional subspace. More generally, in n-space there are TPP polyhedral surfaces not lying in any space of lower dimension, although for higher and higher dimensions, the surfaces must become more and more complicated. This is the primary contribution of Thomas Banchoff's Ph.D. thesis.

Images courtesy of Tom Banchoff. These images may be viewed on Banchoff's World Wide Web Page. WWW address: http://www.geom.umn.edu/~banchoff/



Hypercube

What does a shadow of a four-dimensional cube look like? To draw a two-dimensional shadow of a three-dimensional cube, we start with a parallelogram that is the shadow of a face of the cube, then move the parallelogram along a third direction in the plane and connect the corresponding vertices. Similarly we can obtain a three-dimensional shadow of a four-dimensional cube, or "hypercube," by moving a parallelepiped along a fourth direction in three-space and connecting the corresponding vertices. If we collect the shadows of an ordinary cube as it rotates, then we create an animated film that we learn to interpret as the images of a cube. In a similar way, if the hypercube rotates in four-space, then its shadows in three-space will produce an animation that we can interpret first as shadows of an object moving in three-space, but then some unfamiliar movements occur. With a good deal of experience it is possible to predict the changes that occur in the shadows of a rotating hypercube.

Illustration by Tom Banchoff

recent papers on smooth surfaces with minimal total absolute curvature, and told him about his key result that says there are no smooth examples of this phenomenon in dimensions higher than five. He suggested that Banchoff might find the polyhedral analogues of these theorems, and so he went to work.

Benefits of Doing Laundry

A week later, while Kuiper was still visiting, he was folding his wash in the laundromat, and at the same time trying to come up with an argument to show why there were no TPP polyhedral surfaces in six-dimensional space, when all of a sudden, he saw how to construct one! He made a paper model that could be folded together in six-space and showed it to Kuiper the next day. He was astonished. He said to Banchoff, "What you have here is a gold mine. I'll give you six months to write a thesis about it. If you haven't done so, I'll give the problem to one of my students. It's too good a problem not to be done by somebody." He finished the thesis project in three months, and he also started studying Dutch. He knew where he wanted to do his postdoctoral work!

After receiving his Ph.D. in 1964, he was a Benjamin Peirce Instructor at Harvard for two years. He then spent a year with Kuiper in Amsterdam, and continued to work with him as a colleague up until his death in 1994.

Shortly after taking up his faculty position at Brown in 1967, Banchoff met Charles Strauss, an applied mathematician with special talents in computer graphics, which was then a brand new field. Strauss was looking for new problems for his interactive "three-dimensional blackboard," and it was clear that these new programs could not only show two-dimensional images of complicated three-dimensional objects, but also they could produce rotating shadows of objects from four-dimensional space. Together, Banchoff and Strauss produced a series of computer-animated films, the most notable being "The Hypercube: Projections and Slices," first shown at the International Congress of Mathematicians in Helsinki in 1978. It is a grand tour of a basic four-dimensional object that has never been built and never can be built in our three-dimensional space.

Almost a quarter century has passed since Banchoff began using computers to enhance his interaction with the fourth dimension. Tremendous advances with hardware and software have enabled Banchoff to help all of us see beyond the third dimension

"Right up until the time I got my Ph.D., I had this recurring nightmare that my advisor Professor Chern would run into Dr. Fan and that my name would come up in their conversation. Then Professor Chern would come back and tell me, regretfully, that he had learned that I would never be a mathematician, never, never. As it happens, several years later I saw Dr. Fan at a mathematics meeting in New Orleans. I went over and introduced myself as one of his former students. He tried to place me. "You were in my freshman course?" "No, you're thinking of Jim Livingston." "You were interested in the four-color problem?" "No, that was Jim Wirth." "Ah, yes, 'Banchoff'" He paused, then he said, "Mr. Banchoff, I am happy to see that you have developed into a mature mathematician." ■

Available Films

"The Hypercube: Projections and Slicing," International Film Bureau, 332 South Michigan Ave., Chicago, IL. 60604, (312) 427-4545

"The Hypersphere: Foliation and Projections, and Fronts & Centers," The Great Media Company, PO Box 98, Nicasio, CA 94946 (415) 662-2426