GLOBAL EXISTENCE FOR QUASILINEAR DISPERSIVE EQUATIONS

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1. General strategy

We are faced with a quasilinear problem with a loss of derivative. Fortunately, we can easily get energy estimates which allows us to get local existence of a solution, a criterion for blow up and a control on high energy norms:

Proposition 1.1. *There exists ε_0 such that: i) (local existence) If the initial data to the problem under consideration satisfies*

$$\|U(0)\|_{H^{10}} \le \varepsilon_0,$$

then there exists a unique local solution $U \in C([0,1], H^{10})$ such that

$$\|U\|_{C([0,1]:H^{10})} \le 10\varepsilon_0.$$

ii) (propagation of regularity) If $U \in C([0,T] : H^{10})$ is a solution of the problem satisfying

$$\|U\|_{L^1([0,T]:W^{5,\infty})} \le 100\varepsilon_0, \tag{1.1}$$

for some $T > 0$ and if $U(0) \in H^N$ for some $N \ge 10$, then $U \in C([0,T] : H^N)$ and

$$\|U\|_{C([0,T]:H^N)} \le 2\|U(0)\|_{H^N}.$$

This gives us at the same time the local existence of solution and a nice blow-up criterion: we will be able to extend the solution as long as (1.1) is bounded a priori. We will not dwell much on this question, but refer to [2] for more details.

In our situation, Proposition 1.1 is essentially a consequence of the following simple lemma:

Lemma 1.2. *For each $N \ge 0$, there exists high-energy functionals such that*

$$\|U\|_{H^N}^2 \lesssim \mathcal{E}_N \lesssim \|U\|_{H^N}^2$$

so long as

$$\|U\|_{H^{10}} \lesssim \varepsilon_N.$$

and for any smooth solution of our problem, we have the inequality

$$\partial_t \mathcal{E}_N \lesssim \|U\|_{W^{5,\infty}} \mathcal{E}_N.$$

The construction of energies for the Euler-Maxwell problem relies on variations of the physical energies. On the other hand, we also remark that for some complicated problems, the energies can be very hard to construct.

As a conclusion, we have reduced the task of proving global existence for solutions to the proof of the following proposition:

Proposition 1.3. *Fix $N \ge 10$. There exists $\varepsilon_1 > 0$ such that if $U \in C([0,T] : H^N)$ is a solution on some time-interval $[0,T]$ such that*

$$\|U\|_{L^\infty([0,T]:H^N)} \lesssim \varepsilon_1, \tag{1.2}$$

and that $U(0)$ satisfy proper assumptions, then

$$\|U\|_{L^1([0,T]:W^{5,\infty})} \lesssim \varepsilon_0. \tag{1.3}$$

Thus, we only have to show *a priori* (1.3) under the additional assumption (1.2). If we take N large enough, the imbalance in regularity allows us to consider derivatives ass essentially bounded operators and the main focus is to get a norm which is integrable. This is the purpose of the semi linear analysis, which is the main difficulty.

2. Semilinear analysis

2.1. **Reduction to the boundedness of bilinear operators.** The main purpose of the semi linear analysis is to obtain the key a priori bound (1.3). Sometimes, especially in low dimensions (d=2), it can be replaced by the slightly weaker condition

$$\sup_{T>0} \|U\|_{L^1([T,2T]:W^{5,\infty})} \lesssim \varepsilon_0,$$

but this is only consistent with $\mathcal{E}_N$ growing logarithmically. In dimension 1, one never even gets the latter estimates and one needs to refine the energy method.

The previous analysis was very dependent on the precise structure of the nonlinearity in order to obtain key cancellations, but rather blind to the precise shape of the equilibrium at which we linearize. In contrast, the semilinear analysis will be very strongly dependent on the linearized system (and hence to the precise shape of the equilibrium at which we linearize), but will be mostly independent of the precise nonlinear structure[1].

In order to understand what can be the most important information to control the nonlinear evolution, we first conjugate by the linear flow. This is most conveniently done after diagonalizing the linearized system which leads to recast the equation as follows:

$$(\partial_t + i\Lambda_j)\, U_j = \mathcal{N}_j(\vec{U}_+, \vec{U}_-), \quad 1 \leq j \leq k$$

where Λ_j are the eigenvalues of the linear system and $\mathcal{N}_j$ are explicit but fairly complicated nonlinear functions[2] of $\vec{U}_+ = (U_1, \dots, U_k)$ and $\vec{U}_- = (\overline{U_1}, \dots, \overline{U_k})$.

A typical example is given by semilinear Klein-Gordon phases with quadratic nonlinearities:

$$\Lambda_j = \sqrt{m_j^2 - c_j^2 \Delta}, \qquad \mathcal{N}_j = \sum_{p,q} c_{j,p,q} U_p U_q + \sum_{p,q} d_{j,p,q} \overline{U_p} U_q.$$

When the system is in this form, we can then easily change variables $\vec{U} \to \vec{V}$ to account for the unknown following the linear flow at first order:

$$U_j(x,t) = e^{-it\Lambda_j} V_j(x,t).$$

We remark that in the absence of nonlinear perturbations, we have that $\partial_t V_j \equiv 0$ and the *linear profiles* V_j are constant. However, the presence of the nonlinear terms will force V_j to be time-dependent, but we may still hope that these profiles will only vary little (say, remain in a compact set).

Assuming that $\mathcal{N}$ is quadratic, the Duhamel formula gives an integral equation for V:

$$\widehat{V}_j(\xi,t) = \widehat{V}_j(\xi,0) + \sum_{p,q} \int_{s=0}^{t} \int_{\mathbb{R}^3} e^{is\Phi^{j;p,q}(\xi,\eta)} m_{j;p,q}(\xi,\eta) \widehat{V}_p(\xi-\eta,s) \widehat{V}_q(\eta,s) d\eta ds,$$

$$\Phi^{j;p,q}(\xi,\eta) = \Lambda_j(\xi) - \Lambda_p(\xi-\eta) - \Lambda_q(\eta),$$

[1] A possible exception being the presence of *null-forms* that can sometimes cancel the most delicate part of the nonlinearity. This is very important in particular for pure wave equations and in lower dimensions.

[2] In the sense that $\mathcal{N}_j(0) = 0$. Usually we can in fact take $\mathcal{N}$ to be polynomial in $\vec{U}$, and in dimension 3, often it suffices to consider the quadratic terms.

We thus see that the interaction $V_p \times V_q \to V_j$ is fully described by the phase $\Phi^{j;p,q}$ and the multiplier $m_{j;p,q}$ through the bilinear interaction

$$\mathcal{F}B^{j;p,q}[f,g](\xi) = \int_{s=0}^{t}\int_{\mathbb{R}^3} e^{is\Phi^{j;p,q}(\xi,\eta)} m_{j;p,q}(\xi,\eta)\widehat{f}(\xi-\eta,s)\widehat{g}(\eta,s)d\eta ds \tag{2.1}$$

and these are the operators that will concern us in the semi linear analysis.

These consideration brings us to the following problem:

Find a norm B such that

$$\|e^{it\Lambda_j}V\|_{W^{5,\infty}} \lesssim t^{-1-\epsilon}\|V\|_B$$

and such that the operators corresponding to the relevant interactions $B^{j;p,q}$ are bounded[3]

$$B^{j;p,q} : B \cap H^N \times B \cap H^N \to B.$$

for all j,p,q.

2.2. **Localization.** In order to understand exactly which information we will be using in our space B, we may rewrite the bilinear interaction through its (frequency localized) kernel:

$$\begin{aligned}
B^{j;p,q}_{k,k_1,k_2}[f,g](x) &= P_k\int_{s=0}^{t}\int_{\mathbb{R}^3\times\mathbb{R}^3} K(x,y_1,y_2,s)\cdot P_{k_1}f(y_1,s)\cdot P_{k_2}g(y_2,s)dy_1dy_2ds\\
K(x,y_1,y_2,s) &= \int_{\mathbb{R}^3\times\mathbb{R}^3} m_{j;p,q}(\xi,\eta)\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)e^{is\Phi^{j;p,q}(\xi,\eta)}e^{i\xi[y_1-x]}e^{i\eta[y_2-y_1]}d\eta d\xi,\\
\varphi_k(\xi) &= \varphi(2^{-k}\xi), \qquad \widehat{P_kf}(\xi) = \varphi_k(\xi)\widehat{f}(\xi).
\end{aligned} \tag{2.2}$$

The formula suggests that an important information will be the localization of the functions. Indeed, for example, assume that we consider a point x out side of the light cone $|x| \gg s$. Then, looking at the gradient of the oscillatory phase

$$\begin{aligned}
\nabla_\xi\left[s\Phi^{j;p,q}(\xi,\eta)+\xi[y_1-x]\right] &= s\nabla_\xi\Phi^{j;p,q}(\xi,\eta)+[y_1-x],\\
\nabla_\eta\left[s\Phi^{j;p,q}(\xi,\eta)+\eta[y_2-y_1]\right] &= s\nabla_\eta\Phi^{j;p,q}(\xi,\eta)+[y_2-y_1].
\end{aligned}$$

Since $|\nabla\Phi| \lesssim 1$, we see upon integrating by parts that we have rapid decay unless y_1 has a similar location as x and similarly for y_2. Thus, we see that the interactions are essentially pointwise at scales larger than s (in the sense that whatever is in a ball of radios s is essentially only affected by what is in a ball of twice the size). This is essentially the finite speed of propagation. It allows us to treat any factor x^θ in front of the output informally as a power of t^θ in our estimates.

In addition, we can refine this finite speed of propagation: if on the support of the integral, we have that $|\nabla_\xi\Phi^{j;p,q}| \lesssim c \ll 1$, then either $|x| \le cs$, or $|x-y_1| \le 10cs$. This is very useful when c is small and essentially means that we will not consider the regions where $\nabla_\xi\Phi = 0$. However, the regions where $\nabla_\eta\Phi = 0$ will play a very important role[4].

[3]It would be even better to get a space B such that $T : B \times B \to B$ is bounded. However, this appears difficult in *quasilinear* problems due to the loss of derivative induced by looking at the system as a perturbation of the full problem when the nonlinearity contains derivative of similar order.

[4]A way to understand this discrepancy between the role of derivatives in ξ and η is from the fact that among the worst interactions is the one when two inputs f and g are localized at the origin ($|y_1| \ll s$ and $|y_2| \ll s$ and produce an output located away from the origin: $|x| \gtrsim s$. Indeed, in this "low-low $\to$ high" case, we have to recover a penalization from the large distance of the output location with respect to the origin (because of the norm we want to consider), but this penalization cannot be recovered from information on the inputs. This contradicts $\nabla_\xi\Phi = 0$ but not $\nabla_\eta\Phi = 0$.

Finally a last advantage in keeping track of the localization comes from the fact that we can use it to get time-decay from the kernel:

$$K(x,y_1,y_2,s) =$$
$$\frac{i}{s}\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{is\Phi^{j;p,q}(\xi,\eta)}e^{i\xi[y_1-x]}\varphi_k(\xi)\partial_\eta\left\{\frac{1}{\partial_\eta\Phi^{j;p,q}(\xi,\eta)}m_{j;p,q}(\xi,\eta)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)e^{i\eta[y_2-y_1]}\right\}d\eta d\xi,$$

We see that, essentially, when $\nabla_\eta\Phi^{j;p,q}\neq 0$, this allows to get a factor of $1/s$ at the expense of a factor of y_1-y_2.

A related but different fact is that there is a simple way to obtain integrability of the unknowns U from localization of V through dispersion by the inequality:

$$\|e^{it\Lambda}P_kV\|_{L^\infty}\lesssim t^{-\theta}2^{\beta k}\|x^\theta P_kV\|_{L^2}$$

for some β and $0<\theta<3/2$.

Note that there is another tool we can use to bound the interactions which has to do with time-oscillations: the *normal form transformation.*

2.3. **Relevance of non degenerate space-time resonant surfaces.**

2.3.1. *"Generic" space time resonant sets in* 3*d.* This discussion is also very neatly explained in [1]. In general, the phase-space $\mathbb{R}^3_\xi\times\mathbb{R}^3_\eta$ is of dimension 6. However, according to previous heuristics, the set that matters most is the set where the phase is stationary:

$$\mathcal{R}=\{\nabla_\eta\Phi(\xi,\eta)=0\}\cap\{\Phi(\xi,\eta)=0\}.$$

In general, this should be a co-dimension 4 set in phase space. However, if the problem is *isotropic* as is the case for the Euler-Maxwell problem, all the eigenvalues are also radial and any set built on these will automatically be invariant under the (joint) action of $O(3)$:

$$(\xi,\eta)\in\mathcal{R}\Rightarrow(R\xi,R\eta)\in\mathcal{R},\quad R\in O(3).$$

Therefore, the stationary set is either a point or a union of 2-dimensional spheres. The set of *non degenerate* stationary points

$$\mathcal{R}\cap\{\det\nabla^2_{\eta\eta}\Phi(\xi,\eta)=0\}=\{\Phi(\xi,\eta)=0\}\cap\{\nabla_\eta\Phi(\xi,\eta)=0\}\cap\{\det\nabla^2_{\eta\eta}\Phi(\xi,\eta)=0\}$$

can be expected "generically" to be a strictly smaller manifold surface with similar invariance property, therefore empty. Thus in many three-dimensional cases, we can expect that the "space-time resonant set" is *non degenerate but not empty.* Note however, that if the degenerate stationary set is not empty, then it is the whole stationary set[5] (or at least a sizable portion of it).

2.3.2. *Systems of Klein-Gordon equations.* In the case of Klein-Gordon, we can use the explicit formulas to look at the sets. In this case, it is more convenient to consider the set of degenerate coherent point:

$$\mathcal{D}=\{\nabla_\eta\Phi(\xi,\eta)=0\}\cap\{\det\nabla^2_{\eta\eta}\Phi(\xi,\eta)=0\}$$

since they only involve the last two eigenvalues. In this case, we directly see that $\nabla_\eta\Phi=0$ implies that ξ and η are collinear, in which case, we can diagonalize the Hessian in an appropriate basis to get

$$\begin{aligned}&-\iota_2\nabla^2_{\eta\eta}\Phi(\xi,\eta)=\begin{pmatrix}\lambda_2''(r)&0\\0&\frac{\lambda_2'(r)}{r}\end{pmatrix}+\iota_2\iota_3\begin{pmatrix}\lambda_3''(s)&0\\0&\frac{\lambda_3'(s)}{s}\end{pmatrix},\\&\Lambda_i(\xi)=\lambda_i(|\xi|),\quad r=|\xi-\eta|,\quad s=|\eta|.\end{aligned}$$

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[5]Note also that if this set was included into the slow set: $\{\nabla_\xi\Phi(\xi,\eta)=0\}$, which is also rotationally invariant, then one could get more control on the functions by refining the finite-speed of propagation argument.

Thus we directly see that if $\iota_1\iota_2 = +$, the Hessian is non degenerate. If $\iota_1\iota_2 = -$, we see that $\mathcal{D}$ is equivalent to

$$\lambda_2'(r) = \lambda_3'(s) \quad \text{and} \quad \{\lambda_2''(r) = \lambda_3''(s) \ \text{ or } \ s = r\}\,.$$

Writing that $\lambda_i(r) = \sqrt{m_i^2 + c_i^2 r^2}$, we see that

$$\lambda_i''(r) = \frac{m_i^2}{c_i^4}\left(\frac{\lambda_i'(r)}{r}\right)^3,$$

and we obtain the simpler conditions:

$$\lambda_2'(r) = \lambda_3'(s) \quad \text{and} \quad \left\{\frac{m_2^2}{c_2^4}\frac{1}{r^3} = \frac{m_3^2}{c_3^4}\frac{1}{s^3} \ \text{ or } \ s = r\right\}.$$

If $m_2 = m_3$, then we see that $\lambda_2'(r) = \lambda_3'(r) \Leftrightarrow c_2 r = c_3 s$ which is only consistent with the conditions above if $c_2 = c_3$. But in this case, Φ will never vanish on the set $\{|\xi - \eta| = |\eta|\}$.

Note also that in the case of different masses, it is certainly possible to achieve the conditions above. In this case, we can freely choose Λ_1 so that it vanishes on the set $\mathcal{D}$.

3. General Heuristics

In the following, we give a series of Heuristics which we will follow all along the proofs.

- We will always think of functions as *defined modulo recomposition by a Calderón-Zygmund operator of norm one*. In other words, we "identify" f and $\mathcal{F}^{-1}q(\xi)\widehat{f}(\xi)$ for q a symbol of order 0. This allows to significantly simplify expressions; on the other hand, it requires that our norms be bounded upon recomposition by a Calderón-Zygmund operator. If this is true, since we will only precompose with a finite number of such operators, this assumption is essentially harmless.

  However, this essentially means that we will not be able to access functions in L^1 or with "better decay" (since e.g. this would be broken upon application of the Riesz transform).
- When we study free evolution of a profiles $u = e^{it\Lambda}f$ (at frequency ~ 1), there are always three different regimes:
  (1) profile close to the origin: $u^1 = e^{it\Lambda}f_{\{|x|\le\sqrt{t}\}}$. This part is dependent on very few properties of f and we may assume that $\hat{f}$ is smooth as long as we have a bound on $\|\hat{f}\|_{L^\infty}$.
  (2) profile transitioning away from the origin: $u^2 = e^{it\Lambda}f_{\{\sqrt{t}\le|x|\le t\}}$. For this part, the roughness of f limits the applicability of stationary phase estimates. On the other hand, bounds on norms of xf give some control on the function.
  (3) profile away from the origin: $u^3 = e^{it\Lambda}f_{\{|x|\ge t\}}$. For this part, the dispersion plays little role (in particular the mass situated in a shell $|x| \sim |t|[l, l+4]$ essentially remain in the same shell). Integration by parts is not helpful and we expect that the main control would come from norms on xf. In general this region is easily treated by energy-type estimates.
- We will have to quantify various quantities (for several functions at the same time), most of which will be big. But in almost all cases there will be the following hierarchy:

  $$t \ge |x| \ge |\xi|^{-1} \gg |\xi|.$$

  In the sense that
  (1) since we have nice energy estimates, we can always penalize high frequencies by a very high factor so that whenever it gets comparable to t, we get good control of the term.
  (2) the fact that we can assume $|x| \ge |\xi|^{-1}$ is a consequence of the uncertainty principle; in our case, this can also be seen by Hardy's inequality.
  (3) the fact that $t \ge |x|$ follows from the fact that the other x's are outside of the light cone and therefore when $|x| \gg t$, then we simply recover whatever estimates we had on our initial data.

- We will consider different degrees of precision about our functions. The more imprecise the description, the more robust the estimates are likely to be and the easier the proofs.
  1. **Energy-estimate description**: this is the estimate that is necessary to close the energy estimates and also serves for terms with similar structure:
  $$\|f\|_{EE} := \sup_t \left\{ t^{1+\beta}\|e^{it\Lambda}f\|_{L^\infty} + \|f\|_{H^N} \right\}$$
  2. **weighted-norm description**: this is the estimate that is necessary to close most of the cases outside of the space-time resonant set and to get most additional estimates (such as estimates on $\partial_t f$):
  $$\|f\|_{WN} := \sup_t \left\{ \|x^{1-\beta}|\nabla|^{-2\beta}f\|_{L^2} + t^{1+\beta}\|e^{it\Lambda}f\|_{L^\infty} \right\}$$
  The main feature of this norm is that it allows to make integration by parts thanks to the weighted control.
  3. **sum-space norm**: this is the full norm which we use only in the most difficult cases. It controls all the above descriptions.

4. Linear and bilinear stationary phase (informal)

Here we give two interpretations of how to understand and use oscillatory phases, either in a linear setting (to prove a dispersion inequality), or a bilinear setting (to understand some part of the result of bilinear interactions).

4.1. Understanding dispersion.

Here we give a non rigorous but (in our opinion) inspiring proof of the dispersion inequality. We consider a dispersive operator given by a Fourier multiplier $i\phi$ and a function f whose Fourier transform is localized, say in the ring $\{1/2 \leq |\xi| \leq 2\}$ where the Hessian of ϕ is uniformly strictly positive.

Recall that the Fourier transform intertwine translation and modulation in the sense that

$$g(x) = f(x - x_0), \qquad \widehat{g}(\xi) = \widehat{f}(\xi)e^{-i\xi\cdot x_0}.$$

Building on this, we consider the solution u of

$$(i\partial_t + \phi(i\nabla))\, u = 0, \quad u(0) = f.$$

The solution is of course given in Fourier space as

$$\widehat{u}(\xi, t) = e^{it\phi(\xi)}\widehat{f}(\xi),$$

which we also write as $u = S(t)f$. Now, fix an arbitrary point ξ_0 in the support of $\widehat{f}$ and expand the phase:

$$t\phi(\xi_0 + \eta) = t\phi(\xi_0) + t\nabla\phi(\xi_0)\cdot\eta + t\nabla^2\phi(\xi_0)[\eta, \eta] + O(t|\eta|^3),$$

and thus

$$\widehat{u}(\xi_0 + \eta, t) = e^{it\phi(\xi_0)}e^{it\nabla\phi(\xi_0)\cdot\eta}e^{iO(t|\eta|^2)}\widehat{f}(\xi_0 + \eta).$$

If we restrict the frequency support of $\widehat{f}$ to the set where $|\eta| \leq \varepsilon t^{-\frac{1}{2}}$, we can believe that we may ignore the third exponential in the line above. The first exponential only represents modulation by a uniform (in frequency) factor. Thus, introducing a $t^{-\frac{1}{2}}$-net $\{\xi_0\}$, we see that we may decompose our propagator into propagators S_{ξ_0} such that[6]

$$\widehat{S_{\xi_0}(t)} = e^{it\phi(\xi)}\chi(t^{\frac{1}{2}}(\xi - \xi_0))$$

and each such operator is essentially a translation by $-t\nabla\phi(\xi_0)$.

[6] Here we consider $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\sum_{k\in\mathbb{Z}^d}\chi(x + k) \equiv 1$ for all x.

Now, what about the L^∞-norm of u? How big can it be? This essentially has to do with superposition. Let us introduce a dual $t^{\frac{1}{2}}$-net $\{x_0\}$ in the physical space and for each ξ_0 decompose

$$f_{\xi_0}(x) = \sum_{x_0} \chi(t^{-\frac{1}{2}}(x-x_0))f_{\xi_0}(x) = \sum_{x_0} f_{\xi_0,x_0}(x), \qquad \widehat{f}_{\xi_0}(\xi) = \chi(t^{\frac{1}{2}}(\xi-\xi_0))\widehat{f}(\xi).$$

Now we fix x and t and ask how we can estimate $u(x,t)$. We have seen that

$$u(t) = S(t)f = \sum_{\xi_0} S_{\xi_0}(t)f_{\xi_0} = \sum_{\xi_0,x_0} S_{\xi_0}(t)f_{\xi_0,x_0},$$

and that S_{ξ_0} is essentially the translation by $-t\nabla\phi(\xi_0)$. So, for fixed ξ_0 and x, t, the only piece f_{ξ_0,x_0} that will have a non negligible presence at x will be given by

$$x - t\nabla\phi(\xi_0) = x_0 + O(t^{\frac{1}{2}}).$$

This defines a mapping for each fixed (x,t), $\xi_0 \to x_0$. In fact, this mapping is 1-to-1 since

$$|x_0 - x_1| = t|\nabla\phi(\xi_0) - \nabla\phi(\xi_1)| + O(t^{\frac{1}{2}}) \gtrsim t|\xi_0-\xi_1| + O(t^{\frac{1}{2}}) \gtrsim t^{\frac{1}{2}}$$

where we have used the fact that $\nabla^2\phi$ is strictly positive. Now, we need only estimate crudely

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ix\xi}e^{it\phi(\xi)}\widehat{f}(\xi)d\xi \simeq \sum_{\xi_0}\int_{\mathbb{R}^d} e^{-ix\xi}e^{it\phi(\xi)}\widehat{f}_{\xi_0,x_0}(\xi)d\xi \\ |u(x,t)| &\lesssim \sum_{\xi_0} \|\widehat{f}_{\xi_0,x_0}\|_{L^\infty}\cdot \mathrm{supp}(\widehat{f}_{\xi_0,x_0}) \\ &\lesssim t^{-\frac{d}{2}}\sum_{\xi_0}\|f_{\xi_0,x_0}\|_{L^1} \lesssim t^{-\frac{d}{2}}\sum_{x_0}\|f_{x_0}\|_{L^1} \lesssim t^{-\frac{d}{2}}\|f\|_{L^1}. \end{aligned}$$

Of course, when one has an explicit propagator, one can obtain this estimate much more easily. For example, this is the case for the Schrödinger equation where

$$u(x,t) = \frac{1}{(4\pi i t)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-i\frac{|x-y|^2}{4t}}u(y)dy,$$

but here, we have used only few general assumptions on the dispersion relation. Also, the case of a function with general support in Fourier space is recovered by scaling.

4.2. **Parameterizing the coherent curve.** The goal of this subsection is to explain the relevance of the condition

$$\det[\nabla^2_{\eta\eta}\Phi] \neq 0$$

in the analysis of quadratic interactions of the form

$$\widehat{I}(\xi) = \iint_{\mathbb{R}\times\mathbb{R}^3} \chi(s,\xi,\eta)e^{is\Phi(\xi,\eta)}\widehat{f}(\xi-\eta,s)\widehat{f}(\eta,s)d\eta ds,$$

where χ stands for a smooth localization function and

$$\Phi(\xi,\eta) = \Lambda_1(\xi) - \iota_2\Lambda_2(\xi-\eta) - \iota_3\Lambda_3(\eta), \quad \iota_1,\iota_2 \in \{\pm\}.$$

The advantage of adding smoothness in $\widehat{f}$ is that we can now integrate by parts in η, except of course on the stationary set when $\nabla_\eta\Phi = 0$, which is equivalent to

$$\nabla\Lambda_2(\eta-\xi) = -\iota_1\iota_2\nabla\Lambda_3(\eta).$$

We would like to parameterize the vanishing of $\nabla_\eta\Phi$ by a function[7]

$$\nabla_\eta\Phi(\xi,\eta) \Leftrightarrow \eta = p(\xi).$$

[7]it makes more sense to parameterize η in terms of ξ since ξ is fixed by the output, whereas η is inside the integral.

In fact, it is easier to do the opposite: we let $z(\eta)$ be the function such that

$$\nabla\Lambda_2(z) = -\iota_2\iota_3\nabla\Lambda_3(\eta), \qquad \nabla^2\Lambda_2(z)\cdot\frac{dz}{d\eta} = -\iota_2\iota_3\nabla^2\Lambda_3(\eta).$$

It is always possible to find such a function z (at least locally) if $\nabla^2\Lambda_2$ is invertible (this is the case, e.g. if Λ_2 is strictly convex as in the case of Schrödinger or Klein-Gordon, or more generally if Λ is homogeneous of degree different from 1). Then we find that

$$\nabla_\eta\Phi(\xi,\eta) = 0 \Leftrightarrow \xi = q(\eta) = \eta - z(\eta).$$

It is possible to invert this relation when the Jacobian matrix of the mapping $\eta \mapsto \eta - z(\eta)$ is invertible; but this is

$$\begin{aligned} Id - \frac{dz}{d\eta} &= Id - (-\iota_2\iota_3[\nabla^2\Lambda_2(z)]^{-1}\nabla^2\Lambda_3(\eta)) \\ &= [\nabla^2\Lambda_2(z)]^{-1}\cdot\left\{\nabla^2\Lambda_2(z) + \iota_2\iota_3\nabla^2\Lambda_3(\eta)\right\}. \end{aligned}$$

We see that the first operator on the righthand side is invertible by hypothesis, while the second precisely coincides with $-\iota_2\nabla^2_{\eta\eta}\Phi$.

In addition, if $\nabla^2_{\eta\eta}\Phi$ is invertible, we can precisely estimate $\nabla_\eta\Phi$ in a neighborhood of the set where it vanishes. Indeed

$$\nabla_\eta\Phi(\xi,\eta) = \nabla_\eta(\xi,p(\xi)) + \left[\int_0^1 \nabla^2\Phi(\xi,p(\xi)+s(\eta-p(\xi)))ds\right]\cdot[\eta-p(\xi)] \simeq \nabla^2_{\eta\eta}\Phi(\xi,p(\xi))\cdot[\eta-p(\xi)]$$

and therefore, when the Hessian is non degenerate,

$$|\nabla_\eta\Phi(\xi,\eta)| \simeq |\eta - p(\xi)|. \tag{4.1}$$

In addition, in this set, we can also understand the size of Φ:

$$\begin{aligned} \Phi(\xi,\eta) &= \Phi(\xi,p(\xi)) + [\Phi(\xi,\eta) - \Phi(\xi,p(\xi)] \\ &= \Phi(\xi,p(\xi)) + \left[\int_0^1 \nabla_\eta\Phi(\xi,s\eta+(1-s)p(\xi))ds\right]\cdot[\eta-p(\xi)]. \end{aligned}$$

Therefore, defining

$$\Psi(\xi) = \Phi(\xi,p(\xi)),$$

we see that in the neighborhood of the stationary set,

$$\Phi(\xi,\eta) = \Psi(\xi) + O(|\eta-p(\xi)|^2) \tag{4.2}$$

and in addition,

$$\nabla_\xi\Psi(\xi) = \nabla_\xi\Phi(\xi,p(\xi)) + (D_\xi p)^T\nabla_\eta\Phi(\xi,p(\xi)) = \nabla_\xi\Phi(\xi,p(\xi)) \tag{4.3}$$

is only determined by whether or not $\nabla_\xi\Phi$ vanishes on the stationary set.

5. A new kind of input

Here, we estimate the type of function produced by "space-time" resonance sets.

We consider the following integral which corresponds to inputs that we *need* to be able to accept: for $f,g \in C^\infty_c(\mathbb{R}^3)$ and $T \gg 1$ define

$$I_T(\xi) = \int_{\mathbb{R}\times\mathbb{R}^3} \theta(\frac{s}{T})\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)e^{is\Phi(\xi,\eta)}\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta ds$$

such that $k,k_1,k_2 \simeq 1$ localize in the neighborhood of a space-resonant set. I_T can also be seen in physical space as

$$I_T(x) = \int_{\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3} \theta(\frac{s}{T})K(x,y_1,y_2,s)P_{k_1}f(y_1)P_{k_2}g(y_2)dy_1dy_2ds,$$

where it suffices to look at the case $|y_1|, |y_2| \lesssim 1$.

This is relevant since we have that

$$I(x,t) = \sum_{T\le t} I_T(x)$$

is essentially the result of the interaction of two free waves with profiles f and g.

We are interested in bounding the quantity $\langle x\rangle^\theta I_T$ into various spaces. By finite speed of propagation, we see that this is essentially the same[8] as bounding $T^\theta I_T$ into various spaces and we can hereafter think of T as the only large parameter.

In this case, we can understand the Kernel better. Indeed, setting $\delta_X = T^{-\frac{1}{2}+\delta}$, we see upon integrating by parts in η that

$$\begin{aligned} K_N(x,y_1,y_2,s) &= \int_{\mathbb{R}^3} e^{i\xi[x-y_1]}\varphi_k(\xi)\left\{\int_{\mathbb{R}^3}(1-\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta)))e^{is\Phi(\xi,\eta)}e^{i\eta[y_2-y_1]}\varphi(\xi-\eta)\varphi(\eta)d\eta\right\}d\xi \\ &= O(T^{-100}). \end{aligned}$$

Indeed, we simply write that

$$\begin{aligned} &\int_{\mathbb{R}^3}(1-\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))e^{is\Phi(\xi,\eta)}e^{i\eta[y_2-y_1]}\varphi(\xi-\eta)\varphi(\eta)d\eta \\ =&\frac{i}{s}\int_{\mathbb{R}^3} e^{is\Phi(\xi,\eta)}\mathrm{div}_\eta\left\{\frac{\nabla_\eta\Phi(\xi,\eta)}{|\nabla_\eta\Phi(\xi,\eta)|^2}(1-\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))e^{i\eta[y_2-y_1]}\varphi(\xi-\eta)\varphi(\eta)\right\}d\eta \end{aligned}$$

and we see that the worst case is when the derivative hits either the localization function or the term $1/\partial_\eta\Phi$ which yield a term of size $T^{1-2\delta}$. But this is still tramped by the gain of s^{-1}. Iterating this as many times as we want, we obtain arbitrarily large decay.

Hence we see that we may replace the kernel by

$$K_R(x,y_1,y_2,s) = \int_{\mathbb{R}^3\times\mathbb{R}^3}\theta(\frac{s}{T})e^{i\xi[x-y_1]}\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))e^{is\Phi(\xi,\eta)}e^{i\eta[y_2-y_1]}\varphi_k(\xi)\varphi(\xi-\eta)\varphi(\eta)d\eta d\xi.$$

At this point, we cannot really integrate by parts in ξ anymore since the cut-off function $\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))$ seems too sensitive. On the other hand, by (4.1), we are now localized to a ball of radius $\delta_X \ll 1$ and the kernel starts to have a simpler shape, see e.g. (4.2).

It is now easier to go back to the Fourier side[9] and consider

$$\widetilde{I}_T(\xi) = \int_{\mathbb{R}\times\mathbb{R}^3}\theta(\frac{s}{T})\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))e^{is\Phi(\xi,\eta)}\varphi_k(\xi)\varphi(\xi-\eta)\varphi(\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta ds.$$

At this point, we can still integrate by parts in time. This essentially allows to gain a factor of $(T\Phi)^{-1}$, so we can decompose

$$\begin{aligned} \widetilde{I}_T(\xi) &= \widetilde{I}_T^1(\xi) + \widetilde{I}_T^2(\xi) \\ \widetilde{I}_T^1(\xi) &= \int_{\mathbb{R}\times\mathbb{R}^3}\theta(\frac{s}{T})\varphi(\delta_T^{-1}\Phi(\xi,\eta))\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))e^{is\Phi(\xi,\eta)}\varphi_k(\xi)\varphi(\xi-\eta)\varphi(\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta ds, \\ \widetilde{I}_T^2(\xi) &= \int_{\mathbb{R}\times\mathbb{R}^3}\theta(\frac{s}{T})(1-\varphi(\delta_T^{-1}\Phi(\xi,\eta)))\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))e^{is\Phi(\xi,\eta)}\varphi_k(\xi)\varphi(\xi-\eta)\varphi(\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)d\eta ds, \end{aligned}$$

[8]Indeed, we see that whenever $|x| \gtrsim T$, then the Kernel defining I_T has rapid decay, whereas the case $|x| \ll T$ certainly has smaller contribution.

[9]Indeed, since x is so little constrained, it is harder to bound the function I_T on the physical side, but since ξ is constrained, it is easier to bound it on the Fourier side.

where $\delta_T = T^{-1+\delta}$. Of course, we can now integrate by parts in time in $\widetilde{I}^2_T$ to get

$$\widetilde{I}^2_T(\xi) = -\frac{i}{T}\int_{\mathbb{R}\times\mathbb{R}^3} \theta'(\frac{s}{T})(1-\varphi(\delta_T^{-1}\Phi(\xi,\eta)))\varphi(\delta_X^{-1}\nabla_\eta\Phi(\xi,\eta))e^{is\Phi(\xi,\eta)}\varphi_k(\xi)\varphi(\xi-\eta)\varphi(\eta)\frac{\widehat{f}(\xi-\eta)\widehat{g}(\eta)}{\Phi(\xi,\eta)}d\eta ds.$$

Now, using that $\Phi(\xi,\eta) = \Psi(\xi) + O(T^{-1+2\delta})$ and that $\nabla\Psi(\xi) \simeq A\frac{\xi}{|\xi|}$ on the support of integration of both $\widetilde{I}^1_T$ and $\widetilde{I}^2_T$, we see that we essentially have[10]

$$\widetilde{I}^1_T(\xi) \simeq \quad T\delta_X^3 \mathbf{1}_{\{|\Psi|\le\delta_T\}}(\xi) \quad \simeq T^{-\frac{1}{2}+3\delta}\mathbf{1}_{\{||\xi|-R|\le\delta_T\}}(\xi)$$
$$\widetilde{I}^2_T(\xi) \lesssim \quad \delta_X^3\frac{1}{\Psi(\xi)}\mathbf{1}_{\{|\Psi|\ge\delta_T\}}(\xi) \quad \simeq T^{-\frac{3}{2}+3\delta}\frac{1}{||\xi|-R|}\mathbf{1}_{\{||\xi|-R|\ge\delta_T\}}(\xi).$$

We now see that

$$\|\widetilde{I}_T\|_{L^2} \lesssim T^{-1+5\delta},$$

which means that xI barely fails to be in L^2. However, we can also see that

$$\|\widetilde{I}_T(\xi)\|_{L^1} \lesssim T^{-\frac{3}{2}+5\delta}$$

is quite small, so that

$$\widetilde{I}_T(x) = \mathcal{F}\{\widetilde{I}_T(\xi)\} \simeq T^{-\frac{3}{2}+5\delta}\mathbf{1}_{\{|x|\lesssim T\}}$$

is still in L^1 and in particular is still consistent with $e^{it\Lambda}(\mathcal{F}\widetilde{I}_T)$ having a $t^{-3/2+}$ decay.

Conclusion: Thus we need to allow for inputs produced by these interactions and in particular *not bounded in* $x^{-1}L^2$. Indeed, these inputs were essentially independent of f, g.

Two things are worth noting: First, since we used very few assumptions on f and g, we should be able to carry this analysis over also in the case where f and g are time-dependent and suitably controlled. Second, that we essentially used a bound on $\|\widehat{f}\|_{L^\infty}$ and $\|\widehat{g}\|_{L^\infty}$ and our outputs are certainly consistent with this norm being bounded. Thus it could be advantageous to add this to the information captured by our norm.

References

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[10] Note that since $|\Phi| \lesssim T^{-1}$ on the support of $\widetilde{I}^1_T$, the exponential is essentially equal to 1, and assuming that $\widehat{f},\widehat{g} \gtrsim 1$ on the space-time resonant set, we practically have a bound from below for $\widetilde{I}^1_T$. For $\widetilde{I}^2_T$, it would be slightly more advantages to do one additional integration by parts in time; however, this becomes harder in the nonlinear case when we let f g vary in time.