CUBIC FOURFOLDS, K3 SURFACES, AND RATIONALITY QUESTIONS

BRENDAN HASSETT

This is a survey of the geometry of complex cubic fourfolds with a view toward rationality questions. Smooth cubic surfaces have been known to be rational since the 19th century [Dol05]; cubic threefolds are irrational by the work of Clemens and Griffiths [CG72]. Cubic fourfolds are likely more varied in their behavior. While there are examples known to be rational, we expect that most cubic fourfolds should be irrational. However, no cubic fourfolds are *proven* to be irrational.

Our organizing principle is that progress is likely to be driven by the dialectic between concrete geometric constructions (of rational, stably rational, and unirational parametrizations) and conceptual tools differentiating various classes of cubic fourfolds (Hodge theory, moduli spaces and derived categories, and decompositions of the diagonal). Thus the first section of this paper is devoted to classical examples of rational parametrizations. In section two we focus on Hodge-theoretic classifications of cubic fourfolds with various special geometric structures. These are explained in section three using techniques from moduli theory informed by deep results on K3 surfaces and their derived categories. We return to constructions in the fourth section, focusing on unirational parametrizations of special classes of cubic fourfolds. In the last section, we touch on recent applications of decompositions of the diagonal to rationality questions, and what they mean for cubic fourfolds.

Acknowledgments. This work is partially supported by NSF grant 1551516. I am grateful to N. Addington, A. Auel, H. Nuer, R. Thomas, Y. Tschinkel, and C. Voisin for comments on this survey and J.-L. Colliot-Thélène, J. Harris, B. Mazur, and A. Várilly-Alvarado for discussions informing its content. I would also like to thank the anonymous referee for pointing out work of Crauder and Katz that invalidated the original formulation of Question 32. I am grateful to CIME and CIRM for organizing the school 'Rationality problems in algebraic geometry' where this material was to be presented, and the organizers Rita Pardini and

Gian Pietro Pirola for supporting this survey despite my inability to lecture at the school.

1. Introduction and classical constructions

Throughout, we work over the complex numbers.

1.1. **Basic definitions.** Let X be a smooth projective variety of dimension n. If there exists a birational map $\rho : \mathbb{P}^n \dashrightarrow X$ we say that X is *rational*. It is *stably rational* if $X \times \mathbb{P}^m$ is rational for some $m \geq 0$. If there exists a generically finite map $\rho : \mathbb{P}^n \dashrightarrow X$ we say that X is *unirational*; this is equivalent to the existence of a dominant map from a projective space of arbitrary dimension to X.

A *cubic fourfold* is a smooth cubic hypersurface $X \subset \mathbb{P}^5$, with defining equation

$$F(u, v, w, x, y, z) = 0$$

where $F \in \mathbb{C}[u, v, w, x, y, z]$ is homogeneous of degree three. Cubic hypersurfaces in $\mathbb{P}^5$ are parametrized by

$$\mathbb{P}(\mathbb{C}[u, v, w, x, y, z]_3) \simeq \mathbb{P}^{55}$$

with the smooth ones corresponding to a Zariski open $U \subset \mathbb{P}^{55}$.

Sometimes we will consider singular cubic hypersurfaces; in these cases, we shall make explicit reference to the singularities. The singular cubic hypersurfaces in $\mathbb{P}^5$ are parametrized by an irreducible divisor

$$\Delta := \mathbb{P}^{55} \setminus U.$$

Birationally, Δ is a $\mathbb{P}^{49}$ bundle over $\mathbb{P}^5$, as having a singularity at a point $p \in \mathbb{P}^5$ imposes six independent conditions.

The moduli space of cubic fourfolds is the quotient

$$\mathcal{C} := [U/\operatorname{PGL}_6].$$

This is a Deligne-Mumford stack with quasi-projective coarse moduli space, e.g., by classical results on the automorphisms and invariants of hypersufaces [MFK94, ch. 4.2]. Thus we have

$$\dim(\mathcal{C}) = \dim(U) - \dim(\operatorname{PGL}_6) = 55 - 35 = 20.$$

1.2. **Cubic fourfolds containing two planes.** Fix disjoint projective planes

$$P_1 = \{u = v = w = 0\}, P_2 = \{x = y = z = 0\} \subset \mathbb{P}^5$$

and consider the cubic fourfolds X containing P_1 and P_2. For a concrete equation, consider

$$X = \{ux^2 + vy^2 + wz^2 = u^2x + v^2y + w^2z\}$$

which is in fact smooth! See [HK07, §5] for more discussion of this example.

More generally, fix forms

$$F_1, F_2 \in \mathbb{C}[u, v, w; x, y, z]$$

of bidegree $(1, 2)$ and $(2, 1)$ in the variables $\{u, v, w\}$ and $\{x, y, z\}$. Then the cubic hypersurface

$$X = \{F_1 + F_2 = 0\} \subset \mathbb{P}^5$$

contains P_1 and P_2, and the defining equation of every such hypersurface takes that form. Up to scaling, these form a projective space of dimension 35. The group

$$\{g \in \mathrm{PGL}_6 : g(P_1) = P_1, g(P_2) = P_2\}$$

has dimension 17. Thus the locus of cubic fourfolds containing a pair of disjoint planes has codimension two in $\mathcal{C}$.

The cubic fourfolds of this type are *rational*. Indeed, we construct a birational map as follows: Given points $p_1 \in P_1$ and $p_2 \in P_2$, let $\ell(p_1, p_2)$ be the line containing them. The Bezout Theorem allows us to write

$$\ell(p_1, p_2) \cap X = \begin{cases} \{p_1, p_2, \rho(p_1, p_2)\} \text{ if } \ell(p_1, p_2) \not\subset X \\ \ell(p_1, p_2) \text{ otherwise.} \end{cases}$$

The condition $\ell(p_1, p_2) \subset X$ is expressed by the equations

$$S := \{F_1(u, v, w; x, y, z) = F_2(u, v, w; x, y, z) = 0\} \subset P_{1,[x,y,z]} \times P_{2,[u,v,w]}.$$

Since S is a complete intersection of hypersurfaces of bidegrees $(1, 2)$ and $(2, 1)$ it is a K3 surface, typically with Picard group of rank two. Thus we have a well- defined morphism

$$\begin{array}{rcl} \rho : P_1 \times P_2 \setminus S & \to & X \\ (p_1, p_2) & \mapsto & \rho(p_1, p_2) \end{array}$$

that is birational, as each point of $\mathbb{P}^5 \setminus (P_1 \cup P_2)$ lies on a unique line joining the planes.

We record the linear series inducing this birational parametrization: ρ is given by the forms of bidegree $(2, 2)$ containing S and ρ^{-1} by the quadrics in $\mathbb{P}^5$ containing P_1 and P_2;

1.3. **Cubic fourfolds containing a plane and odd multisections.** Let X be a cubic fourfold containing a plane P. Projection from P gives a quadric surface fibration

$$q : \tilde{X} := \mathrm{Bl}_P(X) \to \mathbb{P}^2$$

with singular fibers over a sextic curve $B \subset \mathbb{P}^2$. If q admits a rational section then $\tilde{X}$ is rational over $K = \mathbb{C}(\mathbb{P}^2)$ and thus over $\mathbb{C}$ as well. The simplest example of such a section is another plane disjoint from P. Another example was found by Tregub [Tre93]: Suppose there is a quartic Veronese surface

$$V \simeq \mathbb{P}^2 \subset X$$

meeting P transversally at three points. Then its proper transform $\tilde{V} \subset \tilde{X}$ is a section of q, giving rationality.

To generalize this, we employ a basic property of quadric surfaces due to Springer (cf. [Has99, Prop. 2.1] and [Swa89]):

> Let $Q \subset \mathbb{P}^3_K$ be a quadric surface smooth over a field K. Suppose there exists an extension L/K of odd degree such that $Q(L) \neq \emptyset$. Then $Q(K) \neq \emptyset$ and Q is rational over K via projection from a rational point.

This applies when there exists a surface $W \subset X$ intersecting the generic fiber of q transversally in an odd number of points. Thus we the following:

Theorem 1. [Has99] *Let X be a cubic fourfold containing a plane P and projective surface W such that*

$$\deg(W) - \langle P, W \rangle$$

is odd. Then X is rational.

The intersection form on the middle cohomology of X is denoted by $\langle , \rangle$.

Theorem 1 gives a countably infinite collection of codimension two subvarieties in $\mathcal{C}$ parametrizing rational cubic fourfolds. Explicit birational maps $\rho : \mathbb{P}^2 \times \mathbb{P}^2 \stackrel{\sim}{\dashrightarrow} X$ can be found in many cases [Has99, §5].

We elaborate the geometry behind Theorem 1: Consider the relative variety of lines of the quadric surface fibration q

$$f : F_1(\tilde{X}/\mathbb{P}^2) \to \mathbb{P}^2.$$

For each $p \in \mathbb{P}^2$, $f^{-1}(p)$ parametrizes the lines contained in the quadric surface $q^{-1}(p)$. When the fiber is smooth, this is a disjoint union of two smooth $\mathbb{P}^1$'s; for $p \in B$, we have a single $\mathbb{P}^1$ with multiplicity two. Thus the Stein factorization

$$f : F_1(\tilde{X}/\mathbb{P}^2) \to S \to \mathbb{P}^2$$

yields a degree two K3 surface—the double cover $S \to \mathbb{P}^2$ branched over B—and a $\mathbb{P}^1$-bundle $r : F_1(\tilde{X}/\mathbb{P}^2) \to S$. The key to the proof is the equivalence of the following conditions (see also [Kuz15, Th. 4.11]):

- the generic fiber of q is rational over K;
- q admits a rational section;
- r admits a rational section.

The resulting birational map $\rho^{-1} : X \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ blows down the lines incident to the section of q, which are parametrized by a surface birational to S.

Cubic fourfolds containing a plane have been re-examined recently from the perspective of twisted K3 surfaces and their derived categories [Kuz10, MS12, Kuz15]. The twisted K3 surface is the pair (S, η), where η is the class in the Brauer group of S arising from r; note that $\eta = 0$ if and only if the three equivalent conditions above hold. Applications of this geometry to rational points may be found in [HVAV11].

Remark 2. The technique of Theorem 1 applies to all smooth projective fourfolds admiting quadric surface fibrations $Y \to P$ over a rational surface P. Having an odd multisection suffices to give rationality.

1.4. **Cubic fourfolds containing quartic scrolls.** A *quartic scroll* is a smooth rational ruled surface $\Sigma \hookrightarrow \mathbb{P}^5$ with degree four, with the rulings embedded as lines. There are two possibilities:

- $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via the linear series $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$
- the Hirzebruch surface $\mathbb{F}_2$ embedded via $|\mathcal{O}_{\mathbb{F}_2}(\xi + f)|$ where f is a fiber and ξ a section at infinity ($f\xi = 1$ and $\xi^2 = 2$).

The second case is a specialization of the first. Note that all scrolls of the first class are projectively equivalent and have equations given by the 2×2 minors of:

$$\begin{pmatrix} u & v & x & y \\ v & w & y & z \end{pmatrix}$$

Lemma 3. *Let Σ be a quartic scroll, realized as the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under the linear series $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$. Then a generic point $p \in \mathbb{P}^5$ lies on a unique secant to Σ. The locus of points on more than one secant equals the Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2$ associated with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ of the second factor.*

Proof. The first assertion follows from a computation with the double point formula [Ful84, §9.3]. For the second, if two secants to Σ, $\ell(s_1, s_2)$ and $\ell(s_3, s_4)$, intersect then $s_1, \ldots, s_4$ are coplanar. But points $s_1, \ldots, s_4 \in \Sigma$ that fail to impose independent conditions on $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$ necessarily have at least three points on a line or all the points on a conic contained in Σ. □

Surfaces in $\mathbb{P}^5$ with 'one apparent double point' have been studied for a long time. See [Edg32] for discussion and additional classical references and [BRS15] for a modern application to cubic fourfolds.

Proposition 4. *If X is a cubic fourfold containing a quartic scroll Σ then X is rational.*

Here is the idea: Consider the linear series of quadrics cutting out Σ. It induces a morphism

$$\mathrm{Bl}_\Sigma(X) \to \mathbb{P}(\Gamma(\mathcal{I}_\Sigma(2))) \simeq \mathbb{P}^5,$$

mapping X birationally to a quadric hypersurace. Thus X is rational.

Remark 5. Here is another approach. Let $R \simeq \mathbb{P}^1$ denote the ruling of Σ; for $r \in R$, let $\ell(r) \subset \Sigma \subset X$ denote the corresponding line. For distinct $r_1, r_2 \in R$, the intersection

$$\mathrm{span}(\ell(r_1), \ell(r_2)) \cap X$$

is a cubic surface containing disjoint lines. Let Y denote the closure

$$\{(x, r_1, r_2) : x \in \mathrm{span}(\ell(r_1), \ell(r_2)) \cap X\} \subset X \times \mathrm{Sym}^2(R) \simeq X \times \mathbb{P}^2.$$

The induced $\pi_2 : Y \to \mathbb{P}^2$ is a cubic surface fibration such that the generic fiber contains two lines. Thus the generic fiber Y_K, $K = \mathbb{C}(\mathbb{P}^2)$, is rational over K and consequently Y is rational over $\mathbb{C}$.

The degree of $\pi_1 : Y \to X$ can be computed as follows: It is the number of secants to Σ through a generic point $p \in X$. There is one such secant by Lemma 3. We will return to this in §4.

Consider the nested Hilbert scheme

$$\mathrm{Scr} = \{\Sigma \subset X \subset \mathbb{P}^5 : \Sigma \text{ quartic scroll}, X \text{ cubic fourfold}\}$$

and let $\pi : \mathrm{Scr} \to \mathbb{P}^{55}$ denote the morphism forgetting Σ. We have $\dim(\mathrm{Scr}) = 56$ so the fibers of π are positive dimensional. In 1940, Morin [Mor40] asserted that the generic fiber of π is one dimensional, deducing (incorrectly!) that the generic cubic fourfold is rational. Fano [Fan43] corrected this a few years later, showing that the generic fiber has dimension two; cubic fourfolds containing a quartic scroll thus form a *divisor* in $\mathcal{C}$. We will develop a conceptual approach to this in §2.1.

1.5. **Cubic fourfolds containing a quintic del Pezzo surface.** Let $T \subset \mathbb{P}^5$ denote a quintic del Pezzo surface, i.e., $T = \mathrm{Bl}_{p_1,p_2,p_3,p_4}(\mathbb{P}^2)$ anti-canonically embedded. Its defining equations are quadrics

$$Q_i = a_{jk}a_{lm} - a_{jl}a_{km} + a_{jm}a_{kl}, \{1, \dots, 5\} = \{i, j, k, l, m\}, j < k < l < m,$$

where the a_{rs} are generic linear forms on $\mathbb{P}^5$. The rational map

$$\begin{array}{rcl} Q : \mathbb{P}^5 & \dashrightarrow & \mathbb{P}^4 \\ [u,v,w,x,y,z] & \mapsto & [Q_1,Q_2,Q_3,Q_4,Q_5] \end{array}$$

contracts each secant of T to a point. Note that a generic $p \in \mathbb{P}^5$ lies on a unique such secant.

Proposition 6. *A cubic fourfold containing a quintic del Pezzo surface is rational.*

Indeed, restricting Q to X yields a *birational* morphism $\mathrm{Bl}_T(X) \to \mathbb{P}^4$.

1.6. **Pfaffian cubic fourfolds.** Recall that if $M = (m_{ij})$ is skew-symmetric $2n \times 2n$ matrix then the determinant

$$\det(M) = \mathrm{Pf}(M)^2,$$

where $\mathrm{Pf}(M)$ is a homogeneous form of degree n in the entries of M, known as its Pfaffian. If the entries of M are taken as linear forms in u, v, w, x, y, z, the resulting hypersurface

$$X = \{\mathrm{Pf}(M) = 0\} \subset \mathbb{P}^5$$

is a Pfaffian cubic fourfold.

We put this on a more systematic footing. Let V denote a six-dimensional vector space and consider the strata

$$\mathrm{Gr}(2,V) \subset \mathrm{Pfaff}(V) \subset \mathbb{P}(\bigwedge^2 V),$$

where $\mathrm{Pfaff}(V)$ parametrizes the rank-four tensors. Note that $\mathrm{Pfaff}(V)$ coincides with the secant variety to $\mathrm{Gr}(2,V)$, which is degenerate, i.e., smaller than the expected dimension. We also have dual picture

$$\mathrm{Gr}(2,V^*) \subset \mathrm{Pfaff}(V^*) \subset \mathbb{P}(\bigwedge^2 V^*).$$

A codimension six subspace $L \subset \mathbb{P}(\bigwedge^2 V)$ corresponds to a codimension nine subspace $L^\perp \subset \mathbb{P}(\bigwedge^2 V^*)$. Let $X = L^\perp \cap \mathrm{Pfaff}(V^*)$ denote the resulting Pfaffian cubic fourfold and $S = L \cap \mathrm{Gr}(2,V)$ the associated degree fourteen K3 surface.

Beauville and Donagi [BD85] (see also [Tre84]) established the following properties, when L is generically chosen:

- X is rational: For each codimension one subspace $W \subset V$, the mapping

$$\begin{array}{rcl} Q_W : X & \dashrightarrow & W \\ [\phi] & \mapsto & \ker(\phi) \cap W \end{array}$$

is birational. Here we interpret $\phi : V \to V^*$ as an antisymmetric linear transformation.

- X contains quartic scrolls: For each point $[P] \in S$, consider
$$\Sigma_P := \{[\phi] \in X : \ker(\phi) \cap P \neq 0\}.$$
We interpret $P \subset V$ as a linear subspace. This is the two-parameter family described by Fano.
- X contains quintic del Pezzo surfaces: For each W, consider
$$T_W := \{[\phi] \in X : \ker(\phi) \subset W\},$$
the indeterminacy of Q_W. This is a five-parameter family.
- The variety $F_1(X)$ of lines on X is isomorphic to $S^{[2]}$, the Hilbert scheme of length two subschemes on S.

Tregub [Tre93] observed the connection between containing a quartic scroll and containing a quintic del Pezzo surface. For the equivalence between containing a quintic del Pezzo surface and the Pfaffian condition, see [Bea00, Prop. 9.2(a)].

Remark 7. Cubic fourfolds X containing disjoint planes P_1 and P_2 admit 'degenerate' quartic scrolls and are therefore limits of Pfaffian cubic fourfolds [Tre93]. As we saw in §1.2, the lines connecting P_1 and P_2 and contained in X are parametrized by a K3 surface
$$S \subset P_1 \times P_2.$$
Given $s \in S$ generic, let ℓ_s denote the corresponding line and $L_i = \mathrm{span}(P_i, \ell_s) \simeq \mathbb{P}^3$. The intersection
$$L_i \cap X = P_i \cup Q_i$$
where Q_i is a quadric surface. The surfaces Q_1 and Q_2 meet along the common ruling ℓ_s, hence $Q_1 \cup_{\ell_s} Q_2$ is a limit of quartic scrolls.

Remark 8 (Limits of Pfaffians). A number of recent papers have explored smooth limits of Pfaffian cubic fourfolds more systematically. For analysis of the intersection between cubic fourfolds containing a plane and limits of the Pfaffian locus, see [ABBVA14]. Auel and Bolognese-Russo-Staglianò [BRS15] have shown that smooth limits of Pfaffian cubic fourfolds are always rational; [BRS15] includes a careful analysis of the topology of the Pfaffian locus in moduli.

1.7. **General geometric properties of cubic hypersurfaces.** Let $\mathrm{Gr}(2, n+1)$ denote the Grassmannian of lines in $\mathbb{P}^n$. We have a tautological exact sequence
$$0 \to S \to \mathcal{O}^{n+1}_{\mathrm{Gr}(2,n+1)} \to Q \to 0$$

where S and Q are the tautological sub- and quotient bundles, of ranks 2 and $n-1$. For a hypersurface $X \subset \mathbb{P}^n$, the variety of lines $F_1(X) \subset \mathrm{Gr}(2, n+1)$ parametrizes lines contained in X. If $X = \{G = 0\}$ for some homogeneous form G of degree $d = \deg(X)$ then $F_1(X) = \{\sigma_G = 0\}$, where

$$\sigma_G \in \Gamma(\mathrm{Gr}(2, n+1), \mathrm{Sym}^d(S^*))$$

is the image of G under the d-th symmetric power of the transpose to $S \hookrightarrow \mathcal{O}^{n+1}_{\mathrm{Gr}(2,n+1)}$.

Proposition 9. [AK77, Th. 1.10] *Let $X \subset \mathbb{P}^n, n \geq 3$, be a smooth cubic hypersurface. Then $F_1(X)$ is smooth of dimension $2n-6$.*

The proof is a local computation on tangent spaces.

Proposition 10. *Let $\ell \subset X \subset \mathbb{P}^n$ be a smooth cubic hypersurface containing a line. Then X admits a degree two unirational parametrization, i.e., a degree two mapping*

$$\rho : \mathbb{P}^{n-1} \dashrightarrow X.$$

Since this result is classical we only sketch the key ideas. Consider the diagram

$$\begin{array}{ccc} & \mathrm{Bl}_\ell(\mathbb{P}^n) & \\ \swarrow & & \searrow \\ \mathbb{P}^n & \dashrightarrow & \mathbb{P}^{n-2} \end{array}$$

where the bottom arrow is projection from ℓ. The right arrow is a $\mathbb{P}^2$ bundle. This induces

$$\begin{array}{ccc} & \mathrm{Bl}_\ell(X) & \\ \swarrow & & \overset{q}{\searrow} \\ X & \dashrightarrow & \mathbb{P}^{n-2} \end{array}$$

where q is a conic bundle. The exceptional divisor $E \simeq \mathbb{P}(N_{\ell/X}) \subset \mathrm{Bl}_\ell(X)$ meets each conic fiber in two points. Thus the base change

$$Y := \mathrm{Bl}_\ell(X) \times_{\mathbb{P}^{n-2}} E \to E$$

has a rational section and we obtain birational equvialences

$$Y \overset{\sim}{\dashrightarrow} \mathbb{P}^1 \times E \overset{\sim}{\dashrightarrow} \mathbb{P}^1 \times \mathbb{P}^{n-3} \times \ell \overset{\sim}{\dashrightarrow} \mathbb{P}^{n-1}.$$

The induced $\rho : Y \dashrightarrow X$ is generically finite of degree two.

2. Special cubic fourfolds

We use the terminology *very general* to mean 'outside the union of a countable collection of Zariski-closed subvarieties'. Throughout this section, X denotes a smooth cubic fourfold over $\mathbb{C}$.

2.1. **Structure of cohomology.** Let X be a cubic fourfold and $h \in H^2(X,\mathbb{Z})$ the Poincaré dual to the hyperplane class, so that $h^4 = \deg(X) = 3$. The Lefschetz hyperplane theorem and Poincaré duality give

$$H^2(X,\mathbb{Z}) = \mathbb{Z}h, \quad H^6(X,\mathbb{Z}) = \mathbb{Z}\frac{h^3}{3}.$$

The Hodge numbers of X take the form

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \end{array}$$

so the Hodge-Riemann bilinear relations imply that $H^4(X,\mathbb{Z})$ is a unimodular lattice under the intersection form $\langle , \rangle$ of signature $(21,2)$. Basic classification results on quadratic forms [Has00, Prop. 2.1.2] imply

$$L := H^4(X,\mathbb{Z})_{\langle,\rangle} \simeq (+1)^{\oplus 21} \oplus (-1)^{\oplus 2}.$$

The primitive cohomology

$$L^0 := \{h^2\}^{\perp} \simeq A_2 \oplus U^{\oplus 2} \oplus E_8^{\oplus 2},$$

where

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and E_8 is the positive definite lattice associated with the corresponding Dynkin diagram. This can be shown using the incidence correspondence between X and its variety of lines $F_1(X)$, which induces the *Abel-Jacobi* mapping [BD85]

$$\alpha : H^4(X,\mathbb{Z}) \to H^2(F_1(X),\mathbb{Z})(-1), \tag{2.1}$$

compatible with Hodge filtrations. (See [AT14, §2] for another approach.) Restricting to primitive cohomology gives an isomorphism

$$\alpha : H^4(X,\mathbb{Z})_{\text{prim}} \xrightarrow{\sim} H^2(F_1(X),\mathbb{Z})_{\text{prim}}(-1).$$

Note that $H^2(F_1(X),\mathbb{Z})$ carries the *Beauville-Bogomolov* form $(,)$; see §3.5 for more discussion. The shift in weights explains the change in signs

$$(\alpha(x_1),\alpha(x_2)) = -\langle x_1, x_2\rangle .$$

2.2. **Special cubic fourfolds defined.** For a very general cubic fourfold X, any algebraic surface $S \subset X$ is homologous to a complete intersection, i.e.,

$$H^{2,2}(X,\mathbb{Z}) := H^4(X,\mathbb{Z}) \cap H^2(\Omega_X^2) \simeq \mathbb{Z}h^2$$

so

$$[S] \equiv nh^2, \quad n = \deg(S)/3.$$

This follows from the Torelli Theorem and the irreducibility of the monodromy action for cubic fourfolds [Voi86]; see § 2.3 below for more details. In particular, X does not contain any quartic scrolls or any surfaces of degree four; this explains why Morin's rationality argument could not be correct.

On the other hand, the integral Hodge conjecture is valid for cubic fourfolds [Voi13, Th. 1.4], so every class

$$\gamma \in H^{2,2}(X,\mathbb{Z})$$

is algebraic, i.e., arises from a codimension two algebraic cycle with integral coefficients. Thus if

$$H^{2,2}(X,\mathbb{Z}) \supsetneq \mathbb{Z}h^2$$

then X admits surfaces that are not homologous to complete intersections.

Definition 11. A cubic fourfold X is *special* if it admits an algebraic surface $S \subset X$ not homologous to a complete intersection. A *labelling* of a special cubic fourfold consists of a rank two saturated sublattice

$$h^2 \in K \subset H^{2,2}(X,\mathbb{Z});$$

its *discriminant* is the determinant of the intersection form on K.

Let $S \subset X$ be a smooth surface. Recall that

$$c_1(T_X) = 3h, \quad c_2(T_X) = 6h^2$$

so the self-intersection

$$\begin{aligned} \langle S,S\rangle &= c_2(N_{S/X}) = c_2(T_X|S) - c_2(T_S) - c_1(T_S)c_1(T_X|S) + c_1(T_S)^2 \\ &= 6H^2 + 3HK_S + K_S^2 - \chi(S), \end{aligned}$$

where $H = h|S$ and $\chi(S)$ is the topological Euler characteristic.

(1) When X contains a plane P we have

$$K_8 = \begin{array}{c|cc} & h^2 & P \\ \hline h^2 & 3 & 1 \\ P & 1 & 3 \end{array}.$$

(2) When X contains a cubic scroll Σ_3, i.e., $\mathrm{Bl}_p(\mathbb{P}^2)$ embedded in $\mathbb{P}^4$, we have

$$K_{12} = \begin{array}{c|cc} & h^2 & \Sigma_3 \\ \hline h^2 & 3 & 3 \\ \Sigma_3 & 3 & 7 \end{array}.$$

(3) When X contains a quartic scroll Σ_4 or a quintic del Pezzo surface T then we have

$$K_{14} = \begin{array}{c|cc} & h^2 & \Sigma_4 \\ \hline h^2 & 3 & 4 \\ \Sigma_4 & 4 & 10 \end{array} \simeq \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 5 \\ T & 5 & 13 \end{array}, \quad T = 3h^2 - \Sigma_4.$$

We return to cubic fourfolds containing two disjoint planes P_1 and P_2. Here we have a rank three lattice of algebraic classes, containing a distinguished rank two lattice:

$$\begin{array}{c|ccc} & h^2 & P_1 & P_2 \\ \hline h^2 & 3 & 1 & 1 \\ P_1 & 1 & 3 & 0 \\ P_2 & 1 & 0 & 3 \end{array} \supset \begin{array}{c|cc} & h^2 & \Sigma_4 \\ \hline h^2 & 3 & 4 \\ \Sigma_4 & 4 & 10 \end{array}, \quad \Sigma_4 = 2h^2 - P_1 - P_2.$$

2.3. **Structural results.** Voisin's Torelli Theorem and the geometric description of the period domains yields a qualitative description of the special cubic fourfolds.

Consider cubic fourfolds X with a prescribed saturated sublattice

$$h^2 \in M \subset L \simeq H^4(X, \mathbb{Z})$$

of algebraic classes. The Hodge-Riemann bilinear relations imply that M is positive definite. Then the Hodge structure on X is determined by $H^1(X, \Omega_X^3) \subset M^\perp \otimes \mathbb{C}$, which is isotropic for $\langle , \rangle$. The relevant period domain is

$$\mathcal{D}_M = \{[\lambda] \in \mathbb{P}(M^\perp \otimes \mathbb{C}) : \langle \lambda, \lambda \rangle = 0\},$$

or at least the connected component with the correct orientation. (See [Mar11, §4] for more discussion of orientations.) The Torelli theorem [Voi86] asserts that the period map

$$\begin{array}{rcl} \tau : \mathcal{C} & \to & \Gamma \backslash \mathcal{D}_{\mathbb{Z}h^2} \\ X & \mapsto & H^1(X, \Omega_X^3) \end{array}$$

is an open immersion; Γ is the group of automorphisms of the primitive cohomology lattice L^0 arising from the monodromy of cubic fourfolds. Cubic fourfolds with additional algebraic cycles, indexed by a saturated sublattice

$$M \subsetneq M' \subset L,$$

correspond to the linear sections of $\mathcal{D}_M$ of codimension $\mathrm{rank}(M'/M)$.

Proposition 12. *Fix a positive definite lattice M of rank r admitting a saturated embedding*

$$h^2 \in M \subset L.$$

If this exists then $M^0 = \{h^2\}^\perp \subset M$ is necessarily even, as it embeds in L^0.

Let $\mathcal{C}_M \subset \mathcal{C}$ denote the cubic fourfolds X admiting algebraic classes with this lattice structure

$$h^2 \in M \subset H^{2,2}(X, \mathbb{Z}) \subset L.$$

Then $\mathcal{C}_M$ has codimension $r-1$, provided it is non-empty.

We can make this considerably more precise in rank two. For each labelling K, pick a generator $K \cap L^0 = \mathbb{Z}v$. Classifying orbits of primitive positive vectors $v \in L^0$ under the automorphisms of this lattice associated with the monodromy representation yields:

Theorem 13. [Has00, §3] *Let $\mathcal{C}_d \subset \mathcal{C}$ denote the special cubic fourfolds admitting a labelling of discriminant d. Then $\mathcal{C}_d$ is non-empty if and only if $d \geq 8$ and $d \equiv 0, 2 \pmod 6$. Moreover, $\mathcal{C}_d$ is an irreducible divisor in $\mathcal{C}$.*

Fix a discriminant d and consider the locus $\mathcal{C}_d \subset \mathcal{C}$. The Torelli Theorem implies that irreducible components of $\mathcal{C}_d$ correspond to saturated rank two sublattices realizations

$$h^2 \in K \subset L$$

up to monodromy. The monodromy of cubic fourfolds acts on L via $\mathrm{Aut}(L, h^2)$, the automorphisms of the lattice L preserving h^2. Standard results on embeddings of lattices imply there is a unique $K \subset L$ modulo $\mathrm{Aut}(L, h^2)$. The monodromy group is an explicit finite index subgroup of $\mathrm{Aut}(L, h^2)$, which still acts transitively on these sublattices. Hence $\mathcal{C}_d$ is irreducible.

The rank two lattices associated with labellings of discrimiant d are:

$$K_d := \begin{cases} \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 1 \\ S & 1 & 2n+1 \end{array} & \text{if } d = 6n+2, n \geq 1 \\ \\ \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 0 \\ S & 0 & 2n \end{array} & \text{if } d = 6n, n \geq 2. \end{cases}$$

The cases $d = 2$ and 6

$$K_2 = \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 1 \\ S & 1 & 1 \end{array} \qquad K_6 = \begin{array}{c|cc} & h^2 & S \\ \hline h^2 & 3 & 0 \\ S & 0 & 2 \end{array} \tag{2.2}$$

correspond to limiting Hodge structures arising from singular cubic fourfolds: the symmetric determinant cubic fourfolds [Has00, §4.4] and the cubic fourfolds with an ordinary double point [Has00, §4.2]. The non-special cohomology lattice $K_d^{\perp}$ is also well-defined for all $(X, K_d) \in \mathcal{C}_d$.

Laza [Laz10], building on work of Looijenga [Loo09], gives precise necessary and sufficient conditions for when the $\mathcal{C}_M$ in Proposition 12 are nonempty:

- M is positive definite and admits a saturated embedding $h^2 \in M \subset L$;
- there exists no sublattice $h^2 \in K \subset M$ with $K \simeq K_2$ or K_6 as in (2.2).

Detailed descriptions of the possible lattices of algebraic classes are given by Mayanskiy [May11]. Furthermore, Laza obtains a characterization of the image of the period map for cubic fourfolds: it is complement of the divisors parametrizing 'special' Hodge structures with a labelling of discriminant 2 or 6.

Remark 14. Li and Zhang [LZ13] have found a beautiful generating series for the degrees of special cubic fourfolds of discriminant d, expressed via modular forms.

We have seen concrete descriptions of surfaces arising in special cubic fourfolds for $d = 8, 12, 14$. Nuer [Nue15, §3] writes down explicit smooth rational surfaces arising in generic special cubic fourfolds of discriminants $d \leq 38$. These are blow-ups of the plane at points in general position, embedded via explicit linear series, e.g.,

(1) For $d = 18$, let S be a generic projection into $\mathbb{P}^5$ of a sextic del Pezzo surface in $\mathbb{P}^6$.
(2) For $d = 20$, let S be a Veronese embedding of $\mathbb{P}^2$.

Question 15. Is the algebraic cohomology of a special cubic fourfold generated by the classes of smooth rational surfaces?

Voisin has shown that the cohomology can be generated by smooth surfaces (see the proof of [Voi14, Th. 5.6]) *or* by possibly singular rational surfaces [Voi07]. Low discriminant examples suggest we might be able to achieve both.

A by-product of Nuer's approach, where it applies, is to prove that the $\mathcal{C}_d$ are unirational. However, for $d \gg 0$ the loci $\mathcal{C}_d$ are of general type [TVA15]. So a different approach is needed in general.

2.4. **Census of rational cubic fourfolds.** Using this framework, we enumerate the smooth cubic fourfolds known to be rational:

(1) cubic fourfolds in $\mathcal{C}_{14}$, the closure of the Pfaffian locus;
(2) cubic fourfolds in $\mathcal{C}_8$, the locus containing a plane P, such that there exists a class W such that $\langle W, (h^2 - P)\rangle$ is odd.

For the second case, note that the discriminant of the lattice $M = \mathbb{Z}h^2 + \mathbb{Z}P + \mathbb{Z}W$ has the same parity as $\langle W, (h^2 - P)\rangle$.

Thus all the cubic fourfolds proven to be rational are parametrized by one divisor $\mathcal{C}_{14}$ and a countably-infinite union of codimension two subvarieties $\mathcal{C}_M \subset \mathcal{C}_8$.

Question 16. Is there a rational (smooth) cubic fourfold not in the enumeration above?

There are conjectural frameworks (see §3.3 and also §3.6) predicting that many special cubic fourfolds should be rational. However, few new examples of cubic fourfolds have been found to support these frameworks.

3. Associated K3 surfaces

3.1. **Motivation.** The motivation for considering associated K3 surfaces comes from the Clemens-Griffiths [CG72] proof of the irrationality of cubic threefolds. Suppose X is a rational threefold. Then we have an isomorphism of polarized Hodge structures

$$H^3(X, \mathbb{Z}) = \oplus_{i=1}^n H^1(C_i, \mathbb{Z})(-1)$$

where the C_i are smooth projective curves. Essentially, the C_i are blown up in the birational map

$$\mathbb{P}^3 \dashrightarrow X.$$

If X is a rational *fourfold* then we can look for the cohomology of surfaces blown up in the birational map

$$\rho : \mathbb{P}^4 \dashrightarrow X.$$

Precisely, if P is a smooth projective fourfold, $S \subset P$ an embedded surface, and $\tilde{P} = \mathrm{Bl}_S(P)$ then we have [Ful84, §6.7]

$$H^4(\tilde{P}, \mathbb{Z}) \simeq H^4(P, \mathbb{Z}) \oplus_\perp H^2(S, \mathbb{Z})(-1).$$

The homomorphism $H^4(P,\mathbb{Z}) \to H^4(\tilde{P},\mathbb{Z})$ is induced by pull-back; the homomorphism $H^2(S,\mathbb{Z})(-1) \to H^4(\tilde{P},\mathbb{Z})$ comes from the composition of pull-back and push-forward

$$\begin{array}{ccc} E = \mathbb{P}(N_{S/P}) & \hookrightarrow & \tilde{P} \\ \downarrow & & \\ S & & \end{array}$$

where $N_{S/P}$ is the normal bundle and E the exceptional divisor. Blowing up points in P contributes Hodge-Tate summands $\mathbb{Z}(-2)$ with negative self-intersection to its middle cohomology; these have the same affect as blowing up a surface (like $\mathbb{P}^2$) with $H^2(S,\mathbb{Z}) \simeq \mathbb{Z}$. Blowing up curves does not affect middle cohomology.

Applying the Weak Factorization Theorem [Wło03, AKMW02]—that every birational map is a composition of blow-ups and blow-downs along smooth centers—we obtain the following:

Proposition 17. *Suppose X is a rational fourfold. Then there exist smooth projective surfaces $S_1,\dots,S_n$ and $T_1,\dots,T_m$ such that we have an isomorphism of Hodge structures*

$$H^4(X,\mathbb{Z}) \oplus (\oplus_{j=1,\dots,m} H^2(T_j,\mathbb{Z})(-1)) \simeq \oplus_{i=1,\dots,n} H^2(S_i,\mathbb{Z})(-1).$$

Unfortunately, it is not clear how to extract a computable invariant from this observation; but see [ABGvB13, Kul08] for work in this direction.

3.2. **Definitions and reduction to lattice equivalences.** In light of examples illustrating how rational cubic fourfolds tend to be entangled with K3 surfaces, it is natural to explore this connection on the level of Hodge structures.

Let (X,K) denote a labelled special cubic fourfold. A polarized K3 surface (S,f) is *associated* with (X,K) if there exists an isomorphism of lattices

$$H^4(X,\mathbb{Z}) \supset K^\perp \xrightarrow{\sim} f^\perp \subset H^2(S,\mathbb{Z})(-1)$$

respecting Hodge structures.

Example 18 (Pfaffians)**.** Let X be a Pfaffian cubic fourfold (see §1.6) and

$$K_{14} = \begin{array}{c|cc} & h^2 & \Sigma_4 \\ \hline h^2 & 3 & 4 \\ \Sigma_4 & 4 & 10 \end{array} \simeq \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 5 \\ T & 5 & 13 \end{array},$$

the lattice containing the classes of the resulting quartic scrolls and quintic del Pezzo surfaces. Let (S,f) be the K3 surface of degree 14

arising from the Pfaffian construction. Then (S, f) is associated with (X, K_{14}).

Example 19 (A suggestive non-example)**.** As we saw in §1.6, a cubic fourfold X containing a plane P gives rise to a degree two K3 surface (S, f). However, this is *not* generally associated with the cubic fourfold. If $K_8 \subset H^4(X, \mathbb{Z})$ is the labelling then

$$K_8^\perp \subset f^\perp$$

as an index two sublattice [vG05, §9.7]. However, when the quadric bundle $q : \mathrm{Bl}_P(X) \to \mathbb{P}^2$ admits a section (so that X is rational), S often admits a polarization g such that (S, g) is associated with some labelling of X. See [Kuz10, Kuz15] for further discussion.

Proposition 20. *The existence of an associated K3 surface depends only on the discriminant of the rank two lattice K.*

Here is an outline of the proof; we refer to [Has00, §5] for details.

Recall the discussion of Theorem 13 in §2.3: For each discriminant $d \equiv 0, 2 \pmod 6$ with $d > 6$, there exists a lattice $K_d^\perp$ such that each special cubic fourfold of discriminant d (X, K) has $K^\perp \simeq K_d^\perp$. Consider the primitive cohomology lattice

$$\Lambda_d := f^\perp \subset H^2(S, \mathbb{Z})(-1)$$

for a polarized K3 surface (S, f) of degree d. The moduli space $\mathcal{N}_d$ of such surfaces is connected, so Λ_d is well-defined up to isomorphism.

We claim (X, K_d) admits an associated K3 surface if and only if there exists an isomorphism of lattices

$$\iota : K_d^\perp \xrightarrow{\sim} \Lambda_d.$$

This is clearly a necessary condition. For sufficiency, given a Hodge structure on Λ_d surjectivity of the Torelli map for K3 surfaces [Siu81] ensures there exists a K3 surface S and a divisor f with that Hodge structure. It remains to show that f can be taken to be a polarization of S, i.e., there are no (-2)-curves orthogonal to f. After twisting and applying ι, any such curve yields an algebraic class $R \in H^{2,2}(X, \mathbb{Z})$ with $\langle R, R\rangle = 2$ and $\langle h^2, R\rangle = 0$. In other words, we obtain a labelling

$$K_6 = \left\langle h^2, R \right\rangle \subset H^4(X, \mathbb{Z}).$$

Such labellings are associated with nodal cubic fourfolds, violating the smoothness of X.

Based on this discussion, it only remains to characterize when the lattice isomorphism exists. Nikulin's theory of discriminant forms [Nik79] yields:

Proposition 21. [Has00, Prop. 5.1.4]*: Let d be a positive integer congruent to 0 or 2 modulo 6. Then there exists an isomorphism*

$$\iota : K_d^{\perp} \xrightarrow{\sim} \Lambda_d(-1)$$

if and only if d is not divisible by $4, 9$ or any odd prime congruent to 2 modulo 3.

Definition 22. An even integer $d > 0$ is *admissible* if it is not divisble by $4, 9$ or any odd prime congruent to 2 modulo 3.

Thus we obtain:

Theorem 23. *A special cubic fourfold (X, K_d) admits an associated K3 surface if and only if d is admissible.*

3.3. **Connections with Kuznetsov's philosophy.** Kuznetsov has proposed a criterion for rationality expressed via derived categories [Kuz10, Conj. 1.1] [Kuz15]: Let X be a cubic fourfold, $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on X, and $\mathcal{A}_X$ the subcategory orthogonal to the exceptional collection $\{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$. Kuznetsov proposes that X is rational if and only if $\mathcal{A}_X$ is equivalent to the derived category of a K3 surface. He verifies this for the known examples of rational cubic fourfolds.

Addington and Thomas [AT14, Th. 1.1] show that the generic $(X, K_d) \in \mathcal{C}_d$ satisfies Kuznetsov's derived category condition precisely when d is admissible. In §3.7 we present some of the geometry behind this result. Thus we find:

> Kuznetsov's conjecture would imply that the generic $(X, K_d) \in \mathcal{C}_d$ for admissible d is rational.

In particular, special cubic fourfolds of discriminants $d = 26, 38, 42, \ldots$ would all be rational!

3.4. **Naturality of associated K3 surfaces.** There are *a priori* many choices for the lattice isomorphism ι so a given cubic fourfold could admit several associated K3 surfaces. Here we will analyze this more precisely. Throughout, assume that d is admissible.

We require a couple variations on $\mathcal{C}_d \subset \mathcal{C}$:

- Let $\mathcal{C}_d^{\nu}$ denote labelled cubic fourfolds, with a saturated lattice $K \ni h^2$ of algebraic classes of rank two and discriminant d.
- Let $\mathcal{C}_d'$ denote pairs consisting of a cubic fourfold X and a saturated embedding of K_d into the algebraic cohomology.

We have natural maps

$$\mathcal{C}_d' \to \mathcal{C}_d^{\nu} \to \mathcal{C}_d.$$

The second arrow is normalization over cubic fourfolds admitting multiple labellings of discriminant d. To analyze the first arrow, note that the K_d admits non-trivial automorphisms fixing h^2 if and only if $6|d$. Thus the first arrow is necessarily an isomorphism unless $6|d$. When $6|d$ the lattice K_d admits an automorphism acting by multiplication by -1 on the orthogonal complement of h^2. $\mathcal{C}'_d$ is irreducible if this involution can be realized in the monodromy group. An analysis of the monodromy group gives:

Proposition 24. [Has00, §5]*: For each admissible $d > 6$, $\mathcal{C}'_d$ is irreducible and admits an open immersion into the moduli space $\mathcal{N}_d$ of polarized K3 surfaces of degree d.*

Corollary 25. *Assume $d > 6$ is admissible. If $d \equiv 2 \pmod 6$ then $\mathcal{C}_d$ is birational to $\mathcal{N}_d$. Otherwise $\mathcal{C}_d$ is birational to a quotient of $\mathcal{N}_d$ by an involution.*

Thus for $d = 42, 78, \ldots$ cubic fourfolds $X \in \mathcal{C}_d$ admit *two* associated K3 surfaces.

Even the open immersions from the double covers

$$j_{\iota,d} : \mathcal{C}'_d \hookrightarrow \mathcal{N}_d$$

are typically not canonical. The possible choices correspond to orbits of the isomorphism

$$\iota : K_d^\perp \xrightarrow{\sim} \Lambda_d(-1)$$

under postcomposition by automorphisms of Λ_d coming from the monodromy of K3 surfaces and precomposition by automorphisms of $K_d^\perp$ coming from the the subgroup of the monodromy group of cubic fourfolds fixing the elements of K_d.

Proposition 26. [Has00, §5.2] *Choices of*

$$j_{\iota,d} : \mathcal{C}'_d \hookrightarrow \mathcal{N}_d$$

are in bijection with the set

$$\{a \in \mathbb{Z}/d\mathbb{Z} : a^2 \equiv 1 \pmod{2d}\}/\pm 1.$$

If d is divisible by r distinct odd primes then there are 2^{r-1} possibilities.

Remark 27. The ambiguity in associated K3 surfaces can be expressed in the language of equivalence of derived categories. Suppose that (S_1, f_1) and (S_2, f_2) are polarized K3 surfaces of degree d, both associated with a special cubic fourfold of discriminant d. This means we have an isomorphism of Hodge structures

$$H^2(S_1, \mathbb{Z})_{\text{prim}} \simeq H^2(S_2, \mathbb{Z})_{\text{prim}}$$

so their transcendental cohomologies are isomorphic. Orlov's Theorem [Orl97, §3] implies S_1 and S_2 are *derived equivalent*, i.e., their bounded derived categories of coherent sheaves are equivalent.

Proposition 26 may be compared with the formula counting derived equivalent K3 surfaces in [HLOY03]. We will revisit this issue in §3.7.

3.5. **Interpreting associated K3 surfaces I.** We offer geometric interpretations of associated K3 surfaces. These are most naturally expressed in terms of moduli spaces of sheaves on K3 surfaces.

Let M be an irreducible holomorphic symplectic variety, i.e., a smooth simply connected projective variety such that $\Gamma(M, \Omega^2_M) = \mathbb{C}\omega$ where ω is everywhere nondegenerate. The cohomology $H^2(M, \mathbb{Z})$ admits a distinguished integral quadratic form $(,)$, called the *Beauville-Bogomolov* form [Bea83]. Examples include:

- K3 surfaces S with $(,)$ the intersection form;
- Hilbert schemes $S^{[n]}$ of length n zero dimensional subschemes on a K3 surface S, with

$$H^2(S^{[n]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta, \quad (\delta, \delta) = -2(n-1), \tag{3.1}$$

  where 2δ parametrizes the non-reduced subschemes.

Example 28. Let (S, f) be a generic degree 14 K3 surface. Then $S^{[2]} \simeq F_1(X) \subset \mathrm{Gr}(2, 6)$ where X is a Pfaffian cubic fourfold (see §1.6). The polarization induced from the Grassmannian is $2f - 5\delta$; note that

$$(2f - 5\delta, 2f - 5\delta) = 4 \cdot 14 - 25 \cdot 2 = 6.$$

The example implies that if $F_1(X) \subset \mathrm{Gr}(2, 6)$ is the variety of lines on an arbitrary cubic fourfold then the polarization $g = \alpha(h^2)$ satisfies

$$(g, g) = 6, \quad \big(g, H^2(F_1(X), \mathbb{Z})\big) = 2\mathbb{Z}.$$

It follows that the Abel-Jacobi map is an isomorphism of *abelian groups*

$$\alpha : H^4(X, \mathbb{Z}) \to H^2(F_1(X), \mathbb{Z})(-1) \tag{3.2}$$

Indeed, α is an isomorphism on primitive cohomology and both

$$\mathbb{Z}h^2 \oplus H^4(X, \mathbb{Z})_{prim} \subset H^4(X, \mathbb{Z})$$

and

$$\mathbb{Z}g \oplus H^2(F_1(X), \mathbb{Z})_{prim} \subset H^2(F(X), \mathbb{Z})$$

have index three as subgroups.

The Pfaffian case is the first of an infinite series of examples:

Theorem 29. [Has00, §6] [Add16, Th. 2] *Fix an integer of the form $d = 2(n^2 + n + 1)/a^2$, where $n > 1$ and $a > 0$ are integers. Let X be*

a cubic fourfold in $\mathcal{C}_d$ with variety of lines $F_1(X)$. Then there exists a polarized K3 surface (S, f) of degree d and a birational map

$$F_1(X) \dashrightarrow S^{[2]}.$$

If $a = 1$ and $X \in \mathcal{C}_d$ in generic then $F_1(X) \simeq S^{[2]}$ with polarization $g = 2f - (2n+1)\delta$.

The first part relies on Verbitsky's global Torelli theorem for hyperkähler manifolds [Mar11]. The last assertion is proven via a degeneration/specialization argument along the nodal cubic fourfolds, which correspond to degree six K3 surfaces (S', f'). We specialize so that $F = S'^{[2]}$ admits involutions not arising from involutions of S'. Thus the deformation space of F admits several divisors parametrizing Hilbert schemes of K3 surfaces.

Since the primitive cohomology of S sits in $H^2(S^{[2]}, \mathbb{Z})$, the Abel-Jacobi map (2.1) explains why S is associated with X. If $3|d$ $(d \neq 6)$ then Theorem 29 and Corollary 25 yield two K3 surfaces S_1 and S_2 such that

$$F_1(X) \simeq S_1^{[2]} \simeq S_2^{[2]}.$$

With a view toward extending this argument, we compute the cohomology of the varieties of lines of special cubic fourfolds. This follows immediately from (3.2):

Proposition 30. *Let (X, K_d) be a special cubic fourfold of discriminant d, $F_1(X) \subset \mathrm{Gr}(2,6)$ its variety of lines, and $g = \alpha(h^2)$ the resulting polarization. Then $\alpha(K_d)$ is saturated in $H^2(F_1(X), \mathbb{Z})$ and*

$$\alpha(K_d) \simeq \begin{cases} \begin{array}{c|cc} & g & T \\ \hline g & 6 & 0 \\ T & 0 & -2n \end{array} & \text{if } d = 6n \\ \\ \begin{array}{c|cc} & g & T \\ \hline g & 6 & 2 \\ T & 2 & -2n \end{array} & \text{if } d = 6n+2. \end{cases}$$

The following example shows that Hilbert schemes are insufficient to explain all associated K3 surfaces:

Example 31. Let $(X, K_{74}) \in \mathcal{C}_{74}$ be a generic point, which admits an associated K3 surface by Theorem 23. There does not exist a K3 surface S with $F_1(X) \simeq S^{[2]}$, even birationally. Indeed, the $H^2(M, \mathbb{Z})$ is a birational invariant of holomorphic symplectic manifolds but

$$\alpha(K_{74}) \simeq \begin{pmatrix} 6 & 2 \\ 2 & -24 \end{pmatrix}$$

is not isomorphic to the Picard lattice of the Hilbert scheme

$$\mathrm{Pic}(S^{[2]}) \simeq \begin{pmatrix} 74 & 0 \\ 0 & -2 \end{pmatrix}.$$

Addington [Add16] gives a systematic discussion of this issue.

3.6. **Derived equivalence and Cremona transformations.** In light of the ambiguity of associated K3 surfaces (see Remark 27) and the general discussion in §3.1, it is natural to seek diagrams

$$\begin{array}{ccccc} & & \mathrm{Bl}_{S_1}(\mathbb{P}^4) & \xrightarrow{\sim} & \mathrm{Bl}_{S_2}(\mathbb{P}^4) \\ & \swarrow^{\beta_1} & & & \searrow^{\beta_2} \\ \mathbb{P}^4 & & & & \mathbb{P}^4 \end{array} \tag{3.3}$$

where β_i is the blow-up along a smooth surface S_i, with S_1 and S_2 derived equivalent but not birational.

Cremona transformations of $\mathbb{P}^4$ with smooth surfaces as their centers have been classified by Crauder and Katz [CK89, §3]; possible centers are either quintic elliptic scrolls or surfaces $S \subset \mathbb{P}^4$ of degree ten given by the vanishing of the 4×4 minors of a 4×5 matrix of linear forms. A generic surface of the latter type admits divisors

	K_S	H
K_S	5	10
H	10	10

where H is the restriction of the hyperplane class from $\mathbb{P}^4$; this lattice admits an involution fixing K_S with $H \mapsto 4K_S - H$. See [Ran88, Prop. 9.18ff.], [Ran91], and [Bak10, p.280] for discussion of these surfaces.

The Crauder-Katz classification therefore precludes diagrams of the form (3.3). We therefore recast our search as follows:

Question 32. Does there exist a diagram

$$\begin{array}{ccc} & X & \\ \swarrow^{\beta_1} & & \searrow^{\beta_2} \\ \mathbb{P}^4 & & \mathbb{P}^4 \end{array}$$

where X is smooth and the β_i are birational projective morphisms, and K3 surfaces S_1 and S_2 such that

- S_1 and S_2 are derived equivalent but not isomorphic;
- S_1 is birational to a center of β_1 but not to any center of β_2;
- S_2 is birational to a center of β_2 but not to any center of β_1?

This could yield counterexamples to the Larsen-Lunts cut-and-paste question on Grothendieck groups, similar to those found by Borisov [Bor14]. Galkin and Shinder [GS14, §7] showed that *if* the class of the affine line were a non-zero divisor in the Grothendieck group then for each rational cubic fourfold X the variety of lines $F_1(X)$ would be birational to $S^{[2]}$ for some K3 surface S. (Note that Borisov shows it *is* a zero divisor.) We have seen (Theorem 29) that this condition holds for infinitely many d.

3.7. **Interpreting associated K3 surfaces II.** Putting Theorem 29 on a general footing requires a larger inventory of varieties arising from K3 surfaces. We shall use fundamental results on moduli spaces of sheaves on K3 surfaces due to Mukai [Muk87], Yoshioka, and others. Let S be a complex projective K3 surface. The *Mukai lattice*

$$\tilde{H}^*(S,\mathbb{Z}) = H^0(S,\mathbb{Z})(-1) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})(1)$$

with unimodular form

$$((r_1, D_1, s_1), (r_2, D_2, s_2)) = -r_1 s_2 + D_1 D_2 - r_2 s_1$$

carries the structure of a Hodge structure of weight two. (The zeroth and fourth cohomology are of type $(1,1)$ and the middle cohomology carries its standard Hodge structure.) Thus we have

$$\tilde{H}^*(S,\mathbb{Z}) \simeq H^2(S,\mathbb{Z}) \oplus_\perp \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Suppose $v = (r, D, s) \in \tilde{H}^*(S,\mathbb{Z})$ is primitive of type $(1,1)$ with $(v,v) \geq 0$. Assume that one of the following holds:

- $r > 0$;
- $r = 0$ and D is ample.

Fixing a polarization h on S, we may consider the moduli space $M_v(S)$ of sheaves Gieseker stable with respect to h. Here r is the rank, D is the first Chern class, and $r+s$ is the Euler characteristic. For h chosen suitably general (see [Yos01, §0] for more discussion), $M_v(S)$ is a projective holomorphic symplectic manifold deformation equivalent to the Hilbert scheme of length $\frac{(v,v)}{2}+1$ zero dimensional subschemes of a K3 surface [Yos01, §8], [Yos00, Th. 0.1]. Thus $H^2(M_v(S),\mathbb{Z})$ comes with a Beauville-Bogomolov form and we have an isomorphism of Hodge structures

$$H^2(M_v(S),\mathbb{Z}) = \begin{cases} v^\perp/\mathbb{Z}v & \text{if } (v,v) = 0 \\ v^\perp & \text{if } (v,v) > 0 \end{cases} \tag{3.4}$$

Example 33. The case of ideal sheaves of length two subschemes is $r = 1$, $D = 0$, and $s = -1$. Here we recover formula (3.1) for $H^2(S^{[2]}, \mathbb{Z}) = H^2(M_{(1,0,-1)}(S), \mathbb{Z})$.

We shall also need recent results of Bayer-Macrì [BM14b, BM14a]: Suppose that M is holomorphic symplectic and birational to $M_v(S)$ for some K3 surface S. Then we may interpret M as a moduli space of objects on the derived category of S with respect to a suitable Bridgeland stability condition [BM14a, Th. 1.2].

Finally, recall Nikulin's approach to lattice classification and embeddings [Nik79]. Given an even unimodular lattice Λ and a primitive nondegenerate sublattice $N \subset \Lambda$, the discriminant group $d(N) := N^*/N$ is equipped with a $(\mathbb{Q}/2\mathbb{Z})$-valued quadratic form q_N, which encodes most of the p-adic invariants of N. The orthogonal complement $N^\perp \subset \Lambda$ has related invariants

$$(d(N^\perp), -q_{N^\perp}) \simeq (d(N), q_N).$$

Conversely, given a pair of nondegenerate even lattices with complementary invariants, there exists a unimodular even lattice containing them as orthogonal complements [Nik79, §12].

Theorem 34. *Let (X, K_d) denote a labelled special cubic fourfold of discriminant d. Then d is admissible if and only if there exists a polarized K3 surface (S, f), a Mukai vector $v = (r, af, s) \in \tilde{H}(S, \mathbb{Z})$, a stability condition σ, and an isomorphism*

$$\varpi : M_v(S) \xrightarrow{\sim} F_1(X)$$

from the moduli space of objects in the derived category stable with respect to σ, inducing an isomorphism between the primitive cohomology of (S, f) and the twist of the non-special cohomology of (X, K_d).

This is essentially due to Addington and Thomas [AT14, Add16].

Proof. Let's first do the reverse direction; this gives us an opportunity to unpack the isomorphisms in the statement. Assume we have the moduli space and isomorphism as described. After perhaps applying a shift and taking duals, we may assume $r \geq 0$ and $a \geq 0$; if $a = 0$ then $v = (1, 0, -1)$, i.e., the Hilbert scheme up to birational equivalence. We still have the computation (3.4) of the cohomology

$$H^2(M_v(S), \mathbb{Z}) = v^\perp \subset \tilde{H}^*(S, \mathbb{Z});$$

see [BM14b, Th. 6.10] for discussion relating moduli spaces of Bridgeland-stable objects and Gieseker-stable sheaves. Thus we obtain a saturated embedding of the primitive cohomology of (S, f)

$$\Lambda_d \hookrightarrow H^2(M_v(S), \mathbb{Z}).$$

The isomorphism ϖ allows us to identify this with a sublattice of $H^2(F(X),\mathbb{Z})$ coinciding with $\alpha(K_d)^\perp$. Basic properties of the Abel-Jacobi map (2.1) imply that (S,f) is associated with (X,K_d), thus d is admissible by Theorem 23.

Now assume d is admissible and consider the lattice $-K_d^\perp$, the orthogonal complement of K_d in the middle cohomology of a cubic fourfold with the intersection form reversed. This is an even lattice of signature $(2,19)$.

If X is a cubic fourfold then there is a natural primitive embedding of lattices [Mar11, §9]

$$H^2(F_1(X),\mathbb{Z}) \hookrightarrow \Lambda$$

where Λ is isomorphic to the Mukai lattice of a K3 surface

$$\Lambda = U^{\oplus 4} \oplus (-E_8)^{\oplus 2}.$$

Here 'natural' means that the monodromy representation on $H^2(F_1(X),\mathbb{Z})$ extends naturally to Λ.

Now consider the orthogonal complement M_d to $-K_d^\perp$ *in the Mukai lattice* Λ. Since d is admissible

$$-K_d^\perp \simeq \Lambda_d \simeq (-d) \oplus U^{\oplus 2} \oplus (-E_8)^{\oplus 2}$$

so $d(-K_d^\perp) \simeq \mathbb{Z}/d\mathbb{Z}$ and $q_{-K_d^\perp}$ takes value $-\frac{1}{d} \pmod{2\mathbb{Z}}$ on one of the generators. Thus $d(M_d) = \mathbb{Z}/d\mathbb{Z}$ and takes value $\frac{1}{d}$ on one of the generators. There is a distinguished lattice with these invariants

$$(d) \oplus U.$$

Kneser's Theorem [Nik79, §13] implies there is a unique such lattice, i.e., $M_d \simeq (d) \oplus U$.

Thus for each generator $\gamma \in d(-K_d^\perp)$ with $(\gamma,\gamma) = -\frac{1}{d} \pmod{2\mathbb{Z}}$, we obtain an isomorphism of Hodge structures

$$H^2(F_1(X),\mathbb{Z}) \subset \Lambda \simeq \tilde{H}^*(S,\mathbb{Z})$$

where (S,f) is a polarized K3 surface of degree d. Here we take f to be one of generators of $U^\perp \subset M_d$. Let $v \in \Lambda$ generate the orthogonal complement to $H^2(F_1(X),\mathbb{Z})$; it follows that $v = (r, af, s) \in \tilde{H}^*(S,\mathbb{Z})$ and after reversing signs we may assume $r \geq 0$.

Consider a moduli space $M_v(S)$ of sheaves stable with respect to suitably generic polarizations on S. Our lattice analysis yields

$$\phi : H^2(F_1(X),\mathbb{Z}) \xrightarrow{\sim} H^2(M_v(S),\mathbb{Z}),$$

an isomorphism of Hodge structures taking $\alpha(K_d^{\perp})$ to the primitive cohomology of S. The Torelli Theorem [Mar11, Cor. 9.9] yields a birational map

$$\varpi_1 : M_v(S) \dashrightarrow F_1(X);$$

since both varieties are holomorphic symplectic, there is a natural induced isomorphism [Huy99, Lem. 2.6]

$$\varpi_1^* : H^2(F_1(X), \mathbb{Z}) \xrightarrow{\sim} H^2(M_v(S), \mathbb{Z}),$$

compatible with Beauville-Bogomolov forms and Hodge structures.

A priori ϕ and ϖ_1^* might differ by an automorphism of the cohomology of $M_v(S)$. If this automorphism permutes the two connected components of the positive cone in $H^2(M_v(S), \mathbb{R})$, we may reverse the sign of ϕ. If it fails to preserve the moving cone, we can apply a sequence of monodromy reflections on $M_v(S)$ until this is the case [Mar11, Thm. 1.5,1.6]. These are analogues to reflections by (-2)-classes on the cohomology of K3 surfaces and are explicitly known for manifolds deformation equivalent to Hilbert schemes on K3 surface; see [HT01, HT09] for the case of dimension four and [Mar11, §9.2.1] for the general picture. In this situation, the reflections correspond to spherical objects in the derived category of S orthogonal to v, thus give rise to autoequivalences on the derived category [ST01, HLOY04]. We use these to modify the stability condition on $M_v(S)$. The resulting

$$\varpi_2^* : H^2(F_1(X), \mathbb{Z}) \xrightarrow{\sim} H^2(M_v(S), \mathbb{Z})$$

differs from ϕ by an automorphism that preserves moving cones, but may not preserve polarizations. Using [BM14a, Th. 1.2(b)], we may modify the stability condition on $M_v(S)$ yet again so that the polarization g on $F_1(X)$ is identified with a polarization on $M_v(S)$. Then the resulting

$$\varpi = \varpi_3 : M_v(S) \dashrightarrow F_1(X);$$

preserves ample cones and thus is an isomorphism. Hence $F_1(X)$ is isomorphic to some moduli space of σ-stable objects over S. □

Remark 35. As suggested by Addington and Thomas [AT14, §7.4], it should be possible to employ stability conditions to show that $\mathcal{A}_X$ is equivalent to the derived category of a K3 surface if and only if X admits an associated K3 surface. (We use the notation of §3.3.)

Remark 36. Two K3 surfaces S_1 and S_2 are derived equivalent if and only if

$$\tilde{H}(S_1, \mathbb{Z}) \simeq \tilde{H}(S_2, \mathbb{Z})$$

as weight two Hodge structures [Orl97, §3]. The proof of Theorem 34 explains why the K3 surfaces associated with a given cubic fourfold are all derived equivalent, as mentioned in Remark 27.

There are other geometric explanations for K3 surfaces associated with special cubic fourfolds. Fix X to be a cubic fourfold not containing a plane. Let $M_3(X)$ denote the moduli space of generalized twisted cubics on X, i.e., closed subschemes arising as flat limits of twisted cubics in projective space. Then $M_3(X)$ is smooth and irreducible of dimension ten [LLSvS15, Thm. A]. Choose $[C] \in M_3(X)$ such that C is a smooth twisted cubic curve and $W := \mathrm{span}(C) \cap X$ is a smooth cubic surface. Then the linear series $|\mathcal{O}_W(C)|$ is two-dimensional, so we have a distinguished subvariety

$$[C] \in \mathbb{P}^2 \subset M_3(X).$$

Then there exists an eight-dimensional irreducible holomorphic symplectic manifold Z and morphisms

$$M_3(X) \xrightarrow{a} Z' \xrightarrow{\sigma} Z$$

where a is an étale-locally trivial $\mathbb{P}^2$ bundle and σ is birational [LLSvS15, Thm. B]. Moreover, Z is deformation equivalent to the Hilbert scheme of length four subschemes on a K3 surface [AL15]. Indeed, if X is Pfaffian with associated K3 surface S then Z is birational to $S^{[4]}$. It would be useful to have a version of Theorem 34 with Z playing the role of $F_1(X)$.

4. Unirational parametrizations

We saw in §1.7 that smooth cubic fourfolds always admit unirational parametrizations of degree two. How common are unirational parametrizations of odd degree?

We review the double point formula [Ful84, §9.3]: Let S' be a nonsingular projective surface, P a nonsingular projective fourfold, and $f : S' \to P$ a morphism with image $S = f(S')$. We assume that $f : S' \to S$ is an isomorphism away from a finite subset of S'; equivalently, S has finitely many singularities with normalization $f : S' \to S$. The *double point class* $\mathbb{D}(f) \in \mathrm{CH}_0(S')$ is given by the formula

$$\begin{aligned} \mathbb{D}(f) &= f^* f_*[S'] - (c(f^*T_P)c(T_{S'})^{-1})_2 \cap [S'] \\ &= f^* f_*[S'] - (c_2(f^*T_P) - c_1(T_{S'})c_1(f^*T_P) + c_1(T_{S'})^2 - c_2(T_{S'})) \end{aligned}$$

We define

$$D_{S \subset P} = \frac{1}{2}([S]_P^2 - c_2(f^*T_P) + c_1(T_{S'})c_1(f^*T_P) - c_1(T_{S'})^2 + c_2(T_{S'}));$$

if S has just transverse double points then $D_{S\subset P}$ is the number of these singularities.

Example 37. (cf. Lemma 3) Let $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^5$ be a quartic scroll, $P = \mathbb{P}^4$, and $f : S' \to \mathbb{P}^4$ a generic projection. Then we have

$$D_{S\subset\mathbb{P}^4} = \frac{1}{2}(16 - 40 + 30 - 8 + 4) = 1$$

double point.

Proposition 38. *Let X be a cubic fourfold and $S \subset X$ a rational surface of degree d. Suppose that S has isolated singularities and smooth normalization S', with invariants $D = \deg(S)$, section genus g, and self-intersection $\langle S, S\rangle$. If*

$$\varrho = \varrho(S, X) := \frac{D(D-2)}{2} + (2 - 2g) - \frac{1}{2}\langle S, S\rangle > 0 \tag{4.1}$$

then X admits a unirational parametrization $\rho : \mathbb{P}^4 \dashrightarrow X$ of degree ϱ.

This draws on ideas from [HT01, §7] and [Voi14, §5].

Proof. We analyze points $x \in X$ such that the projection

$$f = \pi_x : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

maps S birationally to a surface $\hat{S}$ with finitely many singularities, and $f : S \to \hat{S}$ is finite and unramified. Thus S and $\hat{S}$ have the same normalization.

Consider the following conditions:

(1) x is contained in the tangent space to some singular point $s \in S$;
(2) x is contained in the tangent space to some smooth point $s \in S$;
(3) x is contained in a positive-dimensional family of secants to S.

The first condition can be avoided by taking x outside a finite collection of linear subspaces. The second condition can be avoided by taking x outside the tangent variety of S. This cannot concide with a smooth cubic fourfold, which contains at most finitely many two-planes [BHB06, Appendix]. We turn to the third condition. If the secants to S sweep out a subvariety $Y \subsetneq \mathbb{P}^5$ then Y cannot be a smooth cubic fourfold. (The closure of Y contains all the tangent planes to S.) When the secants to S dominate $\mathbb{P}^5$ then the locus of points in $\mathbb{P}^5$ on infinitely many secants has codimension at least two.

Projection from a point on X outside these three loci induces a morphism

$$S \to \hat{S} = f(S)$$

birational and unramified onto its image. Moreover, this image has finitely many singularities that are resolved by normalization.

Let W denote the second symmetric power of S. Since S is rational, W is rational as well. There is a rational map coming from residual intersection

$$\begin{array}{rcl} \rho : W & \dashrightarrow & X \\ s_1 + s_2 & \mapsto & x \end{array}$$

where $\ell(s_1, s_2) \cap X = \{s_1, s_2, x\}$. This is well-defined at the generic point of W as the generic secant to S is not contained in X. (An illustrative special case can be found in Remark 5.)

The degree of ρ is equal to the number of secants to S through a generic point of X. The analysis above shows this equals the number of secants to S through a generic point of $\mathbb{P}^5$. These in turn correspond to the number of double points of $\hat{S}$ arising from *generic* projection to $\mathbb{P}^4$, i.e.,

$$\deg(\rho) = D_{\hat{S}\subset\mathbb{P}^4} - D_{S\subset X}.$$

The double point formula gives

$$\begin{array}{rcl} 2D_{S,X} & = & \langle S, S\rangle - (c_2(T_X|S') - c_1(T_{S'})c_1(T_X|S') + c_1(T_{S'}) - c_2(T_{S'})) \\ 2D_{\hat{S},\mathbb{P}^4} & = & \left\langle \hat{S}, \hat{S} \right\rangle_{\mathbb{P}^4} - (c_2(T_{\mathbb{P}^4}|S') - c_1(T_{S'})c_1(T_{\mathbb{P}^4}|S') + c_1(T_{S'}) - c_2(T_{S'})) \end{array}$$

where $\left\langle \hat{S}, \hat{S} \right\rangle_{\mathbb{P}^4} = D^2$ by Bezout's Theorem. Taking differences (cf.§2.2) yields

$$D_{\hat{S}\subset\mathbb{P}^4} - D_{S\subset X} = \frac{1}{2}(D^2 - 4D + 2Hc_1(T_{S'}) + \langle S, S\rangle),$$

where $H = h|S$. Using the adjunction formula

$$2g - 2 = H^2 + K_{S'}H$$

we obtain (4.1). □

Corollary 39 (Odd degree unirational parametrizations)**.** *Retain the notation of Proposition 38 and assume that S is not homologous to a complete intersection. Consider the discriminant*

$$d = 3\langle S, S\rangle - D^2$$

of

	h^2	S
h^2	3	D
S	D	$\langle S, S\rangle$

.

Then the degree

$$\varrho(S, X) = \frac{d}{2} - 2\langle S, S\rangle + (2 - 2g) + (D^2 - D)$$

has the same parity as $\frac{d}{2}$. Thus the degree of $\rho : \mathbb{P}^4 \dashrightarrow X$ is odd provided d is not divisble by four.

Compare this with Theorem 45 below.

How do we obtain surfaces satisfying the assumptions of Proposition 38? Nuer [Nue15, §3] exhibits *smooth* such surfaces for all $d \leq 38$, thus we obtain

Corollary 40. *A generic cubic fourfold $X \in \mathcal{C}_d$, for $d = 14, 18, 26, 30,$ and 38, admits a unirational parametrization of odd degree.*

There are hueristic constructions of such surfaces in far more examples [HT01, §7]. Let $X \in \mathcal{C}_d$ and consider its variety of lines $F_1(X)$; for simplicity, assume the Picard group of $F_1(X)$ has rank two. Recent work of [BHT15, BM14a] completely characterizes rational curves

$$R \simeq \mathbb{P}^1 \subset F_1(X)$$

associated with extremal birational contractions of $F_1(X)$. The incidence correspondence

$$\begin{array}{ccc} \mathcal{INC} & \longrightarrow & F_1(X) \\ \downarrow & & \\ X & & \end{array}$$

yields

$$S' := \mathcal{INC}|R \to S \subset X,$$

i.e., a ruled surface with smooth normalization.

Question 41. When does the resulting ruled surfaces have isolated singularities? Is this the case when R is a generic rational curve arising as an extremal ray of a birational contraction?

The discussion of [HT01, §7] fails to address the singularity issues, and should be seen as a heuristic approach rather than a rigorous construction. The technical issues are illustrated by the following:

Example 42 (Voisin's counterexample)**.** Assume that X is not special so that $\mathrm{Pic}(F_1(X)) = \mathbb{Z}g$. Let R denote the positive degree generator of the Hodge classes $N_1(X, \mathbb{Z}) \subset H_2(F_1(X), \mathbb{Z})$; the lattice computations in §3.5 imply that

$$g \cdot R = \frac{1}{2}(g, g) = 3.$$

Moreover, there is a two-parameter family of rational curves $\mathbb{P}^1 \subset X, [\mathbb{P}^1] = R$, corresponding to the cubic surfaces $S' \subset X$ singular along a line.

These may be seen as follows: The cubic surfaces singular along some line have codimension seven in the parameter space of all cubic surfaces. However, there is a nine-parameter family of cubic surface

sections of a given cubic fourfold, parametrized by $\mathrm{Gr}(4,6)$. Indeed, for a fixed flag

$$\ell \subset \mathbb{P}^3 \subset \mathbb{P}^5$$

a tangent space computation shows that the cubic fourfolds

$$\{X : X \cap \mathbb{P}^3 \text{ singular along the line } \ell\}$$

dominate the moduli space $\mathcal{C}$.

Let $S' \subset \mathbb{P}^3$ be a cubic surface singular along a line and $X \supset S'$ a smooth cubic fourfold. Since the generic point of X does not lie on a secant line of S', it cannot be used to produce a unirational parametrization of X. The reasoning for Proposition 38 and formula (4.1) is not valid, as S' has non-isolated singularities.

Nevertheless, the machinery developed here indicates where to look for unirational parameterizations of odd degree:

Example 43 ($d = 42$ case)**.** Let $X \in \mathcal{C}_{42}$ be generic. By Theorem 29, $F_1(X) \simeq T^{[2]}$ where (T, f) is K3 surface of degree 42 and $g = 2f - 9\delta$. Take R to be one of the rulings of the divisor in $T^{[2]}$ parametrizing non-reduced subschemes, i.e., those subschemes supported at a prescribed point of T. Note that $R \cdot g = 9$ so the ruled surface S associated with the incidence correspondence has numerical invariants:

	h^2	S
h^2	3	9
S	9	41

Assuming S has isolated singularities, we have $D_{S\subset X} = 8$ and X admits a unirational parametrization of degree $\varrho(S \subset X) = 13$. **Challenge:** Verify the singularity assumption for some $X \in \mathcal{C}_{42}$.

5. Decomposition of the diagonal

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. A *decomposition of the diagonal* of X is a rational equivalence in $\mathrm{CH}^n(X \times X)$

$$N\Delta_X \equiv N\{x \times X\} + Z,$$

where N is a non-zero integer, $x \in X(\mathbb{C})$, and Z is supported on $X \times D$ for some subvariety $D \subsetneq X$. Let $N(X)$ denote the smallest positive integer N occuring in a decomposition of the diagonal, which coincides with the greatest common divisor of such integers. If $N(X) = 1$ we say X admits an *integral decomposition of the diagonal*.

Proposition 44. [ACTP13, Lem. 1.3] [Voi15a, Lem. 4.6] *X admits a decomposition of the diagonal if and only if*

$$A_0 = \{P \in \mathrm{CH}_0 : \deg(P) = 0\}$$

is universally N-torsion for some positive integer N, i.e., for each extension $F/\mathbb{C}$ we have $NA_0(X_F) = 0$. Moreover, $N(X)$ is the annihilator of the torsion.

We sketch this for the convenience of the reader: Basic properties of Chow groups give the equivalence of the decomposition of $N\Delta_X$ with $NA_0(X_{\mathbb{C}(X)}) = 0$. Indeed, $A_0(X_{\mathbb{C}(X)})$ is the inverse limit of $A_0(X \times U)$ for all open $U \subset X$. Conversely, taking the basechange of a decomposition of the diagonal to the extension F gives that $A_0(X_F)$ is annihilated by N.

We recall situations where we have decompositions of the diagonal:

Rationally connected varieties. Suppose X is rationally connected and choose $\beta \in H_2(X, \mathbb{Z})$ such that the evaluation

$$M_{0,2}(X, \beta) \to X \times X$$

is dominant. Fix an irreducible component M of $M_{0,2}(X, \beta) \times_{X\times X} \mathbb{C}(X \times X)$. Then $N(X)$ divides the index $\iota(M)$. Indeed, each effective zero-cycle $Z \subset M$ corresponds to $|Z|$ conjugate rational curves joining generic $x_1, x_2 \in X$. Together these give $|Z|x_1 = |Z|x_2$ in $\mathrm{CH}_0(X_{\mathbb{C}(X)})$. Thus we obtain a decomposition of the diagonal. See [CT05, Prop. 11] for more details.

Unirational varieties. If $\rho : \mathbb{P}^n \dashrightarrow X$ has degree ϱ then $N(X)|\varrho$. Thus $N(X)$ divides the greatest common divisor of the degrees of unirational parametrizations of X. A cubic hypersurface X of dimension at least two admits a degree two unirational parametrization (see Prop 10), so $N(X)|2$. We saw in §4 that many classes of special cubic fourfolds admit odd degree unirational parametrizations. In these cases, we obtain *integral* decompositions of the diagonal.

Rational and stably rational varieties. The case of rational varieties follows from our analysis of unirational parametrizations. For the stably rational case, it suffices to observe that

$$A_0(Y) \simeq A_0(Y \times \mathbb{P}^1)$$

and use the equivalence of Proposition 44 (see [Voi15a, Prop. 4.7] for details). Here we obtain an *integral* decomposition of the diagonal.

Remarkably, at least half of the special cubic fourfolds admit integral decompositions of the diagonal:

Theorem 45. [Voi14, Th. 5.6] *A special cubic fourfold of discriminant* $d \equiv 2 \pmod 4$ *admits an integral decomposition of the diagonal.*

This suggests the following question:

Question 46. Do special cubic fourfolds of discriminant $d \equiv 2 \pmod 4$ always admit unirational parametrizations of odd degree? Are they stably rational?

Cubic fourfolds do satisfy a universal cohomological condition that follows from an integral decomposition of the diagonal: They admit universally trivial unramified H^3. This was proved first for cubic fourfolds containing a plane ($d = 8$) using deep properties of quadratic forms [ACTP13], then in general by Voisin [Voi15b, Exam. 3.2].

Question 47. Is there a cubic fourfold X with

$$K_d = H^{2,2}(X, \mathbb{Z}), \quad d \equiv 0 \pmod 4,$$

and admitting an integral decomposition of the diagonal? A unirational parametriziation of odd degree?

References

[ABBVA14] Asher Auel, Marcello Bernardara, Michele Bolognesi, and Anthony Várilly-Alvarado. Cubic fourfolds containing a plane and a quintic del Pezzo surface. *Algebr. Geom.*, 1(2):181–193, 2014.

[ABGvB13] Asher Auel, Christian Böhning, and Hans-Christian Graf von Bothmer. The transcendental lattice of the sextic Fermat surface. *Math. Res. Lett.*, 20(6):1017–1031, 2013.

[ACTP13] Asher Auel, Jean-Louis Colliot-Thélène, and Raman Parimala. Universal unramified cohomology of cubic fourfolds containing a plane. In *Brauer Groups and Obstruction Problems: Moduli Spaces and Arithmetic*. 2013. Proceedings of a workshop at the American Institute of Mathematics.

[Add16] Nicolas Addington. On two rationality conjectures for cubic fourfolds. *Math. Res. Lett.*, 23(1):1–13, 2016.

[AK77] Allen B. Altman and Steven L. Kleiman. Foundations of the theory of Fano schemes. *Compositio Math.*, 34(1):3–47, 1977.

[AKMW02] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.*, 15(3):531–572 (electronic), 2002.

[AL15] Nicolas Addington and Manfred Lehn. On the symplectic eightfold associated to a Pfaffian cubic fourfold. *J. Reine Angew. Math.*, 2015.

[AT14] Nicolas Addington and Richard Thomas. Hodge theory and derived categories of cubic fourfolds. *Duke Math. J.*, 163(10):1885–1927, 2014.

[Bak10] H. F. Baker. *Principles of geometry. Volume 6. Introduction to the theory of algebraic surfaces and higher loci*. Cambridge Library Collection. Cambridge University Press, Cambridge, 2010. Reprint of the 1933 original.

[BD85] Arnaud Beauville and Ron Donagi. La variété des droites d'une hypersurface cubique de dimension 4. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(14):703–706, 1985.

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.

[Bea00] Arnaud Beauville. Determinantal hypersurfaces. *Michigan Math. J.*, 48:39–64, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[BHB06] T. D. Browning and D. R. Heath-Brown. The density of rational points on non-singular hypersurfaces. II. *Proc. London Math. Soc. (3)*, 93(2):273–303, 2006. With an appendix by J. M. Starr.

[BHT15] Arend Bayer, Brendan Hassett, and Yuri Tschinkel. Mori cones of holomorphic symplectic varieties of K3 type. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):941–950, 2015.

[BM14a] Arend Bayer and Emanuele Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. *Invent. Math.*, 198(3):505–590, 2014.

[BM14b] Arend Bayer and Emanuele Macrì. Projectivity and birational geometry of Bridgeland moduli spaces. *J. Amer. Math. Soc.*, 27(3):707–752, 2014.

[Bor14] Lev Borisov. Class of the affine line is a zero divisor in the Grothendieck ring, 2014. arXiv:1412.6194.

[BRS15] Michele Bolognesi, Francesco Russo, and Giovanni Staglianò. Some loci of rational cubic fourfolds, 2015. arXiv:1504.05863.

[CG72] C. Herbert Clemens and Phillip A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.

[CK89] Bruce Crauder and Sheldon Katz. Cremona transformations with smooth irreducible fundamental locus. *Amer. J. Math.*, 111(2):289–307, 1989.

[CT05] Jean-Louis Colliot-Thélène. Un théorème de finitude pour le groupe de Chow des zéro-cycles d'un groupe algébrique linéaire sur un corps p-adique. *Invent. Math.*, 159(3):589–606, 2005.

[Dol05] Igor V. Dolgachev. Luigi Cremona and cubic surfaces. In *Luigi Cremona (1830–1903) (Italian)*, volume 36 of *Incontr. Studio*, pages 55–70. Istituto Lombardo di Scienze e Lettere, Milan, 2005.

[Edg32] W. L. Edge. The number of apparent double points of certain loci. *Math. Proc. Cambridge Philos. Soc.*, 28:285–299, 1932.

[Fan43] Gino Fano. Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4° ordine. *Comment. Math. Helv.*, 15:71–80, 1943.

[Ful84] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[GS14] Sergey Galkin and Evgeny Shinder. The Fano variety of lines and rationality problem for a cubic hypersurface, 2014. arXiv:1405.5154.

[Has99] Brendan Hassett. Some rational cubic fourfolds. *J. Algebraic Geom.*, 8(1):103–114, 1999.

[Has00] Brendan Hassett. Special cubic fourfolds. *Compositio Math.*, 120(1):1–23, 2000.

[HK07] Klaus Hulek and Remke Kloosterman. The L-series of a cubic fourfold. *Manuscripta Math.*, 124(3):391–407, 2007.

[HLOY03] Shinobu Hosono, Bong H. Lian, Keiji Oguiso, and Shing-Tung Yau. Fourier-Mukai partners of a $K3$ surface of Picard number one. In *Vector bundles and representation theory (Columbia, MO, 2002)*, volume 322 of *Contemp. Math.*, pages 43–55. Amer. Math. Soc., Providence, RI, 2003.

[HLOY04] Shinobu Hosono, Bong H. Lian, Keiji Oguiso, and Shing-Tung Yau. Autoequivalences of derived category of a $K3$ surface and monodromy transformations. *J. Algebraic Geom.*, 13(3):513–545, 2004.

[HT01] B. Hassett and Y. Tschinkel. Rational curves on holomorphic symplectic fourfolds. *Geom. Funct. Anal.*, 11(6):1201–1228, 2001.

[HT09] Brendan Hassett and Yuri Tschinkel. Moving and ample cones of holomorphic symplectic fourfolds. *Geom. Funct. Anal.*, 19(4):1065–1080, 2009.

[Huy99] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999.

[HVAV11] Brendan Hassett, Anthony Várilly-Alvarado, and Patrick Varilly. Transcendental obstructions to weak approximation on general K3 surfaces. *Adv. Math.*, 228(3):1377–1404, 2011.

[Kul08] Vik. S. Kulikov. A remark on the nonrationality of a generic cubic fourfold. *Mat. Zametki*, 83(1):61–68, 2008.

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 219–243. Birkhäuser Boston, Inc., Boston, MA, 2010.

[Kuz15] Alexander Kuznetsov. Derived categories view on rationality problems. Lecture notes for the CIME–CIRM summer school, Levico Terme, 2015. arXiv:1509.09115.

[Laz10] Radu Laza. The moduli space of cubic fourfolds via the period map. *Ann. of Math. (2)*, 172(1):673–711, 2010.

[LLSvS15] Christian Lehn, Manfred Lehn, Christoph Sorger, and Duco van Straten. Twisted cubics on cubic fourfolds. *J. Reine Angew. Math.*, 2015.

[Loo09] Eduard Looijenga. The period map for cubic fourfolds. *Invent. Math.*, 177(1):213–233, 2009.

[LZ13] Zhiyuan Li and Letao Zhang. Modular forms and special cubic fourfolds. *Adv. Math.*, 245:315–326, 2013.

[Mar11] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 257–322. Springer, Heidelberg, 2011.

[May11] Evgeny Mayanskiy. Intersection lattices of cubic fourfolds, 2011. arXiv:1112.0806.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*

(2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, third edition, 1994.

[Mor40] Ugo Morin. Sulla razionalità dell'ipersuperficie cubica generale dello spazio lineare S_5. *Rend. Sem. Mat. Univ. Padova*, 11:108–112, 1940.

[MS12] Emanuele Macrì and Paolo Stellari. Fano varieties of cubic fourfolds containing a plane. *Math. Ann.*, 354(3):1147–1176, 2012.

[Muk87] S. Mukai. On the moduli space of bundles on $K3$ surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987.

[Nik79] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979.

[Nue15] Howard Nuer. Unirationality of moduli spaces of special cubic fourfolds and $K3$ surfaces, 2015. arXiv:1503.05256v1.

[Orl97] D. O. Orlov. Equivalences of derived categories and $K3$ surfaces. *J. Math. Sci. (New York)*, 84(5):1361–1381, 1997. Algebraic geometry, 7.

[Ran88] Kristian Ranestad. *On smooth surfaces of degree 10 in the projective fourspace*. PhD thesis, University of Oslo, 1988.

[Ran91] Kristian Ranestad. Surfaces of degree 10 in the projective fourspace. In *Problems in the theory of surfaces and their classification (Cortona, 1988)*, Sympos. Math., XXXII, pages 271–307. Academic Press, London, 1991.

[Siu81] Yum Tong Siu. A simple proof of the surjectivity of the period map of $K3$ surfaces. *Manuscripta Math.*, 35(3):311–321, 1981.

[ST01] Paul Seidel and Richard Thomas. Braid group actions on derived categories of coherent sheaves. *Duke Math. J.*, 108(1):37–108, 2001.

[Swa89] Richard G. Swan. Zero cycles on quadric hypersurfaces. *Proc. Amer. Math. Soc.*, 107(1):43–46, 1989.

[Tre84] S. L. Tregub. Three constructions of rationality of a cubic fourfold. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (3):8–14, 1984.

[Tre93] S. L. Tregub. Two remarks on four-dimensional cubics. *Uspekhi Mat. Nauk*, 48(2(290)):201–202, 1993.

[TVA15] Sho Tanimoto and Anthony Várilly-Alvarado. Kodaira dimension of moduli of special cubic fourfolds, 2015. arXiv:1509.01562v1.

[vG05] Bert van Geemen. Some remarks on Brauer groups of $K3$ surfaces. *Adv. Math.*, 197(1):222–247, 2005.

[Voi86] Claire Voisin. Théorème de Torelli pour les cubiques de $\mathbf{P}^5$. *Invent. Math.*, 86(3):577–601, 1986.

[Voi07] Claire Voisin. Some aspects of the Hodge conjecture. *Jpn. J. Math.*, 2(2):261–296, 2007.

[Voi13] Claire Voisin. Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal. *J. Algebraic Geom.*, 22(1):141–174, 2013.

[Voi14] Claire Voisin. On the universal CH_0 group of cubic hypersurfaces. *J. Eur. Math. Soc. (JEMS)*, 2014. To appear, arXiv:1407.7261v2.

[Voi15a] Claire Voisin. Stable birational invariants and the Lüroth problem, 2015. Preprint at `http://webusers.imj-prg.fr/~claire.voisin/`.

[Voi15b] Claire Voisin. Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.*, 201(1):207–237, 2015.

[Wło03] Jarosław Włodarczyk. Toroidal varieties and the weak factorization theorem. *Invent. Math.*, 154(2):223–331, 2003.

[Yos00] Kōta Yoshioka. Irreducibility of moduli spaces of vector bundles on $K3$ surfaces, 2000. arXiv:9907001v2.

[Yos01] Kōta Yoshioka. Moduli spaces of stable sheaves on abelian surfaces. *Math. Ann.*, 321(4):817–884, 2001.

Department of Mathematics, Brown University, Box 1917, 151 Thayer Street, Providence, RI 02912, USA

E-mail address: bhassett@math.brown.edu