Clay Mathematics Proceedings
Volume 8, 2009

Rational surfaces over nonclosed fields

Brendan Hassett

Abstract. This paper is based on lectures given at the Clay Summer School on Arithmetic Geometry in July 2006.

These notes offer an introduction to the birational geometry of algebraic surfaces, emphasizing the aspects useful for arithmetic. The first three sections are explicitly devoted to birational questions, with a special focus on rational surfaces. We explain the special rôle these play in the larger classification theory. The geometry of rational ruled surfaces and Del Pezzo surfaces is studied in substantial detail. We extend this theory to geometrically rational surfaces over non-closed fields, enumerating the minimal surfaces and describing their geometric properties. This gives essentially the complete classification of rational surfaces up to birational equivalence.

The final two sections focus on singular Del Pezzo surfaces, universal torsors, and their algebraic realizations through Cox rings. Current techniques for counting rational points (on rational surfaces over number fields) often work better for singular surfaces than for smooth surfaces. The actual enumeration of the rational points often boils down to counting integral points on the universal torsor. Universal torsors were first employed in the (ongoing) search for effective criteria for when rational surfaces over number fields admit rational points.

It might seem that these last two topics are far removed from birational geometry, at least the classical formulation for surfaces. However, singularities and finite-generation questions play a central rôle in the minimal model program. And the challenges arising from working over non-closed fields help highlight structural characteristics of this program that usually are only apparent over $\mathbb{C}$ in higher dimensions. Indeed, these notes may be regarded as an arithmetically motivated introduction to modern birational geometry.

In general, the prerequisites for these notes are a good understanding of algebraic geometry at the level of Hartshorne [**Har77**]. Some general understanding of descent is needed to appreciate the applications to non-closed fields. Readers

2000 *Mathematics Subject Classification.* Primary 14J26; Secondary 14G05, 14J20.

Key words and phrases. rational surfaces, Del Pezzo surfaces, minimal models.

Supported in part by NSF Grants #0134259 and #0554491, the Clay Mathematics Institute, and the University of Göttingen.

interested in applications to positive characteristic would benefit from some exposure to étale cohomology at the level of Milne [**Mil80**]. There is one place where we do not fully observe these prerequisites: The discussion of the Cone Theorem is not self-contained although we do sketch the main ideas. Thankfully, a number of books ([**CKM88**], [**KM98**], [**Rei97**], [**Kol96**],[**Mat02**]) give good introductory accounts of this important topic.

Finally, we should indicate how this account relates to others in the literature. The general approach taken to the geometry of surfaces over algebraically closed fields owes much to Beauville's book [**Bea96**] and Reid's lecture notes [**Rei97**]. The extensions to non-closed fields draw from Kollár's book [**Kol96**]. Readers interested in details of the Galois action on the lines of a Del Pezzo surface and its implications for arithmetic should consult Manin's classic book [**Man74**] and the more recent survey [**MT86**]. The books [**CKM88, KM98, Mat02**] offer a good introduction to modern birational geometry; [**Laz04**] has a comprehensive account of linear series. We have made no effort to explain how universal torsors and Cox rings are used for the descent of rational points; the recent book of Skorobogatov [**Sko01**] does a fine job covering this material.

I am grateful to Anthony Várilly-Alvarado, Michael Joyce, Ambrus Pál, and other members of the summer school for helpful comments.

1. Rational surfaces over algebraically closed fields

Let k be an algebraically closed field. Throughout, a *variety* will designate an integral separated scheme of finite type over k.

1.1. Classical example: Cubic surfaces. Here a *cubic surface* means a smooth cubic hypersurface $X \subset \mathbb{P}^3$. We recall a well-known construction for such surfaces:

Let $p_1, \dots, p_6 \in \mathbb{P}^2$ be points in the projective plane in *general position*, i.e.,

- the points are distinct;
- no three of the points are collinear;
- the six points do not lie on a plane conic.

Consider the vector space of homogeneous cubics vanishing at these points; it is an exercise to show this has dimension four

$$I_{p_1,\dots,p_6}(3) = \langle F_0, F_1, F_2, F_3 \rangle$$

and has no additional basepoints.

The resulting linear series gives a rational map

$$\begin{array}{rcl} \rho : \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \\ [x_0, x_1, x_2] & \mapsto & [F_0, F_1, F_2, F_3] \end{array}$$

that is not well-defined at $p_1, \dots, p_6$. Consider the blow-up

$$\beta : X := \mathrm{Bl}_{p_1,\dots,p_6}\mathbb{P}^2 \to \mathbb{P}^2$$

with exceptional divisors

$$E_1, \dots, E_6.$$

Blowing up the base scheme of a linear series resolves its indeterminacy, so we obtain a morphism

$$j : X \to \mathbb{P}^3$$

with $j = \rho \circ \beta$.

PROPOSITION 1.1. *The morphism j gives a closed embedding of X in $\mathbb{P}^3$.*

We leave the proof as an exercise.

Given this, we may describe the image of j quite easily. The first step is to analyze the *Picard group* $\mathrm{Pic}(X)$ and its associated intersection form

$$\begin{array}{ccc} \mathrm{Pic}(X) \times \mathrm{Pic}(X) & \rightarrow & \mathbb{Z} \\ (D_1, D_2) & \mapsto & D_1 \cdot D_2 \end{array}.$$

We recall what happens to the intersection form under blow-ups. Let $\beta : Y \to P$ be the blow-up of a smooth surface at a point, with exceptional divisor E. Then we have an orthogonal direct-sum decomposition

$$\mathrm{Pic}(Y) = \mathrm{Pic}(P) \oplus_{\perp} \mathbb{Z}E, \quad E \cdot E = E^2 = -1,$$

where the inclusion $\mathrm{Pic}(P) \hookrightarrow \mathrm{Pic}(Y)$ is induced by β^*.

Returning to our particular situation, we have

$$\mathrm{Pic}(X) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_6.$$

Here L is the pullback of the hyperplane from $\mathbb{P}^2$ with $L^2 = L \cdot L = 1$ and $L \cdot E_a = 0$ for each a. We also have $E_a \cdot E_b = 0$ for $a \neq b$.

Since j is induced by the linear series of cubics with simple basepoints at $p_1, \ldots, p_6$, we have

$$j^* \mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_X(3L - E_1 - \cdots - E_6)$$

so that

$$\deg(j(X)) = (3L - E_1 - \cdots - E_6)^2 = 9 - 6 = 3.$$

This proves that the image is a smooth cubic surface. The images of the exceptional divisors $E_1, \ldots, E_6$ have degree

$$E_i \cdot (3L - E_1 - \cdots - E_6) = 1$$

and thus are lines on our cubic surface.

PROPOSITION 1.2. *The cubic surface $j(X) \subset \mathbb{P}^3$ contains the following 27 lines:*

- *the exceptional curves E_a;*
- *proper transforms of lines through p_a and p_b, with class $L - E_a - E_b$;*
- *proper transforms of conics through five basepoints p_a, p_b, p_c, p_d, p_e, with class $2L - E_a - E_b - E_c - E_d - E_e$.*

This beautiful analysis leaves open a number of classification questions:

(1) Does every cubic surface arise as the blow-up of $\mathbb{P}^2$ in six points in general linear position?
(2) Are there exactly 27 lines on a cubic surface?

To address these, we introduce some general geometric definitions:

DEFINITION 1.3. Let Y be a smooth projective surface with canonical class K_Y, i.e., the divisor class associated with the differential two-forms $\Omega^2_Y = \bigwedge^2 \Omega^1_Y$. We say that Y is a *Del Pezzo surface* if $-K_Y$ is ample, i.e., there exists an embedding $Y \subset \mathbb{P}^N$ such that $\mathcal{O}_{\mathbb{P}^N}(1)|Y = \mathcal{O}_Y(-rK_Y)$ for some $r > 0$.

Note that if Y is Del Pezzo then $K_Y^2 > 0$.

REMARK 1.4. Let $X \subset \mathbb{P}^3$ be a cubic surface and H the hyperplane class on $\mathbb{P}^3$. Adjunction

$$K_X = (K_{\mathbb{P}^3} + X)|_X = (-4H + 3H)|_X = -H|_X$$

implies that any cubic surface is a Del Pezzo surface.

DEFINITION 1.5. Let Y be a smooth projective surface. A (-1)-curve is a smooth rational curve $E \subset Y$ with $E^2 = -1$.

Of course, exceptional divisors are the main examples. We also have the following characterization.

PROPOSITION 1.6. *Let Y be a smooth projective surface. Let $E \subset Y$ be an irreducible curve with*

$$E^2 < 0, \quad K_Y \cdot E < 0.$$

Then E is a (-1)-curve. In particular, on a Del Pezzo surface every irreducible curve with $E^2 < 0$ is a (-1)-curve.

PROOF. Let $p_a(E)$ denote the arithmetic genus of E. Since E is an irreducible curve we know that $p_a(E) \geq 0$ with equality if and only if $E \simeq \mathbb{P}^1$. Combining this with the adjunction formula, we obtain

$$-2 \leq 2p_a(E) - 2 = E \cdot (K_Y + E).$$

Thus $E^2 = -1$, $K_Y \cdot E = -1$, and E is a smooth rational curve. □

REMARK 1.7. The lines on a cubic surface are precisely its (-1)-curves. Indeed, if $\ell \subset X$ is a line then the genus formula gives

$$-2 = 2g(\ell) - 2 = \ell^2 + K_X \cdot \ell = \ell^2 - 1.$$

Suppose then that

$$\ell = aL - b_1E_1 - \cdots - b_6E_6$$

is a line on a cubic surface. Then the following equations must be satisfied

$$\begin{aligned} 1 &= -K_X \cdot \ell = 3a - b_1 - b_2 - b_3 - b_4 - b_5 - b_6 \\ -1 &= \ell^2 = a^2 - b_1^2 - b_2^2 - b_3^2 - b_4^2 - b_5^2 - b_6^2 \end{aligned}$$

and these can be solved explicitly. There are precisely 27 solutions; see Exercise 1.1.6 and [**Man74**, 26.2], especially for the connection with root systems. Thus the cubic surfaces arising as blow-ups of $\mathbb{P}^2$ in six points in general position have precisely 27 lines.

We extend this analysis to all smooth cubic surfaces:

THEOREM 1.8. *Let $\mathbb{P}^{19} = \mathbb{P}(\mathrm{Sym}^3(k^4))$ parametrize all cubic surfaces and let*

$$Z = \{(X, \ell) : X \text{ cubic surface, } \ell \subset X \text{ line }\} \subset \mathbb{P}^{19} \times \mathbb{G}(1,3)$$

denote the incidence correspondence. Let $U \subset \mathbb{P}^{19}$ denote the locus of smooth cubic surfaces and

$$\pi_1 : Z_U := Z \times_{\mathbb{P}^{19}} U \to U$$

the projection. Then π_1 is a finite étale morphism.

Since U is connected the degree of π_1 is constant, and we conclude

COROLLARY 1.9. *Every smooth cubic surface has 27 lines.*

PROOF. (cf. [**Mum95**, p. 173 ff.]) We claim that Z is proper and irreducible of dimension 19: Each line ℓ is contained in a $20 - 4 = 16$-dimensional linear series of cubic surfaces, so the projection $Z \to \mathbb{G}(1,3)$ is a $\mathbb{P}^{15}$ bundle. Consequently, π_1 is a proper morphism. In particular, for each one-parameter family of lines in cubic surfaces (X_t, ℓ_t), the flat limit

$$\lim_{t \to 0} (X_t, \ell_t)$$

is also a line in a cubic surface.

Let $\mathcal{N}_{\ell/X}$ denote the normal bundle of a line ℓ in a smooth cubic surface X. We have $\mathcal{N}_{\ell/X} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ so that

$$h^0(\mathcal{N}_{\ell/X}) = h^1(\mathcal{N}_{\ell/X}) = 0.$$

Recall that $H^0(\mathcal{N}_{\ell/X})$ (resp. $H^1(\mathcal{N}_{\ell/X})$) is the tangent space (resp. obstruction space) of the scheme of lines on X at ℓ. It follows then that Z_U is smooth of relative dimension zero over U, i.e., π_1 is étale. Furthermore, proper étale morphisms are finite. □

One further piece of information can be extracted from this result: The intersections of the 27 lines are constant over all the cubic surfaces. This means that every cubic surface X contains a pair (and even a sextuple!) of pairwise disjoint lines (cf. Exercise 1.1.3).

PROPOSITION 1.10. *Let X be a smooth cubic surface containing disjoint lines E_1 and E_2. Let $\ell_1, \ldots, \ell_5$ denote the lines in X meeting E_1 and E_2. There is a birational morphism*

$$\begin{array}{rcl} \varphi : X & \to & \mathbb{P}^1 \times \mathbb{P}^1 \\ x & \mapsto & (p_{E_1}(x), p_{E_2}(x)) \end{array}$$

where $p_{E_i} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ is projection from E_i. This contracts $\ell_1, \ldots, \ell_5$ to distinct points $q_1, \ldots, q_5 \in \mathbb{P}^1 \times \mathbb{P}^1$ satisfying the following genericity conditions:

- *no pair of them lie on a ruling of $\mathbb{P}^1 \times \mathbb{P}^1$;*
- *no four of them lie on a curve of bidegree $(1, 1)$;*

The inverse φ^{-1} is given by the linear system of forms of bidegree $(2, 2)$ through $q_1, \ldots, q_5$.

We leave this as an exercise.

COROLLARY 1.11. *Every smooth cubic surface is isomorphic to $\mathbb{P}^2$ blown up at six points.*

PROOF. We first verify that $\mathrm{Bl}_{q_1,\ldots,q_5}\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathbb{P}^2$ blown up at six points. Indeed, we can realize $\mathbb{P}^1 \times \mathbb{P}^1$ as a smooth quadric $Q \subset \mathbb{P}^3$, so that the fibers of each projection are lines on Q. Let $q \in Q$ be any point and R_1 and R_2 the two rulings passing though q. Projection from q

$$p_q : Q \dashrightarrow \mathbb{P}^2$$

lifts to a morphism

$$\mathrm{Bl}_q Q \to \mathbb{P}^2$$

contracting the proper transforms of R_1 and R_2. □

Before concluding, we draw two morals from this story:

- (-1)-curves govern much of the geometry of a Del Pezzo surface;
- classifying (-1)-curves is a crucial step in classifying the surfaces.

Exercises.

Exercise 1.1.1. Show that six distinct points on the plane impose independent conditions on cubics if no four of the points are collinear. Show that the resulting linear system has base scheme equal to these six points.

Exercise 1.1.2. Give a careful proof of Proposition 1.1.

Exercise 1.1.3. Verify that the 27 curves described in Proposition 1.2 are in fact lines on the cubic surface. Check that each of these has self-intersection -1. Show that

(1) each line is intersected by ten other lines;
(2) any pair of disjoint lines is intersected by five lines;
(3) each line is contained in a collection of six pairwise disjoint lines.

Exercise 1.1.4. Prove Proposition 1.10.

Exercise 1.1.5. Let X be a smooth cubic surface. Show that the intersection form on $K_X^{\perp} \subset \mathrm{Pic}(X)$ is isomorphic to

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6
ρ_1	-2	1	0	0	0	0
ρ_2	1	-2	1	0	0	0
ρ_3	0	1	-2	1	0	1
ρ_4	0	0	1	-2	1	0
ρ_5	0	0	0	1	-2	0
ρ_6	0	0	1	0	0	-2

.

Up to sign, this is the Cartan matrix associated to the root system $\mathbf{E}_6$.

Exercise 1.1.6. Consider a line on a cubic surface $\ell \subset X$, and the associated class $\lambda = 3\ell + K_X \in K_X^{\perp}$. Verify that $\lambda^2 = -12$ and $\lambda \cdot \eta \equiv 0 \pmod 3$ for each $\eta \in K_X^{\perp} \subset \mathrm{Pic}(X)$. Deduce that there are a finite number of lines on a cubic surface.

1.2. The structure of birational morphisms of surfaces. Our first task is to show that all (-1)-curves arise as exceptional curves of blow-ups:

Theorem 1.12 (Castelnuovo contraction criterion). [**Har77**, V.5.7] *Let X be a smooth projective surface and $E \subset X$ a (-1)-curve. Then there exists a smooth projective surface Y and a morphism $\beta : X \to Y$ contracting E to a point $y \in Y$, so that X is isomorphic to $\mathrm{Bl}_y Y$. Each morphism $\psi : X \to Z$ contracting E admits a factorization*

$$\psi : X \xrightarrow{\beta} Y \to Z.$$

Proof. (Sketch) Let H be a very ample divisor on X such that

$$H^1(X, \mathcal{O}_X(H)) = 0.$$

Set $\mathcal{L} = \mathcal{O}_X(H + (H \cdot E)E)$ so that $\mathcal{L}|E \simeq \mathcal{O}_E$. For each $n > 0$ we have the inclusion

$$\mathcal{O}_X(nH) \hookrightarrow \mathcal{L}^n = \mathcal{O}_X(nH + n(H \cdot E)E)$$

which is an isomorphism away from E. Thus the sections in the image of

$$\Gamma(X, \mathcal{O}_X(nH)) \hookrightarrow \Gamma(X, \mathcal{L}^n)$$

induce an embedding of $X \setminus E$.

We claim that $\mathcal{L}$ is globally generated, so we have a morphism

$$\beta : X \to Y := \mathrm{Proj}(\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^n)).$$

Since $\mathcal{L}|E$ is trivial, β necessarily contracts E to a point; β is an isomorphism away from E.

Here is the idea: Since $\mathcal{L}|E$ is globally generated, it suffices to show that the restriction

$$\Gamma(X, \mathcal{L}) \to \Gamma(E, \mathcal{L}|E) \simeq \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$$

is surjective. Taking the long exact sequence associated to

$$0 \to \mathcal{L}(-E) \to \mathcal{L} \to \mathcal{L}|E \to 0,$$

we are reduced to showing that $H^1(X, \mathcal{L}(-E)) = 0$. Indeed, we can show inductively that $H^1(X, \mathcal{O}_X(H + aE)) = 0$ for $a = 1, \dots, H \cdot E - 1$: The exact sequence

$$0 \to \mathcal{O}_X(H + (a-1)E) \to \mathcal{O}_X(H + aE) \to \mathcal{O}_E(H \cdot E - a) \to 0$$

expresses $\mathcal{O}_X(H + aE)$ as an extension of sheaves with vanishing H^1.

The trickiest bit is to check that Y is smooth and β is the blow-up of a point of Y. The necessary local computation can be found in [**Bea96**, II.17] or [**Har77**, pp. 415].

For the factorization step, the standard isomorphism

$$\mathrm{Pic}(X) = \beta^*\mathrm{Pic}(Y) \oplus \mathbb{Z}E$$

identifies $\beta^*\mathrm{Pic}(Y)$ with line bundles on X restricting to zero along E. Moreover, the induced map

$$\Gamma(Y, \mathcal{M}') \to \Gamma(X, \beta^*\mathcal{M}')$$

is an isomorphism. Suppose that $\mathcal{M}$ is very ample on Z so that ψ is induced by certain sections of $\psi^*\mathcal{M}$. However, $\mathcal{M} = \beta^*\mathcal{M}'$ for some $\mathcal{M}'$ on Y and the relevant sections of $\psi^*\mathcal{M}$ come from sections of $\mathcal{M}'$. □

THEOREM 1.13. *Let $\phi : X \to Y$ be a birational morphism of smooth projective surfaces. Then there exists a factorization*

$$X = X_0 \stackrel{\beta_1}{\to} X_1 \to \cdots \to X_{r-1} \stackrel{\beta_r}{\to} X_r = Y$$

where each β_j is a blow-up of a point on X_j. (If ϕ is an isomorphism we take $X_0 = X_r$.)

PROOF. We assume ϕ is not an isomorphism. Hence it is ramified and the induced map

$$\phi^*\Omega^2_Y \to \Omega^2_X$$

is not an isomorphism. Since these sheaves are invertible, we can therefore write

$$\phi^*\Omega^2_Y = \Omega^2_X(-(m_1E_1 + \cdots + m_rE_r)),$$

where the E_i are irreducible *ϕ-exceptional curves*, i.e., ϕ contracts E_i to a point in Y. Since $\phi^*K_Y|_{E_i}$ is trivial we have $\phi^*K_Y \cdot E_i = 0$. The multiplicity m_i is positive because ϕ is ramified along E_i. In divisorial notation, we obtain the *discrepancy formula*:

$$K_X = \phi^*K_Y + \sum m_iE_i, \quad m_i > 0. \tag{1.1}$$

By the Hodge index theorem [**Har77**, V.1.9], the intersection form on

$$\Lambda = \mathbb{Z}E_1 + \cdots + \mathbb{Z}E_r$$

is negative definite, so in particular each $E_i^2 < 0$. We claim that $K_X \cdot E_i < 0$ for some i; then Proposition 1.6 guarantees that E_i is a (-1)-curve.

We know that

$$(\sum m_i E_i)^2 = \sum_i m_i E_j \cdot (\sum_j m_j E_j) = \sum_i m_i E_i \cdot K_X$$

is negative, because the intersection form on Λ is negative definite. Hence some $E_i \cdot K_X$ must be negative.

Using the Castelnuovo criterion we contract E_i

$$X = X_0 \to X_1,$$

so that X_0 is the blow-up of X_1 at a point. Moreover, ϕ factors through X_1. This factorization process terminates because the exceptional locus of ϕ has a finite number of irreducible components. □

DEFINITION 1.14. A smooth projective surface X is *minimal* if every birational morphism $\phi : X \to Y$ to a smooth variety is an isomorphism.

Theorem 1.13 says that X is minimal if and only if it has no (-1)-curves.

Exercises.

EXERCISE 1.2.1. Let X be the blow-up of $\mathbb{P}^2$ at $[0,0,1], [0,1,0], [1,0,0]$. Realize X in $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ using the bihomogeneous equations

$$x_0 y_0 = x_1 y_1 = x_2 y_2.$$

Verify that the proper transforms of the lines $x_0 = 0, x_1 = 0, x_2 = 0$ are (-1)-curves and write down explicit linear series contracting each one individually.

EXERCISE 1.2.2. Let X be a cubic surface, realized as $\mathbb{P}^2$ blown up at six points. Describe a basepoint-free linear series on X contracting the six curves

$$2L - E_a - E_b - E_c - E_d - E_e.$$

What is the image of the corresponding morphism $X \to Y$?

1.3. Relative minimality and ruled surfaces. Let $f : X \to B$ denote a dominant morphism from a smooth projective surface to a variety. We say that X is *minimal relative to* f if there exists no commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

where ϕ is birational and Y is smooth. In analogy to Theorem 1.13, X is minimal relative to f if and only if there are no (-1)-curves in the fibers of f.

A *ruled surface* is a morphism $f : X \to B$ from a smooth projective surface to a smooth curve whose generic fiber is rational; it is *minimal* if it is minimal relative to f. If f is smooth then each fiber is isomorphic to $\mathbb{P}^1$; in this case, $f : X \to B$ is called a $\mathbb{P}^1$-*bundle*.

PROPOSITION 1.15. *Let X be a smooth projective surface and $f : X \to B$ a $\mathbb{P}^1$-bundle. Then each $b \in B$ admits an étale-open neighborhood $U \to B$ and an isomorphism:*

$$\begin{array}{ccc} X \times_B U & \xrightarrow{\sim} & \mathbb{P}^1 \times U \\ & \searrow \quad \swarrow & \\ & U & \end{array}$$

PROOF. Since f is smooth it admits a multisection $M \subset X$ with $f|M$ unbranched over b; let $U \subset M$ denote the open set where f is unramified. The pullback $g : X' := X \times_B M \to M$ admits the canonical diagonal section Σ. Consider the direct images of $\mathcal{O}_{X'}(\Sigma)$. Cohomology and base change implies that $\mathbb{R}^1 g_* \mathcal{O}_{X'}(\Sigma)$ is trivial and $\mathcal{E} := g_* \mathcal{O}_{X'}(\Sigma)$ has rank two. Under these conditions cohomology commutes with base change, so a fiber-by-fiber analysis shows that

$$g^* g_* \mathcal{O}_{X'}(\Sigma) \to \mathcal{O}_{X'}(\Sigma)$$

is surjective and the induced morphism

$$X' \to \mathbb{P}(\mathcal{E})$$

is an isomorphism over M. □

THEOREM 1.16. *Let $f : X \to B$ be a minimal ruled surface. Then X is a $\mathbb{P}^1$-bundle over B.*

Before proving this, we'll require a preliminary result.

LEMMA 1.17. Let F denote the class of a fiber of f. Consider a fiber of f with irreducible components $E_1, \dots, E_r$. Then we have $E_i^2 < 0$ and $F \cdot E_i = 0$ for each i and $K_X \cdot E_i < 0$ for some i. In particular, each reducible fiber contains a (-1)-curve.

PROOF. Each fiber of f is numerically equivalent to F, i.e., has the same intersection numbers with curves in X. Since these fibers are generally disjoint from the E_i, we have $F \cdot E_i = 0$ for each i and $F \cdot F = 0$.

Express $F = \sum_{i=1}^r m_i E_i$ where $m_i > 0$ is the multiplicity of the fiber along E_i. Note that F is connected, e.g., by Stein factorization. Thus each E_i meets some E_j and

$$E_i \cdot \sum_{j \neq i} m_j E_j = E_i \cdot (F - m_i E_i) > 0.$$

It follows that $E_i \cdot E_i < 0$.

Finally, $K_X \cdot F = -2$ by adjunction, so $K_X \cdot E_i < 0$ for some index. □

PROOF. (Theorem 1.16)

The key point is to show that the fibers of f are all isomorphic to $\mathbb{P}^1$. Since f is a dominant morphism from a nonsingular surface to a nonsingular curve, it is flat with fibers of arithmetic genus zero. Each fiber is a Cartier divisor on X and thus has no embedded points. Under the assumptions of Theorem 1.16, each fiber of f is irreducible. We also have that each fiber has multiplicity one. Indeed, writing $F = mE$ we have

$$-2 = K_X \cdot F = m K_X \cdot E$$

so $m = 1, 2$. However, if $m = 2$ then adjunction yields $2g(E) - 2 = E^2 + K_X E = -1$, which is absurd. Thus each fiber of f is isomorphic to $\mathbb{P}^1$, and in particular f is smooth. □

We record one additional fact for future reference, whose proof is left as an exercise:

PROPOSITION 1.18. *Let $E_1, \dots, E_r$ be the components of a fiber of a ruled surface; let F denote the class of the fiber. The induced intersection form on*

$$(\mathbb{Z}E_1 + \cdots + \mathbb{Z}E_r)/\mathbb{Z}F$$

is negative definite and unimodular.

We now pursue a finer analysis of the structure of ruled surfaces.

PROPOSITION 1.19. *Let C be a variety defined over a field K such that $C_{\bar{K}} \simeq \mathbb{P}^1_{\bar{K}}$. Then there exists a closed embedding $C \hookrightarrow \mathbb{P}^2$ as a plane conic. There is a quadratic extension K'/K such that $C_{K'} \simeq \mathbb{P}^1_{K'}$.*

We leave the proof as an exercise.

We apply Proposition 1.19 in the case where K is the function field of the base B and C is the generic fiber f:

$$\begin{array}{ccc} C & \subset & X \\ {\scriptstyle f_\circ}\downarrow & & \downarrow {\scriptstyle f} \\ \mathrm{Spec}(k(B)) & \rightarrow & B \end{array}$$

The Tsen-Lang theorem says that every quadratic form in ≥ 3 variables over $k(B)$ represents zero, so $C(k(B)) \neq \emptyset$. Each rational point corresponds to a section $\mathrm{Spec}(k(B)) \rightarrow C$ of $f_\circ$, and thus to a rational map from B to X. Since X is proper, this extends uniquely to a section $s : B \rightarrow X$ of f. We have proven the following:

PROPOSITION 1.20. *Let $f : X \rightarrow B$ be a ruled surface. There exists a section $s : B \rightarrow X$ of f.*

Combining this with the argument for Proposition 1.15, we obtain

COROLLARY 1.21 (Classification of ruled surfaces). *Every minimal ruled surface $f : X \rightarrow B$ is isomorphic to $\mathbb{P}(\mathcal{E})$ for some rank-two vector bundle $\mathcal{E}$ on B.*

Combining this with Grothendieck's classification of vector bundles on $\mathbb{P}^1$ gives:

COROLLARY 1.22. *Every ruled surface $f : X \rightarrow \mathbb{P}^1$, minimal relative to f, is isomorphic to a* Hirzebruch surface

$$\mathbb{F}_d := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d)), \quad d \geq 0.$$

In particular, ruled surfaces over $\mathbb{P}^1$ are rational.

See Exercise 1.3.3 for more details of the argument.

Exercises.

EXERCISE 1.3.1. Prove Proposition 1.19. *Hint:* Note that Ω^1_C is an invertible sheaf on C defined over K and coincides with $\mathcal{O}_{\mathbb{P}^1}(-2)$ over $C_{\bar{K}}$. Use the sections of the dual $(\Omega^1_C)^*$ to embed C.

EXERCISE 1.3.2. Prove Proposition 1.18. *Hint:* Use the mechanism of the proof of Theorem 1.16.

Exercise 1.3.3. Give a detailed proof for Corollary 1.22. For the classification assertion, show that each vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ decomposes as

$$\mathcal{E} \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i), \quad a_1 \leq a_2 \leq \cdots \leq a_r.$$

To establish rationality, exhibit a nonempty open subset $U \subset \mathbb{P}^1$ such that $\mathcal{E}|U \simeq \mathcal{O}_U^{\oplus r}$.

Exercise 1.3.4. For each $d \geq 0$, show there exists a diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\beta}\swarrow & & \searrow{\scriptstyle\gamma} \\ \mathbb{F}_d & & \mathbb{F}_{d+1} \end{array}$$

where β (resp. γ) is the blow-up of $\mathbb{F}_d$ (resp. $\mathbb{F}_{d+1}$) at a suitable point.

2. Effective cones and classification

From the modern point of view, the presence of (-1)-curves is controlled by how the effective cone and the canonical class interact. In this section, we develop technical tools for analyzing this interaction. We continue to work over an algebraically closed field k.

2.1. Cones of curves and divisors. Let X be a smooth projective complex variety, $N_1(X,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$ the sublattice generated by homology classes of algebraic curves in X, and $N^1(X,\mathbb{Z}) \subset H^2(X,\mathbb{Z})$ the *Néron-Severi group* parametrizing homology classes of divisors in X.

We can extend these definitions to fields of positive characteristic: Consider the Chow group of dimension (resp. codimension) one cycles in X; two cycles are *numerically equivalent* if their intersections with any divisor (resp. curve) are equal. We define $N_1(X,\mathbb{Z})$ (resp. $N^1(X,\mathbb{Z})$) as the quotient of the corresponding Chow group by the cycles numerically equivalent to zero. The rank of $N^1(X,\mathbb{Z})$ is bounded by the second (étale) Betti number of X; see [**Mil80**, V.3.28] for the surface case.

Definition 2.1. A Cartier divisor D on a variety X is *nef* (numerically eventually free or numerically effective) if $D \cdot C \geq 0$ for each curve $C \subset X$.

Here is the main example: A Cartier divisor D is *semiample* if ND is basepoint-free for some $N \in \mathbb{N}$. Since ND remains basepoint-free when restricted to curves $C \subset X$, we have $D \cdot C = \deg D|C \geq 0$.

We have the monoid of effective curves

$$\mathrm{NE}_1(X,\mathbb{Z}) = \{[D] \in N_1(X,\mathbb{Z}) : D \text{ effective sum of curves }\}$$

and the associated closed cone

$$\overline{\mathrm{NE}}_1(X) = \text{ smallest closed cone containing } \mathrm{NE}_1(X,\mathbb{Z}) \subset N_1(X,\mathbb{R}),$$

as well as the monoid of effective divisors

$$\mathrm{NE}^1(X,\mathbb{Z}) = \{[D] \in N^1(X,\mathbb{Z}) : D \text{ effective divisor }\}$$

and the associated cone of *pseudo-effective divisors*

$$\overline{\mathrm{NE}}^1(X) = \text{ smallest closed cone containing } \mathrm{NE}^1(X,\mathbb{Z}) \subset N^1(X,\mathbb{R}).$$

We also have the *nef cone* $\overline{\mathrm{NM}}^1(X) \subset N^1(X,\mathbb{R})$ and $\mathrm{NM}^1_\circ(X)$ its interior. Note that $\overline{\mathrm{NM}}^1(X)$ and $\overline{\mathrm{NE}}_1(X)$ are dual in the sense that

$$\overline{\mathrm{NM}}^1(X) = \{D \in N^1(X,\mathbb{R}) : D \cdot C \geq 0 \text{ for each } C \in \overline{\mathrm{NE}}_1(X)\}.$$

These cones are governed by the following general results:

THEOREM 2.2. *Let X be a proper variety and D a Cartier divisor on X.*

(1) *(Nakai criterion) D is ample if and only if $D^{\dim(Z)} \cdot Z > 0$ for each closed subvariety $Z \subset X$.*
(2) *(Kleiman criterion) Assume X is projective. Then D is ample if and only if $D \in \mathrm{NM}^1_\circ(X)$, i.e., $D \cdot C > 0$ for each nonzero class C in the closure of the cone of curves on X.*

It is not difficult to verify that an ample divisor necessarily satisfies these conditions; we leave this as an exercise. We refer the reader to [**KM98**, §1.5] for proofs that these conditions are sufficient in general and to [**Har77**, §V.1] for Nakai's criterion in the special case of smooth projective surfaces.

THEOREM 2.3. *Let X be a smooth projective variety. A divisor $D \in N^1(X,\mathbb{Z})$ lies in the pseudoeffective cone $\overline{\mathrm{NE}}^1(X)$ if and only if, for each ample H and rational $\epsilon > 0$, some multiple of $D + \epsilon H$ is effective. It lies in the interior*

$$\mathrm{NE}^1_\circ(X) \subset \overline{\mathrm{NE}}^1(X)$$

if and only if there exists an $N \gg 0$ so that

$$ND = A + E$$

where A is ample and E is effective. Such divisors are said to be big.

PROOF. First, any divisor of the form $H + E$ is in the interior of the pseudoeffective cone. If B is an arbitrary divisor then $nH + B$ is very ample for some $n > 0$, and $n(H + E) + B$ is effective.

Conversely, let D lie in the interior of $\overline{\mathrm{NE}}^1(X)$. Consider

$$D - \mathrm{NM}^1_\circ(X) \subset N^1(X,\mathbb{R}),$$

i.e., the cone of anti-ample divisor classes translated so that the vertex is at D. Note that the ample cone of X is open, so we can pick a

$$B \in N^1(X,\mathbb{Q}) \cap (D - \mathrm{NM}^1_\circ(X))$$

and $m > 0$ so that $E := mB$ is an effective divisor. Express

$$B = D - tA$$

for A ample and $t \in \mathbb{Q}_{>0}$. Thus

$$D = \frac{1}{m}E + tA$$

and clearing denominators gives the desired result.

If D is not in $\overline{\mathrm{NE}}^1(X)$ then, for each H ample, there exists an $\epsilon > 0$ so that $D + \epsilon H$ is not effective. Conversely, if $D \in \overline{\mathrm{NE}}^1(X)$ then $D + \epsilon H \in \mathrm{NE}^1_\circ(X)$ and we can write

$$N(D + \epsilon H) = A + E$$

where $N \gg 0$, A is ample, and E is effective. □

COROLLARY 2.4. *Let X be a smooth projective surface. A nef divisor D is big if and only if $D^2 > 0$. Indeed, any divisor D (not necessarily nef) such that $D^2 > 0$ and $D \cdot H > 0$ for some ample divisor H is big.*

In fact, the analogous statement is true in all dimensions [**KM98**, 2.61].

PROOF. If D is big then we can express $D = A + E$, where A is an ample $\mathbb{Q}$-divisor and E is an effective $\mathbb{Q}$-divisor. We expand

$$D^2 = D \cdot (A + E) \geq D \cdot A = A \cdot A + A \cdot E > 0.$$

Conversely, if $D^2 > 0$ then the Riemann-Roch formula implies either

$$h^0(\mathcal{O}_X(mD)) \geq \frac{D^2}{2} m^2, \quad m \gg 0,$$

or

$$h^2(\mathcal{O}_X(mD)) \geq \frac{D^2}{2} m^2, \quad m \gg 0.$$

The latter possibility would imply $K_X - mD$ is effective for $m \gg 0$, which is incompatible with D being nef. (Actually, we only need that $D \cdot H > 0$ for some ample divisor H.) Given A very ample, a straightforward dimension count shows that $h^0(\mathcal{O}_X(mD - A))$ remains positive for $m \gg 0$, i.e., that D can be expressed as a sum of an ample and an effective divisor. □

Exercises.

EXERCISE 2.1.1. Let X denote the blow-up of $\mathbb{P}^2$ at a point. Give examples of big divisors D on X with $D^2 < 0$.

EXERCISE 2.1.2. Let X be a smooth projective variety. Verify that the conditions of the Nakai and Kleiman criteria are necessary for a divisor to be ample. When X is a surface, deduce the sufficiency of the Kleiman criterion from the Nakai criterion.

EXERCISE 2.1.3. The *volume* of a Cartier divisor D on an n-dimensional projective variety X is defined [**Laz04**, 2.2.31] as

$$\mathrm{vol}(D) = \limsup_{m \to \infty} h^0(X, \mathcal{O}_X(mD))/(m^n/n!).$$

When X is a smooth surface, show that D is big if and only if $\mathrm{vol}(D) > 0$.

2.2. Examples of effective cones of surfaces. For surfaces that are obtained by blowing up the plane $\beta : X \to \mathbb{P}^2$, we write L for the pullback of the line class on $\mathbb{P}^2$ and $E_1, E_2, \ldots$ for the exceptional curves.

(1) Let f_1 and f_2 denote the rulings of $X = \mathbb{P}^1 \times \mathbb{P}^1$, so that $\mathrm{Pic}(X) = \mathbb{Z}f_1 + \mathbb{Z}f_2$. Then we have
$$\overline{\mathrm{NE}}_1(X) = \{a_1 f_1 + a_2 f_2 : a_1, a_2 \geq 0\}$$
and
$$\mathrm{NE}_1^{\circ}(X) = \{a_1 f_1 + a_2 f_2 : a_1, a_2 > 0\}.$$
(2) If X is $\mathbb{P}^2$ blown up at one point then
$$\overline{\mathrm{NE}}_1(X) = \{aE + b(L - E) : a, b \geq 0\}.$$
(3) If X is $\mathbb{P}^2$ blown up at two points then
$$\overline{\mathrm{NE}}_1(X) = \{a_1 E_1 + a_2 E_2 + b(L - E_1 - E_2) : a_1, a_2, b \geq 0\}.$$

(4) If X is $\mathbb{P}^2$ blown up at three non-collinear points then

$$\overline{\mathrm{NE}}_1(X) = \langle L - E_1 - E_2, L - E_1 - E_3, L - E_2 - E_3, E_1, E_2, E_3 \rangle ,$$

i.e., the cone generated by the designated divisors.

(5) If X is a cubic surface with lines $\ell_1, \ldots, \ell_{27}$ then

$$\overline{\mathrm{NE}}_1(X) = \langle \ell_1, \ldots, \ell_{27} \rangle .$$

We describe a technique for verifying these claims, using the crucial fact that curves and divisors coincide on surfaces, i.e., $\overline{\mathrm{NE}}_1(X) = \overline{\mathrm{NE}}^1(X)$. To decide whether a collection of irreducible curves $\Gamma = \{C_1, \ldots, C_N\}$ generates $\overline{\mathrm{NE}}_1(X)$, it suffices to

- Compute a set of generators Ξ for the dual cone $\langle \Gamma \rangle^*$; there are computer programs like PORTA [**CL97**] and Polymake [**GJ00**] which can extract Ξ from Γ.
- Check that each $A_i \in \Xi$ can be written $A_i = \sum m_{ij} C_j, m_{ij} \geq 0$.

Here is why this works: If D is effective then we can write

$$D = M + F, \quad F = \sum_j n_j C_j, n_j \geq 0,$$

where M is effective with no support at $C_1, \ldots, C_N$. (Here F is the portion of the fixed part of D supported in Γ.) In particular, $M \cdot C_j \geq 0$ for each j, i.e., $M \in \langle \Gamma \rangle^*$. But then $M = \sum_i a_i A_i$ with $a_i \geq 0$. Thus we have

$$M = \sum_{ij} a_i m_{ij} C_j$$

and D is an effective sum of the C_j.

Example 2.5. For $X = \mathrm{Bl}_{p_1,p_2}\mathbb{P}^2$ take $\Gamma = \{E_1, E_2, L - E_1 - E_2\}$, which generates a simplicial cone. The dual generators are

$$\Xi = \{L - E_1, L - E_2, L\}$$

and we can write

$$L - E_1 = (L - E_1 - E_2) + E_2, \quad L - E_2 = (L - E_1 - E_2) + E_1,$$
$$L = (L - E_1 - E_2) + E_1 + E_2.$$

It follows that Γ generates $\overline{\mathrm{NE}}_1(X)$.

Exercises.

Exercise 2.2.1. Verify each of our claims about the generators of the effective cone.

2.3. Extremal rays. We'll need the following general notion from convex geometry:

Definition 2.6. Given a closed cone $\mathcal{C} \subset \mathbb{R}^n$, an element $R \in \mathcal{C}$ generates an *extremal ray* if for each representation

$$R = D_1 + D_2, \quad D_1, D_2 \in \mathcal{C}$$

we have $D_1, D_2 \in \mathbb{R}_{\geq 0} R$.

We will conflate the element R and the ray $\mathbb{R}_{\geq 0}R$. For a polyhedral cone, i.e., one generated by a finite number of elements

$$\mathcal{C} = \langle C_1, \dots, C_N \rangle = \mathbb{R}_{\geq 0}C_1 + \cdots + \mathbb{R}_{\geq 0}C_N,$$

the extremal rays correspond to the irredundant generators. On the other hand, for the cone over the unit circle

$$\{(x, y, z) : x^2 + y^2 - z^2 \leq 0\} \subset \mathbb{R}^3$$

each point of the circle yields an extremal ray.

Our main examples of extremal rays are (-1)-curves:

Proposition 2.7. *Let X be a smooth projective surface and E a (-1)-curve. Then E is extremal in $\overline{\mathrm{NE}}_1(X)$.*

If $\beta : X \to Y$ is the blow-down of E then

$$\beta_*\overline{\mathrm{NE}}_1(X) = \overline{\mathrm{NE}}_1(Y),$$

hence faces of $\overline{\mathrm{NE}}_1(Y)$ correspond to faces of $\overline{\mathrm{NE}}_1(X)$ containing E.

Proof. If we could express $E = D_1 + D_2$ with $D_1, D_2 \in \overline{\mathrm{NE}}_1(X)$ not in $\mathbb{R}_{\geq 0}E$, then

$$0 = \beta_*D_1 + \beta_*D_2$$

for nonzero $\beta_*D_i \in \overline{\mathrm{NE}}_1(Y)$. This contradicts the fact that $\overline{\mathrm{NE}}_1(Y)$ is strongly convex, i.e., that the origin is extremal.

The inclusion

$$\beta_*\overline{\mathrm{NE}}_1(X) \subset \overline{\mathrm{NE}}_1(Y)$$

is clear because the image of an effective divisor is effective. On the other hand, suppose that D is effective on Y. Since Y is nonsingular, D is a Cartier divisor and β^*D is a well-defined effective Cartier divisor. The projection formula $\beta_*\beta^*D = D$ shows that $D \in \beta_*\overline{\mathrm{NE}}_1(X)$. □

Corollary 2.8. *Let X be Del Pezzo and $\beta : X \to Y$ a blow-down morphism. Then Y is Del Pezzo.*

Proof. Indeed, we have the discrepancy formula

$$\beta^*K_Y = K_X - E,$$

where E is the exceptional curve. Since $-K_X$ is positive on $\overline{\mathrm{NE}}_1(X) \setminus \{0\}$, $-K_Y$ is positive on $\overline{\mathrm{NE}}_1(Y) \setminus \{0\}$. A direct argument that $-K_Y$ is ample can be extracted from the proof of the Castelnuovo Contraction Criterion (Theorem 1.12). □

Definition 2.9. Let X be a smooth projective surface. The *positive cone* $\mathcal{C}$ denotes the component of

$$\{D : D^2 > 0\} \subset N^1(X, \mathbb{R})$$

containing the hyperplane class. (The Hodge index theorem implies this has two connected components.) Let $\overline{\mathcal{C}}$ denote its closure.

We can formulate a more general version of Proposition 2.7, which complements Theorem 2.3 and Corollary 2.4:

Proposition 2.10. [**Kol96**, II.4] *Let X be a smooth projective surface. Then each irreducible curve D with $D^2 \leq 0$ lies in the boundary $\partial\overline{\mathrm{NE}}_1(X)$. Furthermore,*

$$\overline{\mathrm{NE}}_1(X) = \overline{\mathcal{C}} + \sum_D \mathbb{R}_{\geq 0} D \tag{2.1}$$

where the sum is taken over irreducible curves D with $D^2 < 0$.

Proof. Corollary 2.4 implies $\overline{\mathrm{NE}}_1(X) \supset \overline{\mathcal{C}}$, and $\overline{\mathrm{NE}}_1(X) \supset \overline{\mathcal{C}} + \sum_D \mathbb{R}_{\geq 0} D$ follows immediately. It remains to establish the reverse inclusion.

First, suppose that D is irreducible with $D^2 \leq 0$. Let H be a very ample divisor of X. Then for each rational $\epsilon > 0$ we claim that $D - \epsilon H$ fails to be in the effective cone. Indeed, if $D - \epsilon H$ were effective then it could be expressed as a nonnegative linear combination of irreducible curves, some different from D,

$$D - \epsilon H \equiv c_0 D + \sum_j c_j D_j, \quad c_0 \in [0,1), c_j > 0.$$

Regrouping terms, for some $\epsilon' > 0$ we obtain

$$D - \epsilon' H \equiv \sum_j c'_j D_j, \quad c'_j > 0.$$

However, this contradicts the fact that

$$D \cdot (D - \epsilon' H) < 0.$$

For D irreducible with $D^2 < 0$ consider the closed cone

$$V = \left\langle z \in \overline{\mathrm{NE}}_1(X) : z \cdot D \geq 0 \right\rangle \subset \overline{\mathrm{NE}}_1(X),$$

which contains all effective divisors without support along D. Thus $\overline{\mathrm{NE}}_1(X)$ is the smallest convex cone containing V and D. Since $D \notin V$, it follows that D is extremal in $\overline{\mathrm{NE}}_1(X)$.

Conversely, suppose that Z is extremal with $Z^2 < 0$. There is necessarily some irreducible curve C such that $C \cdot Z < 0$. Let Z_i denote a sequence of effective $\mathbb{Q}$-divisors approaching Z. Since $Z_i \cdot C < 0$ for $i \gg 0$, we must have that $C^2 < 0$. Moreover C appears in Z_i with coefficient c_i and $c = \lim c_i > 0$. Thus $Z - cC$ is pseudoeffective and $C \in \mathbb{R}_{\geq 0} Z$ by extremality. □

For special classes of surfaces, the negative extremal rays are necessarily (-1)-curves or (-2)-curves, i.e., smooth rational curves with self-intersection -2:

Corollary 2.11. [**Kol96**, II.4.14] *Suppose X is a smooth projective surface with $-K_X$ nef. Then the sum in expression (2.1) can be taken over D with $D^2 = -1$ or -2 and $D \simeq \mathbb{P}^1$.*

Proof. Since $K_X \cdot D \leq 0$ and $D^2 < 0$ then the adjunction formula

$$2g(D) - 2 = D^2 + K_X \cdot D$$

allows only the possibilities listed. □

Exercises.

Exercise 2.3.1. Classify extremal rays and describe decomposition (2.1) for:

- $\mathbb{P}^2$ blown up at two points or three non-collinear points;
- the Hirzebruch surfaces $\mathbb{F}_d$.

2.4. Structural results on the cone of curves I. The closed cone of effective curves has a very nice structure in the region where the canonical class fails to be nef. There are two different approaches to these structural results. The first emphasizes vanishing theorems (for higher cohomology) on line bundles and the resulting implications for linear series, e.g., basepoint-freeness. You can find details of this approach in references such as [**Rei97**, D] and [**CKM88**]. One significant disadvantage is that the reliance on Kodaira-type vanishing makes generalization to positive characteristic problematic. The second approach emphasizes the geometric properties of the curves themselves, especially the bend-and-break technique of Mori. This approach is taken in Mori's original papers, as well as in [**Kol96**, II.4,III.1].

Since both approaches are important for applications, we will sketch the key elements of each, referring to the literature for complete arguments.

THEOREM 2.12 (Cone Theorem). [**Rei97**, D.3.2] [**KM98**, Thm. 3.7] [**CKM88**, 4.7] *Let X be a smooth projective surface with canonical class K_X. There exists a countable collection of $R_i \in \overline{\mathrm{NE}}_1(X) \cap N_1(X, \mathbb{Z})$ with $K_X \cdot R_i < 0$ such that*

$$\overline{\mathrm{NE}}_1(X) = \overline{\mathrm{NE}}_1(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0} R_i,$$

where the first term is the intersection of $\overline{\mathrm{NE}}_1(X)$ with the halfplane $\{v \in N_1(X, \mathbb{R}) : v \cdot K_X \geq 0\}$. Given any ample divisor H and $\epsilon > 0$, there exists a finite number of R_i satisfying $(K_X + \epsilon H) \cdot R_i \leq 0$.

COROLLARY 2.13. *Let X be a Del Pezzo surface. Then $\overline{\mathrm{NE}}_1(X)$ is a finite rational polyhedral cone.*

What's even more remarkable is that the extremal rays can be interpreted geometrically. The following theorem should be understood as a far-reaching extension of the Castelnuovo contraction criterion (Theorem 1.12):

THEOREM 2.14 (Contraction Theorem). [**Rei97**, D.4] [**KM98**, Theorem 3.7] *Let X be a smooth projective surface and R a generator of an extremal ray with $K_X \cdot R < 0$. There exists a morphism $\phi : X \to Y$ to a smooth projective variety, with the following properties:*

1. *$\phi_* R = 0$ and ϕ contracts those curves with classes in the ray $\mathbb{R}_{\geq 0} R$;*
2. *ϕ has relative Picard rank one and $\mathrm{Pic}(Y)$ can be identified with $R^\perp \subset \mathrm{Pic}(X)$.*

PROOF. The proofs of Theorems 2.12 and 2.14 are intertwined. We can only offer a sketch of the arguments required. Some of these work only in characteristic zero, but we will make clear which ones.

Suppose we want to analyze the part of the effective cone along which K_X is negative. Fix an ample divisor H, which is necessarily positive along $\overline{\mathrm{NE}}_1(X)$. Which divisors $\tau K_X + H, \tau \in [0, 1]$, are nef? Consider the *nef threshold*

$$t = \sup\{\tau \in \mathbb{R} : \tau K_X + H \text{ nef }\},$$

i.e., $(tK_X + H)^\perp$ is a supporting hyperplane of $\overline{\mathrm{NE}}_1(X)$ provided $tK_X + H$ is nonzero. If we choose H suitably general, we can assume this hyperplane meets $\overline{\mathrm{NE}}_1(X)$ in an extremal ray. (Of course, for special H it might cut out a higher-dimensional face.)

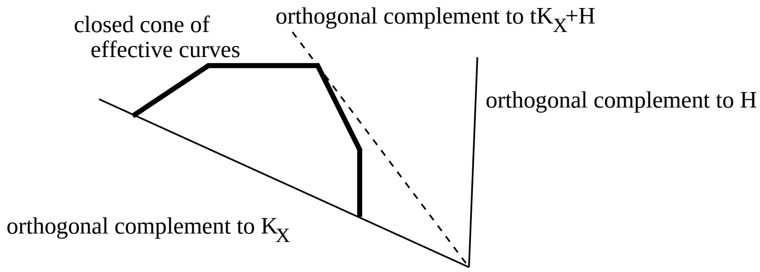

Figure 1. Finding a supporting hyperplane of the cone of curves (drawn in the projectivization of $N_1(X, \mathbb{R})$)

The first step is the rationality of the nef threshold. The most straightforward proof [**Rei97**, D.3.1] uses the Riemann-Roch formula and Kodaira vanishing, and thus is valid only in characteristic zero:

Lemma 2.15 (Rationality). The nef threshold is rational.

Thus the K_X-negative extremal rays of the cone of effective curves are determined by linear inequalities with rational coefficients. These rays here can be chosen to be integral.

The second step is to show that the ($\mathbb{Q}$-)divisor $D := tK_X + H$ is semiample. Then the resulting morphism $\phi : X \to Y$ will contract precisely the extremal rays in the face supported by the hyperplane $(tK_X + H)^{\perp}$, which gives the contraction theorem.

Lemma 2.16 (Basepoint-freeness). Let D be a nef $\mathbb{Q}$-divisor such that $D = tK_X + H$ for H ample and $t > 0$. Then D is semiample.

Proof. Since D is nef we have $D^2 \geq 0$. If $D^2 > 0$ then the Nakai criterion (Theorem 2.2) implies D is ample unless there exists an irreducible curve E with $D \cdot E = 0$. The Hodge index theorem implies $E^2 < 0$. Since $K_X \cdot E < 0$, Proposition 1.6 implies that E is a (-1)-curve. The desired contraction exists by the Castelnuovo Criterion (Theorem 1.12).

Now suppose $D^2 = 0$. If D is numerically equivalent to zero then $-K_X$ is ample. In this situation, D is semiample if and only if it is torsion, which is a consequence of the following lemma.

Lemma 2.17. Suppose that X is a smooth projective surface and $-K_X$ is nef and big, e.g., X is a Del Pezzo surface. Then

- $H^2(X, \mathcal{O}_X) = 0$ and $\mathrm{Pic}(X)$ is smooth;
- $H^1(X, \mathcal{O}_X) = 0$ and the identity component of $\mathrm{Pic}(X)$ is trivial.

Proof. Since some positive multiple of $-K_X$ is effective, no positive multiple of K_X is effective. Thus

$$h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X)) = 0$$

and $\mathrm{Pic}(X)$ is smooth. The identity component has dimension $q = h^1(X, \mathcal{O}_X)$.

Let $b_i(X)$ denote the ith Betti number of X; in positive characteristic, we define these using étale cohomology [**Mil80**]. Recall the formulas [**Mil80**, III.4.18,V.3.12]

$b_1(X) = 2q$ and

$$c_2(X) = \chi(X) = b_0(X) - b_1(X) + b_2(X) - b_3(X) + b_4(X) = 2 - 4q + b_2(X).$$

Noether's formula

$$12\chi(\mathcal{O}_X) = c_1(X)^2 + c_2(X)$$

and the fact that $c_1(X)^2 = K_X^2 > 0$ imply

$$12(1-q) > 2 - 4q + b_2(X).$$

Consequently

$$10 > 8q + b_2(X)$$

and thus $q = 0$ or $q = b_2(X) = 1$. To exclude the last case, observe that if the identity component of the Picard group is positive dimensional then so is the Albanese variety. (Indeed, these abelian varieties are dual to each other.) Furthermore, the Albanese map $X \to \mathrm{Alb}(X)$ [**Lan59**, II.3] is a dominating morphism to an elliptic curve. The classes of a fiber and the pullback of an ample divisor from the Albanese are necessarily independent; thus $b_2(X) \geq 2$. □

We return to the proof of Lemma 2.16. If D is not numerically trivial then $K_X \cdot D < 0$ and Riemann-Roch imply that $h^0(X, \mathcal{O}_X(mD))$ grows at least linearly in m. And since Corollary 2.4 ensures that D is not big, $h^0(\mathcal{O}_X(mD))$ cannot be a quadratic function of m. Decompose D into a moving and a fixed part

$$D = M + F, \quad M^2 \geq 0, \ M \cdot F \geq 0.$$

Note that $D \cdot F = M \cdot F + F^2 \geq 0$ (since D is nef), $M^2 = 0$ (as M is not big), and $F^2 \leq 0$ (since F is not big). On the other hand,

$$0 = D^2 = 2M \cdot F + F^2 \geq M \cdot F$$

so $M \cdot F = 0$ and $F^2 = 0$ as well. The Hodge index theorem implies that M and F are proportional in the Néron-Severi group, provided they are numerically nontrivial. In particular, if $F \neq 0$ then $K_X \cdot F < 0$ and $h^0(F, \mathcal{O}_X(mF))$ grows linearly in m, contradicting the fact that F is fixed. Thus $D = M$ is moving with perhaps isolated basepoints, the number of which is bounded by $M^2 = 0$. We conclude that D is basepoint-free. □

This completes the proof of Theorem 2.14.

REMARK 2.18. This argument yields another result we shall use later: Let X be a smooth projective surface with $-K_X$ nef and big. Assume that D is a nef line bundle on X with $D^2 = 0$. Then D is semiample.

The third step is to bound the denominator of the nef threshold (cf. [**Rei97**, D.3.1] and [**CKM88**, 12.12]):

LEMMA 2.19 (Bounding denominators). *Assume the nef threshold is rational. Then its denominator is ≤ 3.*

PROOF. Again, the argument is a case-by-case analysis of $D = tK_X + H$. If $D^2 > 0$ then D is orthogonal to a (-1)-curve and $t \in \mathbb{Z}$. If D is numerically trivial then $-K_X$ is ample and Lemma 2.17 implies $\chi(\mathcal{O}_X) = 1$. Express $-K_X = rL$, where L is a primitive ample divisor and $r \in \mathbb{N}$. It suffices to show that $r \leq 3$. Noether's formula and the argument for Lemma 2.17 give

$$12 = r^2L^2 + c_2(X) = r^2L^2 + 2 + b_2(X)$$

so the only possibilities are $r = 1, 2, 3$.

It remains to consider the situation where $D^2 = 0$ but $D \not\equiv 0$. As we've seen, $D \cdot K_X < 0$ in this case. Furthermore, $t'K_X + H$ is never effective for $t' > t$. Here we claim $2t \in \mathbb{N}$. Otherwise, there exist $m, n \in \mathbb{N}$ with $m \gg 0$ such that

$$mt = n + \alpha, \quad 1/2 < \alpha < 1.$$

Thus $nK_X + mH$ is ample, $\Gamma(\mathcal{O}_X(-nK_X - mH)) = 0$, and

$$H^2(\mathcal{O}_X((n+1)K_X + mH)) = 0$$

by Serre duality. We deduce

$$\begin{aligned} h^0(\mathcal{O}_X((n+1)K_X + mH)) &= \chi(\mathcal{O}_X) + \tfrac{1}{2}((n+1)K_X + mH) \cdot (nK_X + mH) \\ &= \chi(\mathcal{O}_X) + \tfrac{1}{2}(-\alpha(1-\alpha)K_X^2 + m(1-2\alpha)D \cdot K_X + m^2 D^2) \\ &= \text{constant} + m \cdot \text{positive number} \end{aligned}$$

which is positive for m sufficiently large. Thus $mH + (n+1)K_X$ is effective, a contradiction. □

To complete the proof of Theorem 2.12, we show that the K_X-negative extremal rays have no accumulation points and for any ample H there are finitely many such rays in the region $(K_X + \epsilon H) \leq 0$. Let $H_1, \dots, H_d$ denote ample divisors forming a basis for the Néron-Severi group such that

$$H = a_1 H_1 + \dots + a_d H_d, \quad a_1, \dots, a_d \in \mathbb{Q}_{>0}.$$

Let t_j denote the nef threshold of H_j. Consider the local coordinate functions

$$b_j(\gamma) = \frac{H_j \cdot \gamma}{-K_X \cdot \gamma}$$

on the open subset of $\mathbb{P}(N_1(X, \mathbb{R}))$ where $K_X \neq 0$. For K_X-negative extremal rays, $b_j \geq t_j$; Lemmas 2.15 and 2.19 imply these are rational numbers with denominators dividing six. It follows that these rays have no accumulation points. The extremal rays with $(NK_X + H) \cdot R_i \leq 0$ for some $N \in \mathbb{N}$ have coordinates satisfying

$$a_1 b_1 + \dots + a_d b_d \leq N.$$

Since the a_j and b_j are positive rational numbers with bounded denominators, there are at most finitely many possibilities. □

The structure of the contraction morphism $\phi : X \to Y$ depends on the intersection properties of irreducible curves E generating our extremal ray:

Case $E^2 < 0$: Proposition 1.6 ensures E is a (-1)-curve and ϕ is the blow-down of E.

Case $E^2 = 0$: By adjunction, $E \simeq \mathbb{P}^1$ and $\phi : X \to Y$ fibers X over a curve with generic fiber $\mathbb{P}^1$. The extremality implies all fibers are irreducible and reduced, thus every fiber of ϕ is a projective line and we have a minimal ruled surface.

Case $E^2 > 0$: Corollary 2.4 implies E is big. Since f contracts E and all its deformations, ϕ is constant. Thus $\mathrm{Pic}(X) = \mathbb{Z}$ and X is Del Pezzo.

Since the first case cannot occur when X is minimal, we obtain:

COROLLARY 2.20. *Let X be a minimal smooth projective surface. Then one of the following conditions holds:*

- *K_X is nef;*
- *X is a $\mathbb{P}^1$-bundle over a curve B;*
- *X is Del Pezzo with* $\mathrm{Pic}(X) = \mathbb{Z}$.

In the first instance, X is the unique minimal smooth projective surface in its birational equivalence class.

PROOF. We only have to establish uniqueness: Let X' be another minimal smooth projective surface birational to X. Choose a factorization

$$\begin{array}{ccc} & Y & \\ {\scriptstyle\phi'}\swarrow & & \searrow{\scriptstyle\phi} \\ X' & & X \end{array}$$

where Y is smooth projective and the morphisms are birational. Indeed, ϕ and ϕ' are sequences of contractions of (-1)-curves (see Theorem 1.13). Express $K_Y = \phi^*K_X + F$ where F is an effective divisor with support equal to the exceptional locus of ϕ. If $E \subset Y$ is a (-1)-curve contracted by ϕ' then

$$-1 = K_Y \cdot E \geq F \cdot E$$

as K_X is nef. Thus E is contained in the support of F and is contracted by ϕ'. Since each ϕ'-exceptional divisor is ϕ-exceptional, we have a factorization

$$\phi : Y \to X' \to X.$$

Since X' is minimal, it must equal X. □

REMARK 2.21 (Relative version). Given a morphism $f : X \to B$ to a variety, we can also consider the relative cone of effective curves

$$\overline{\mathrm{NE}}_1(f : X \to B) = \{D \in \overline{\mathrm{NE}}_1(X) : f_*D = 0\}.$$

When B is smooth, this is the intersection of $\overline{\mathrm{NE}}_1(X)$ with the orthogonal complement to $f^*\mathrm{Pic}(B)$. The Cone Theorem 2.12 describes its structure in the region where the canonical divisor K_X is negative. There is a relative version of the Contraction Theorem giving contractions over B

$$\begin{array}{ccc} X & \stackrel{\phi}{\longrightarrow} & Y \\ & \searrow \quad \swarrow & \\ & B & \end{array} .$$

The classification of ruled surfaces (Theorem 1.16) is a prime example.

Exercises.

EXERCISE 2.4.1. Let X be a smooth projective surface with K_X nef. Show that X is not rational.

EXERCISE 2.4.2. Let $p_1, \dots, p_8 \in \mathbb{P}^2$ be general points. Let p_0 be the last basepoint of the pencil of cubic curves containing these points. Show that $X = \mathrm{Bl}_{p_0,p_1,\dots,p_8}\mathbb{P}^2$ has an infinite number of (-1)-curves.

Hints: Let $E_0, \dots, E_8$ denote the exceptional divisors. Consider the elliptic fibration

$$\eta : X \to \mathbb{P}^1$$

induced by the linear series $|f|$ with $f = -K_X = 3L - E_0 - \cdots - E_8$. Verify that sections of η are all (-1)-curves. Designate E_0 as the zero section of η and let $\sigma_i : \mathbb{P}^1 \to X, i = 1, \dots, 8$ denote the sections associated with $E_1, \dots, E_8$. Given sections $\sigma, \sigma' : \mathbb{P}^1 \to X$ we have

$$\begin{aligned} [(\sigma + \sigma')(\mathbb{P}^1)] \quad &= \quad [\sigma(\mathbb{P}^1)] + [\sigma'(\mathbb{P}^1)] - [E_0] + w(\sigma, \sigma')[f] \\ & \quad\quad w(\sigma, \sigma') = -[\sigma(\mathbb{P}^1) - E_0] \cdot [\sigma'(\mathbb{P}^1) - E_0]. \end{aligned}$$

Use this to analyze $m_1\sigma_1 + \cdots + m_8\sigma_8$.

2.5. Structural results on the cone of curves II. Our discussion of the Cone and Contraction Theorems is missing one crucial element: We have not shown that the extremal rays are generated by classes of *rational* curves on X. Another issue is that we cited arguments for the rationality of extremal rays relying on vanishing theorems; these do not readily extend to positive characteristic. These gaps can be filled using Mori's 'bend-and-break' technique:

THEOREM 2.22 (Bend-and-break). [**Kol96**, II.5.14] *Let X be a smooth projective variety, C a smooth projective curve, and $f : C \to X$ a morphism. Let M be a nef $\mathbb{R}$-divisor. Assume that $-K_X \cdot C > 0$. Then for each $x \in f(C)$ there is a rational curve $x \in D_x \subset X$ such that*

$$M \cdot D_x \leq 2\dim(X)\frac{M \cdot C}{-K_X \cdot C}, \quad -K_X \cdot D_x \leq \dim(X) + 1.$$

This is a deep result that we will not prove here. The main idea is to use the fact that the anticanonical class is negative to show that f admits deformations $f_t : C \to X$ whose images still contain x. This strategy works beautifully provided C has genus zero, but in higher genus it is necessary to reduce modulo p and precompose f with the Frobenius map. Then we consider limits of $f_t(C) \subset X$ as $t \to 0$; these necessarily contain rational curves $x \in D' \subset X$. We can iterate this strategy until we obtain a rational curve $D_x \ni x$ with fairly small anticanonical degree, i.e., $-K_X \cdot D_x \leq \dim(X) + 1$.

We still have not mentioned the rôle of the divisor M. This is crucial in applications to the cone of curves:

THEOREM 2.23 (Cone Theorem bis). [**Kol96**, III.1.2] *Let X be a smooth projective surface with canonical class K_X. There exists a countable collection of rational curves $L_i \subset X$ with $0 < -K_X \cdot L_i \leq 3$ such that*

$$\overline{\mathrm{NE}}_1(X) = \overline{\mathrm{NE}}_1(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0}[L_i]. \tag{2.2}$$

Given any ample divisor H and $\epsilon > 0$, there exists a finite number of L_i satisfying $(K_X + \epsilon H) \cdot L_i \leq 0$.

PROOF. We offer a sketch proof following [**Kol96**]: Let M be an $\mathbb{R}$-divisor corresponding to a supporting hyperplane of a K_X-negative extremal ray $R \in \overline{\mathrm{NE}}_1(X)$. Thus $M \cdot R = 0$ and $M \cdot \gamma > 0$ for $\gamma \in \overline{\mathrm{NE}}_1(X)$ with $\gamma \notin \mathbb{R}_{\geq 0}R$; in particular, M is a nef $\mathbb{R}$-divisor. We show that M is a supporting hyperplane of the closure W of the cone associated to the right-hand side of (2.2). (We refer the reader to [**Kol96**, III.1] for the argument that the right-hand side of (2.2) defines a closed cone.)

Assume this is not the case. Rescaling M if necessary, we may assume that $M \cdot D \geq 1.$ for each irreducible curve $D \subset X$ with $[D] \in W$. Consider the functional

$$\begin{array}{rcl} \phi : N_1(X, \mathbb{R})_{K_X<0} & \to & \mathbb{R} \\ \gamma & \mapsto & -M \cdot \gamma / K_X \cdot \gamma \end{array}$$

which is nonnegative on $\overline{\mathrm{NE}}_1(X)_{K_X<0}$ and positive away from $\mathbb{R}_{\geq 0}R$. Choose a sequence of effective curves with real coefficients approaching R

$$C_i = \sum_j a_{ij}C_{ij}, a_{ij} > 0, \quad \lim_{i\to\infty} C_i = R.$$

For each i, there exists an index j such that $K_X \cdot C_{ij} < 0$ and $\phi(C_{ij}) \leq \phi(C_i)$. And we have $\lim_{i\to\infty} \phi(C_i) = \phi(R) = 0$.

On the other hand, bend-and-break yields rational curves L_i such that $-K_X \cdot L_i \leq 3$ and

$$M \cdot L_i \leq 4\phi(C_{ij}) < 4\phi(C_i).$$

The left-hand side is bounded from below by 1 while the right-hand side approaches zero, so we obtain a contradiction. □

2.6. Classification of surfaces.

THEOREM 2.24. *If X is a Del Pezzo surface with* $\mathrm{Pic}(X) = \mathbb{Z}$ *then* $X \simeq \mathbb{P}^2$.

PROOF. Our argument follows [**Kol96**, III.3.7]. We first offer a short proof in characteristic zero: Let L be a line bundle generating $\mathrm{Pic}(X)$ with $L \cdot K_X < 0$. Lemma 2.17 ensures that $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$, so by Hodge theory we have $b_1(X) = 0$ and $b_2(X) = 1$. Poincaré duality then implies $L \cdot L = 1$. Noether's formula

$$12\chi(\mathcal{O}_X) = c_1^2(X) + c_2(X)$$

implies $c_1^2(X) = K_X^2 = 9$. We conclude that $K_X = -3L$ and $\chi(L) = 3$. Since $h^2(X, L) = h^0(X, K_X - L) = 0$, we have $h^0(X, L) \geq 3$. Moreover, the members of the corresponding linear series are integral curves of genus zero, i.e., $\mathbb{P}^1$'s. A straightforward inductive argument shows that L is basepoint-free and thus induces a degree-one morphism $X \to \mathbb{P}^2$, i.e., $X \simeq \mathbb{P}^2$.

We only used characteristic zero to show that $K_X = -3L$. Suppose then that $K_X = -rL$ for some $r \in \mathbb{N}$, where L is a generator of $\mathrm{Pic}(X)$. We have already seen in the proof of Lemma 2.19 that $r = 1, 2, 3$. If $r = 2$ then

$$2g(L) - 2 = L^2 + K_X L = -L^2$$

so $L^2 = 2$ and $g(L) = 0$; Riemann-Roch then gives $\chi(X, L) = 4$. Arguing as above, L is basepoint-free and defines a morphism $\phi : X \to \mathbb{P}^n$ for $n \geq 3$. The image is a quadric surface or a plane, and the latter possibility would contradict nondegeneracy. However, a quadric surface cannot be a rank-one Del Pezzo surface.

Finally, suppose that $r = 1$. The Cone Theorem (Theorem 2.23) implies the existence of a rational curve $f : \mathbb{P}^1 \to X$ with $\deg f^*(-K_X) = \deg f^*L \leq 3$. We have $f_*[\mathbb{P}^1] = mL$ for some $m \in \mathbb{N}$ with $mL^2 \leq 3$, and consequently $K_X^2 \leq 3$. Since every curve in X has positive self-intersection, we can deduce a contradiction from the following fact:

LEMMA 2.25. Let Y be a Del Pezzo surface with $K_Y^2 \leq 4$. Then Y contains a (-1)-curve.

Such curves C are called *lines* because $-K_Y \cdot C = 1$.

There are two general approaches to this. The most direct (see [**Kol96**, III.3.6] or Exercise 2.6.2) is to express Y as a hypersurface in a suitable weighted projective space, i.e., as a cubic surface in $\mathbb{P}^3$ (when $K_Y^2 = 3$), a quartic surface in $\mathbb{P}(1, 1, 1, 2)$ (when $K_Y^2 = 2$), or as a sextic surface in $\mathbb{P}(1, 1, 2, 3)$ (when $K_Y^2 = 1$). Proving this entails a fair amount of *ad hoc* analysis of linear series. Another approach (cf. [**Isk79**]) involves showing that Y lifts to characteristic zero and using the classification tools available there. □

Theorem 2.26 (Castelnuovo's Criterion). [**Bea96**, V.6] [**Kol96**, III.2.4] *Let X be a smooth projective minimal surface. Then X is rational if and only if*

$$q(X) = h^1(\mathcal{O}_X) = 0, \quad P_2(X) := h^0(X, \mathcal{O}_X(2K_X)) = 0.$$

Proof. The necessity of the numerical conditions is clear, as $P_2(X)$ and $q(X)$ are birational invariants of smooth projective varieties. For sufficiency, we may assume that X is minimal and falls into one of the three categories of Proposition 2.20. The third case (where X is Del Pezzo with $\mathrm{Pic}(X) = \mathbb{Z}$) is rational by Theorem 2.24. In the second case (where X is ruled over a curve B), the assumption $q(X) = 0$ implies that B has genus zero. Corollary 1.22 yields that X is rational. Finally, suppose that K_X is nef, so in particular $K_X^2 \geq 0$. We know that K_X is not effective; if $\Gamma(X, \mathcal{O}_X(K_X)) \neq 0$ then $\Gamma(X, \mathcal{O}_X(2K_X)) \neq 0$. Thus

$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 1$$

and

$$\chi(\mathcal{O}_X(-K_X)) = K_X^2 + 1 \geq 1.$$

Since $h^2(\mathcal{O}_X(-K_X)) = h^0(\mathcal{O}_X(2K_X)) = 0$ we conclude $h^0(\mathcal{O}_X(-K_X)) > 0$, i.e., $-K_X$ is effective. As K_X is nef, the only possibility is K_X trivial, a contradiction. □

Corollary 2.27. *Del Pezzo surfaces are rational.*

Proof. Let X be a Del Pezzo surface. Since $-K_X$ is ample we have that $P_2(X) = 0$. Lemma 2.17 gives $h^1(\mathcal{O}_X) = 0$. □

Corollary 2.28. *Each Del Pezzo surface X is isomorphic to one of the following:*

- $\mathbb{P}^1 \times \mathbb{P}^1$*;*
- *a blow-up of* $\mathbb{P}^2$ *at eight or fewer points.*

Proof. By Corollary 2.8, we just need to show that minimal Del Pezzo surfaces X are either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Our previous analysis implies X is $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_d$. But then X contains a rational curve of self-intersection $-d$, so $d = 0, 1$ by Proposition 1.6. □

Remark 2.29. The classification of complex surfaces goes back to the work of Castelnuovo and Enriques in the late 19th and early 20th centuries. The extension to positive characteristic is largely due to Zariski, who first proved the Castelnuovo rationality criterion in this context [**Zar58a, Zar58b**].

Exercises.

Exercise 2.6.1. Suppose that X is a surface such that K_X is not nef and $\mathrm{Pic}(X)$ has rank at least three. Then X contains a (-1)-curve.

Exercise 2.6.2. Let Y be a Del Pezzo surface with $K_Y^2 = 3$ (resp. $K_Y^2 = 4$). Show that $-K_Y$ is very ample and the image under $|-K_Y|$ is a cubic surface (resp. complete intersection of two quadric hypersurfaces.) Conclude that Y contains a line (cf. Corollary 1.9).

3. Classifying surfaces over non-closed fields

Let k be a perfect field with algebraic closure $\bar{k}$ and Galois group $G = \mathrm{Gal}(\bar{k}/k)$. Let X be a smooth projective surface over k so that

$$\bar{X} = X_{\bar{k}} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$$

is connected. We use $\mathrm{Pic}(X)$ to denote line bundles on X defined over k.

3.1. Minimal surfaces.

Definition 3.1. A smooth projective surface X over k is *minimal* if any birational morphism over k to a smooth surface

$$\phi : X \to Y$$

is an isomorphism.

Theorem 3.2. *X is minimal if and only if $\bar{X}$ admits no Galois-invariant collection of pairwise disjoint (-1)-curves.*

Proof. Suppose X is not minimal and admits a birational morphism $\phi : X \to Y$. By Theorem 1.13, $\bar{X}$ admits a (-1)-curve E contracted by ϕ. Since ϕ is birational there are only a finite number of such curves, so let $E_1, \dots, E_r$ denote the curves in the Galois orbit of E. As we saw in the proof of Theorem 1.13, the intersection form on $\mathbb{Z}E_1 + \cdots + \mathbb{Z}E_r$ is negative definite, thus the matrix

$$\begin{pmatrix} E_i^2 & E_iE_j \\ E_iE_j & E_j^2 \end{pmatrix}, \quad i \neq j,$$

has positive determinant. It follows that $E_i \cdot E_j < 1$, which gives the disjointness.

Conversely, let $E_1, \dots, E_r$ denote a Galois-invariant collection of pairwise disjoint (-1)-curves. Let H be an ample divisor on X. Since $H \cdot E_i = H \cdot E_j$ for each i, j, the divisor

$$H' = H + \sum_{j=1}^{r} (H \cdot E_i) E_i$$

is also Galois-invariant. We just take $Y = \mathrm{Proj}(\bigoplus_{n \geq 0} \Gamma(X, nH'))$, as in the proof of Theorem 1.12. □

Remark 3.3 (Galois-invariant classes versus divisors defined over k). Not every element $L \in \mathrm{Pic}(\bar{X})^G$ comes from a line bundle defined over k. Applying the Hochschild-Serre spectral sequence [**Mil80**, III.2.20], we find

$$H^1(X, \mathcal{O}_X^*) = \ker \left(H^0_G H^1(\bar{X}, \mathcal{O}_{\bar{X}}^*) \xrightarrow{d_2^{01}} H^2_G H^0(\bar{X}, \mathcal{O}_{\bar{X}}^*) \right)$$

which yields

$$\mathrm{Pic}(X) = \ker \left(\mathrm{Pic}(\bar{X})^G \xrightarrow{d_2^{01}} \mathrm{Br}(k) \right).$$

Since $\mathrm{Br}(k)$ is torsion, some power NL with $N > 0$ is defined.

On the other hand, when $X(k) \neq \emptyset$ the homomorphism d_2^{01} is trivial. Indeed, the spectral sequence shows that the image of d_2^{01} lies in the kernel of the homomorphism

$$s^* : \mathrm{Br}(k) \to \mathrm{Br}(X) = H^2(X, \mathcal{O}_X^*)$$

induced by the structure map $s : X \to \mathrm{Spec}(k)$. Each rational point $x : \mathrm{Spec}(k) \to X$ induces $x^* : \mathrm{Br}(X) \to \mathrm{Br}(k)$, a left-inverse of s^*. Thus s^* is injective and d_2^{01} is trivial. (See [**CTS87**] for a comprehensive discussion.)

EXAMPLE 3.4. Suppose we have a cubic surface X with six conjugate disjoint lines $E_1, \dots, E_6$. Does it follow that X is the blow-up of $\mathbb{P}^2$ at six conjugate points?

The divisor class $-K_X + E_1 + \cdots + E_6 = 3L$ is definitely defined over k. The corresponding linear series gives a morphism

$$X \to Y \subset \mathbb{P}^9$$

blowing down $E_1, \dots, E_6$; here $\bar{Y} \simeq \mathbb{P}^2_{\bar{k}}$ is embedded via the cubic Veronese embedding. This is an example of a *Brauer-Severi variety*, i.e., a variety Y such that $\bar{Y} \simeq \mathbb{P}^{\dim(Y)}_{\bar{k}}$. Moreover, the invariant class

$$L \in \mathrm{Pic}(\bar{X})^G$$

comes from $\mathrm{Pic}(X)$ if and only if $Y \simeq \mathbb{P}^2_k$. A diagram-chase shows that $d_2^{01}(L) \in \mathrm{Br}(k)$ vanishes if and only if $[Y] \in \mathrm{Br}(k)$ is trivial.

Exercises.

EXERCISE 3.1.1. Let Y be a Brauer-Severi surface. Show there exists a smooth cubic surface X admitting a birational morphism $\phi : X \to Y$. *Hint:* A generic vector field on Y vanishes at three Galois-conjugate points. Blow up along two such collections of points.

EXERCISE 3.1.2 (Degree seven Del Pezzo surfaces). Let X be a surface such that $\bar{X} \simeq \mathrm{Bl}_{p_1,p_2}(\mathbb{P}^2)$. Show there exists a birational morphism $X \to \mathbb{P}^2$, obtained by blowing up a pair of Galois-conjugate points.

EXERCISE 3.1.3 (Some degree eight Del Pezzo surfaces). Let X be a surface such that $\bar{X} \simeq \mathrm{Bl}_p(\mathbb{P}^2)$. Show that X is isomorphic to $\mathrm{Bl}_p(\mathbb{P}^2)$ over k.

EXERCISE 3.1.4 (Degree five Del Pezzo surfaces). [**Sko93**] [**SD72**] Let X be a surface such that $\bar{X} \simeq \mathrm{Bl}_{p_1,p_2,p_3,p_4}(\mathbb{P}^2)$, where the points are distinct and no three are collinear.

(1) Show that the four points are projectively equivalent to

$$[1,0,0],[0,1,0],[0,0,1],[1,1,1]$$

over $\bar{k}$.

(2) Show that sections of $-K_X$ embed X as a quintic surface in $\mathbb{P}^5$.

(3) Show that this surface is cut out by five quadrics. *Hint:* It suffices to verify this on passage to $\bar{k}$.

(4) Choose generic $Q_0, Q_1, Q_2 \in I_X(2)$. Verify that

$$Q_0 \cap Q_1 \cap Q_2 = X \cup W,$$

where $\bar{W}$ is isomorphic to $\mathrm{Bl}_p\mathbb{P}^2$.

(5) Using Exercise 3.1.3, show that the exceptional divisor $E \subset W$ is defined over k and intersects X in one point.

Conclude that $X(k) \neq \emptyset$.

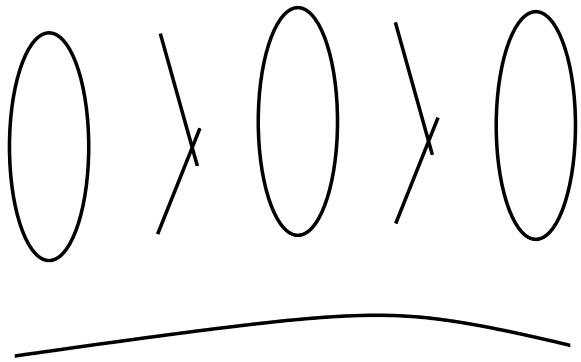

FIGURE 2. Degenerate fibers of a conic bundle

3.2. Conic bundles. Our treatment owes a great deal to Iskovskikh [**Isk79**].

DEFINITION 3.5. A *conic bundle* is a dominant morphism $f : X \to C$ from a smooth projective minimal surface X to a smooth curve, so that the generic fiber is a smooth curve of genus zero.

Of course, over an algebraically closed field this is the same as a minimal ruled surface. However, without the Tsen-Lang Theorem we cannot construct a section of f defined over k.

Proposition 1.19 does still apply: It guarantees that each smooth fiber of f is a plane conic and splits over a quadratic extension. It follows that there exists a *bisection* of f, i.e., an irreducible curve $D \subset C$ so that $f|_D : D \to C$ has degree two. Indeed, intersect the generic fiber (realized as a plane conic) with a line and take the closure in X.

THEOREM 3.6. *Let $f : X \to C$ be a conic bundle. Then any reducible fibers of $\bar{X} \to \bar{C}$ consist of two (-1)-curves intersecting in one point, conjugate under the Galois action.*

PROOF. Suppose F is a reducible fiber of $\bar{X}$; designate the field of definition of $f(F) \in C$ by $k_1 \supset k$. The existence of a reducible fiber guarantees that $\bar{X}$ is not relatively minimal and F contains a (-1)-curve (Theorem 1.16). Let $E_1, \dots, E_r$ be the Galois-orbit under the action of $\mathrm{Gal}(\bar{k}_1/k_1)$; the E_i are *not* pairwise disjoint by minimality (Theorem 3.2).

We claim that the only combinatorial possibility is $r = 2$, $E_1 \cdot E_2 = 1$, and $E_1^2 = E_2^2 = -1$. Write $T = E_1 \cup \dots \cup E_r$ and set $n = E_i \cdot (\sum_{j \neq i} E_j)$, i.e., the number of points of intersection of each component with the other components. We can compute the arithmetic genus using

$$2p_a(T) - 2 = -2r + rn.$$

Since F has arithmetic genus zero $p_a(T) \leq 0$ and $n = 1$, i.e., each connected component of T consists of two (-1)-curves meeting at one point. Reordering indices if needed, let $E_1 \cup E_2$ denote one of these components.

By Proposition 1.18, if $E_1, \ldots, E_{r+s}$ are the irreducible components of F then the intersection form on

$$(\bigoplus_{j=1}^{r+s} \mathbb{Z}E_j)/\mathbb{Z}F$$

is negative definite. However, we have $(E_1 + E_2)^2 = -1 + 2 - 1 = 0$ so necessarily $F = E_1 + E_2$. This proves the claim and the result. □

We have seen (Proposition 1.19) that the generic fiber of $f : X \to C$ admits a natural realization as a smooth plane conic. This is obtained using sections of the dual to the differential one-forms. We can extend this over all of C using the *relative dualizing sheaf*

$$\omega_f = \Omega^2_X \otimes (f^*\Omega^1_C)^{-1}.$$

We have natural homomorphisms

$$\Omega^1_{f^{-1}(p)} \to \omega_{f^{-1}(p)} = \omega_f|_{f^{-1}(p)},$$

where the first arrow is an isomorphism wherever f is smooth.

Corollary 3.7 (Conic bundles really are conic bundles). *Let $f : X \to C$ be a conic bundle with relative dualizing sheaf ω_f. Then we have an embedding over C*

$$\begin{array}{ccc} X & \stackrel{j}{\hookrightarrow} & \mathbb{P}(f_*\omega_f^{-1}) \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

realizing each fiber of X as a plane conic.

Proof. We use the classification of fibers in Theorem 3.6. For the smooth fibers, the anticanonical embedding has already been discussed in Proposition 1.19. For the reducible fibers, the anticanonical sheaf is very ample, realizing the fiber as a union of two distinct lines in $\mathbb{P}^2$.

Thus for each $p \in C$, $\omega_f^{-1}|_{f^{-1}(p)}$ is very ample and has no higher cohomology. Cohomology and base change gives that $f_*\omega_f^{-1}$ is locally free of rank three and has cohomology commuting with base extension. Thus we obtain a closed embedding over C

$$j : X \hookrightarrow \mathbb{P}(f_*\omega_f^{-1})$$

in a $\mathbb{P}^2$-bundle over C. □

Definition 3.8. A *rational conic bundle* is a conic bundle $f : X \to C$ over a curve of genus zero.

3.3. Analysis of Néron-Severi lattices. We analyze the Néron-Severi group of rational conic bundles $f : X \to \mathbb{P}^1$. Note that K_X is defined over the base field.

Theorem 3.6 and Corollary 1.22 imply that $\bar{X}$ is a blow-up of a Hirzebruch surface at r points in distinct fibers:

$$\begin{array}{ccc} \bar{X} & \longrightarrow & \mathbb{F}_d \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^1 & \end{array}.$$

The corresponding reducible fibers of $\bar{X} \to \mathbb{P}^1$ are denoted

$$E_1 \cup E'_1, E_2 \cup E'_2, \ldots, E_r \cup E'_r,$$

so that $E_i + E_i' = F$ for each i.

There are a number of natural lattices to consider. We have the relative Néron-Severi lattice

$$N^1(\bar{X} \to \mathbb{P}^1, \mathbb{Z}) = \{D \in N^1(\bar{X}, \mathbb{Z}) : f_* D = 0\} = \mathbb{Z}F + \mathbb{Z}E_1 + \cdots + \mathbb{Z}E_r,$$

the quotient lattice

$$N^1(\bar{X} \to \mathbb{P}^1, \mathbb{Z})/\mathbb{Z}F = (\mathbb{Z}E_1 + \mathbb{Z}E_2 + \cdots + \mathbb{Z}E_r + \mathbb{Z}F)/\mathbb{Z}F$$

with matrix

$$\begin{array}{c|cccc} & E_1 & E_2 & \dots & E_r \\ \hline E_1 & -1 & 0 & \dots & 0 \\ E_2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_r & 0 & 0 & \dots & -1 \end{array},$$

and the image Λ of the orthogonal complement $K_X^{\perp}$. This is generated by

$$\rho_1 = E_1' - E_2, \rho_2 = E_1 - E_2, \rho_3 = E_2 - E_3, \dots, \rho_r = E_{r-1} - E_r$$

with intersection matrix

$$\begin{array}{c|ccccccc} & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \dots & \rho_{r-1} & \rho_r \\ \hline \rho_1 & -2 & 0 & 1 & 0 & \dots & 0 & 0 \\ \rho_2 & 0 & -2 & 1 & 0 & \ddots & \ddots & \vdots \\ \rho_3 & 1 & 1 & -2 & 1 & \ddots & \ddots & \vdots \\ \rho_4 & 0 & 0 & 1 & -2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ \rho_{r-1} & 0 & \ddots & \ddots & \ddots & 1 & -2 & 1 \\ \rho_r & 0 & \dots & \dots & 0 & 0 & 1 & -2 \end{array}$$

Up to sign, this is the Cartan matrix associated to the root system $\mathbf{D}_r$.

Recall the traditional description of $\mathbf{D}_r$: Consider

$$\mathbb{Z}^r = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_r$$

with the standard pairing $e_i \cdot e_j = \delta_{ij}$. Consider the index two sublattice

$$M = \{m_1 e_1 + \cdots + m_r e_r : m_1 + \ldots + m_r \equiv 0 \pmod 2\} \subset \mathbb{Z}^r$$

with generators

$$\{-e_1 - e_2, e_1 - e_2, e_2 - e_3, \dots, e_{r-1} - e_r\}.$$

The Weyl group $W(\mathbf{D}_r)$ acts on M via reflections associated to the roots $\{\pm e_i \pm e_j\}$. It can be identified with signed $r \times r$ permutation matrices with determinant equal to the sign of the permutation. It is thus a semidirect product

$$W(\mathbf{D}_r) = (\mathbb{Z}/2\mathbb{Z})^{r-1} \rtimes \mathfrak{S}_r,$$

where the first group should be interpreted as the diagonal matrices with entries ± 1 and determinant 1 and the second group as the permutation matrices. Each element of $W(\mathbf{D}_r)$ is thus classified by the induced permutation of signed coordinate vectors

$$\{e_1, e_1' = -e_1, \dots, e_r, e_r' = -e_r\}.$$

Identifying E_i with e_i and E_i' with e_i', we obtain isomorphisms of lattices

$$\begin{array}{ccc} M & \simeq & -\Lambda \\ \downarrow & & \downarrow \\ \mathbb{Z}^r & \simeq & -N^1(f : \bar{X} \to \mathbb{P}^1, \mathbb{Z})/\mathbb{Z}F \end{array}$$

where the vertical arrows are inclusions of index-two subgroups. The Galois action of $G = \mathrm{Gal}(\bar{k}/k)$ on $\mathrm{Pic}(\bar{X})$ induces actions on both Λ and $N^1(\bar{X} \to \mathbb{P}^1, \mathbb{Z})/\mathbb{Z}F$. It is worthwhile to compare these to the action of $W(\mathbf{D}_r)$ on M and $\mathbb{Z}^r$.

Exercises.

Exercise 3.3.1. Let $\phi : X \to Y$ be a birational extremal contraction of smooth projective surfaces, i.e., a contraction of a collection of pairwise disjoint (-1)-curves. Show that

$$\Lambda = K_{\bar{X}}^{\perp} \cap N^1(\phi : \bar{X} \to \bar{Y}, \mathbb{Z})$$

is isomorphic to the lattice

$$\mathbb{Z}\rho_1 + \cdots + \mathbb{Z}\rho_{r-1}$$

with intersections

$$\rho_i \cdot \rho_j = \begin{cases} -2 & \text{if } i = j \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}.$$

This is the Cartan matrix for $\mathbf{A}_{r-1}$. Interpret the action of the Weyl group $W(\mathbf{A}_{r-1}) \simeq \mathfrak{S}_r$ in terms of the geometry of ϕ.

3.4. Classification of minimal rational surfaces over general fields. This is due to Manin [**Man66**] and Iskovskikh [**Isk79**]; another proof can be found in [**Kol96**, III.2].

Theorem 3.9. *Let X be a smooth projective minimal surface with $\bar{X}$ rational. Then X is one of the following:*

- $\mathbb{P}^2$*;*
- $X \subset \mathbb{P}^3$ *a smooth quadric with* $\mathrm{Pic}(X) = \mathbb{Z}$*;*
- *a Del Pezzo surface with* $\mathrm{Pic}(X) = \mathbb{Z}K_X$*;*
- *a conic bundle* $f : X \to C$ *over a rational curve, with* $\mathrm{Pic}(X) \simeq \mathbb{Z} \oplus \mathbb{Z}$*.*

Notice that the third case includes Brauer-Severi surfaces.

Thus if Y is a smooth projective rational surface over k then there exists a birational morphism $\phi : Y \to X$ defined over k, where X is one of the surfaces listed in Theorem 3.9.

Proof. Since $\bar{X}$ is rational $K_{\bar{X}}$ cannot be nef (Exercise 2.4.1), and there exists an irreducible curve $L \subset \bar{X}$ such that $K_{\bar{X}} \cdot L < 0$. In particular, $\overline{\mathrm{NE}}_1(\bar{X})$ admits $K_{\bar{X}}$-negative extremal rays. By the Cone Theorem 2.23, elements of $\overline{\mathrm{NE}}_1(\bar{X})$ can be expressed as

$$C + \sum a_i [L_i], \quad a_i > 0, \tag{3.1}$$

where $C \in \overline{\mathrm{NE}}_1(\bar{X})$ satisfies $C \cdot K_{\bar{X}} \geq 0$ and the L_i are rational curves generating $K_{\bar{X}}$-negative extremal rays. Of course, the Galois group G acts on $\mathrm{Pic}(\bar{X})$ and on the $K_{\bar{X}}$-negative extremal rays. Thus for elements of $\overline{\mathrm{NE}}_1(\bar{X})$ the two parts of (3.1) can be taken to be G-invariant.

Let $\overline{\mathrm{NE}}_1(\bar{X})^G$ denote the closure of the Galois-invariant effective cone in the real vector space spanned by Galois-invariant curve classes. Since $\overline{\mathrm{NE}}_1(\bar{X})^G$ has K_X-negative curves, it necessarily admits a K_X-negative extremal ray Z. This need not be extremal in $\overline{\mathrm{NE}}_1(\bar{X})$, but it does lie in some face of that cone, which we analyze. Since Z is extremal and K_X-negative, it must be proportional to the average over the orbit of a single extremal ray of $\bar{X}$:

$$Z = aE, \quad E = \sum_{j=1}^{n} L_j, \quad L_j = g_j L, g_j \in G.$$

In other words, the minimal face of $\overline{\mathrm{NE}}_1(\bar{X})$ containing Z is spanned by the Galois orbit of one extremal ray.

Assume first that $\mathrm{Pic}(X) \simeq \mathbb{Z}$, generated by some ample divisor H defined over k. Then $-K_X = rH$ for some positive integer r and X is Del Pezzo. When $r > 1$ we necessarily have $\bar{X} \simeq \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ by Corollary 2.28. In the first instance, X is a Brauer-Severi surface with a line H defined over the ground field; thus $\Gamma(\mathcal{O}_X(H))$ gives an isomorphism $X \simeq \mathbb{P}^2$. This is the first case of the theorem. In the second instance, the line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1)$ is defined over the ground field. Its global sections give an embedding $X \hookrightarrow \mathbb{P}^3$ whose image is a quadric surface. This is the second case of the theorem. Finally, if $-K_X$ generates $\mathrm{Pic}(X)$ then we are in the third case of the theorem.

Now assume that $\mathrm{Pic}(X)$ has higher rank, so in particular $E \in \partial\overline{\mathrm{NE}}_1(\bar{X})^G$. It follows that $E^2 \leq 0$. Indeed, if $E^2 > 0$ then E is big by Corollary 2.4 and thus lies in the interior of the effective cone by Theorem 2.3. (And there are some extremal L whose Galois orbits do not lie in any proper face of the cone of curves.)

Suppose now that $E^2 < 0$, which implies that $L^2 < 0$. As before, Proposition 1.6 implies L is a (-1)-curve. Furthermore, $L \cap L_j = \emptyset$ when $L \neq L_j$; indeed, if the Galois conjugates were nondisjoint then their sum would have nonnegative self-intersection. Theorem 3.2 implies that X is not minimal, a contradiction.

Suppose next that $E^2 = 0$. If $L^2 < 0$ then we would still have that L is a (-1)-curve. Since $E^2 = 0$ each curve meets precisely one of its Galois conjugates, transversely at one point. Thus the orbit of L decomposes as

$$\{L_1, L_1'\}, \{L_2, L_2'\}, \ldots, \{L_r, L_r'\},$$

where $L_i \cdot L_i' = 1$ and all other pairs of (-1)-curves are disjoint. Write $F_i = L_i + L_i'$ so that $F_i \cdot F_m = 0$ for each $i, m = 1, \ldots, r$; the Hodge index theorem implies that $F_1 = F_2 = \ldots = F_r$ and $E = rF_i$ for each i. Contracting E (or equivalently, $L_1, L_1', \ldots, L_r'$) gives a morphism

$$f : X \to C$$

whose generic fibers are smooth conics and with $r > 0$ degenerate fibers consisting of reducible singular conics. This is the conic bundle case of the theorem.

Finally, suppose that $E^2 = 0$ and $L^2 = 0$. Then each Galois conjugate of L is necessarily disjoint from L, so the Hodge index theorem argument above shows that $[L]$ is Galois-invariant. Contracting L gives a conic bundle $f : X \to C$ without degenerate fibers. □

Remark 3.10. This almost completes the birational classification of rational surfaces. It remains to enumerate birational equivalences among the surfaces listed in Theorem 3.9. This enumeration can be found in [**MT86**, 3.1.1, 3.3.2]

Exercises.

Exercise 3.4.1. To get a feeling for the difficulties involved, show that if X is minimal then $\bar{X} \neq \mathrm{Bl}_{p_1,\ldots,p_9}\mathbb{P}^2$. Specify a Galois action on $\mathrm{Pic}(\bar{X})$, in particular, a finite group acting linearly, preserving the intersection form, and fixing K_X. Consider the orbits of the (-1)-curves under this action. Convince yourself there is an orbit consisting of either

- disjoint (-1)-curves; or
- r disjoint pairs of (-1)-curves, with each pair meeting transversely at one point.

3.5. An application: Rational points over function fields. Our next result is due to Manin and Colliot-Thélène [**CT87**]. For more context and discussion, see [**Kol96**, IV.6]:

Theorem 3.11. *Let B be a smooth curve over $\mathbb{C}$ with function field $k = \mathbb{C}(B)$. Suppose that X is a smooth projective surface over k with $\bar{X}$ rational. Then $X(k) \neq \emptyset$.*

Of course, this result can be obtained from the Graber-Harris-Starr Theorem [**GHS03**]. However, we will present it using our classification techniques.

Proof. We first reduce to the case where X is minimal. Suppose we have a birational morphism $\phi : X \to Y$ to a smooth projective surface. We can factor ϕ as a sequence

$$X = X_0 \to X_1 \to X_2 \to \cdots \to X_r = Y$$

where each intermediate morphism is the blow-up of a Galois-orbit of points.

Suppose $x \in X_i(k)$ is a rational point. If x is contained in the center of the blow-up $\beta_i : X_{i-1} \to X_i$ then the exceptional divisor $E \subset X_{i-1}$ is rational over k and isomorphic to $\mathbb{P}^1$. It follows that $E(k) \neq \emptyset$ and $X_{i-1}(k) \neq \emptyset$. If x is disjoint from the center of $X_{i-1} \to X_i$ then x lies in the open subset $U \subset X_{i-1}$ over which β_i is an isomorphism. Thus $\beta_i^{-1}(x)$ is a rational point of X_i.

We consider the minimal cases one by one. The case $X = \mathbb{P}^2$ is straightforward. The case of a quadric surface $Q \subset \mathbb{P}^3$ follows from the Tsen-Lang Theorem.

We address the cases of Del Pezzo surfaces of degree $d = K_X^2$. Del Pezzo surfaces with certain degrees *always* have rational points. We assume without proof standard results on anticanonical linear series $|-K_X|$ and embeddings of X in projective space:

$d = 7$ There is no minimal Del Pezzo surface in this degree—see Exercise 3.1.2.

$d = 8$ ($\bar{X} \simeq \mathrm{Bl}_p\mathbb{P}^2$) There is no minimal Del Pezzo surface of the type—see Exercise 3.1.3.

$d = 5$ X always has a rational point—see Exercise 3.1.4.

$d = 1$ $\bar{X} \simeq \mathrm{Bl}_{p_1,\ldots,p_8}\mathbb{P}^2$ in this case and

$$-K_{\bar{X}} = 3L - E_1 - \cdots - E_8.$$

In this situation, $|-K_X|$ is the pencil of cubics $C_t, t \in \mathbb{P}^1$, passing through $p_1, \ldots, p_8$. The base locus of this pencil on $\mathbb{P}^2$ consists of nine points, i.e., $p_1, \ldots, p_8$ and one additional point p_0. The basis locus of $|-K_X|$ on X is just the point p_0. Since $-K_X$ is defined over k, the unique basepoint $p_0 \in X(k)$.

We address the remaining cases using the classification results of §2.6. Since $k = \mathbb{C}(B)$, we can use the following variant of the Tsen-Lang Theorem:

THEOREM 3.12. *Let k be the function field of a curve defined over an algebraically closed field. Let $F_1, \dots, F_r \in k[x_0, \dots, x_n]$ be nonconstant weighted homogeneous polynomials, with weighted degrees satisfying*

$$\deg(F_1) + \cdots + \deg(F_r) \le n.$$

Then the system $F_1 = \cdots = F_r = 0$ admits a nontrivial solution over k.

$d = 3$ Here X is a cubic surface in $\mathbb{P}^3$ and the result follows from Tsen-Lang.

$d = 9$ X is a Brauer-Severi surface. However, Exercise 3.1.1 allows us to blow up X to obtain a cubic surface, which has rational points by the previous case. (The reader knowledgeable in central simple algebras can prove $\mathrm{Br}(\mathbb{C}(B)) = 0$ using properties of the reduced norm.)

$d = 4$ Here X is a complete intersection of two quadrics in $\mathbb{P}^4$ and our variant of the Tsen-Lang Theorem applies.

$d = 2$ Here $|-K_X|$ induces a morphism

$$X \to \mathbb{P}^2$$

of degree two, branched over a quartic plane curve. It follows that X is a hypersurface of degree four in the weighted projective space $\mathbb{P}(2, 1, 1, 1)$ of the form $w^2 = f(x, y, z)$. An application of the Tsen-Lang Theorem gives our result.

$d = 6$ It suffices to show there exists a quadratic extension k'/k over which rational points are dense on X. Then after blowing up two suitable conjugate points we obtain a degree-four Del Pezzo surface, which has a rational point.

From our analysis of the effective cone of X in §2.2, there are two nef divisors $L, L' \in \mathrm{Pic}(\bar{X})$ so that

$$L^2 = (L')^2 = 1, -K_X \cdot L = -K_X \cdot L' = 3.$$

Indeed, we take $L' = 2L - E_1 - E_2 - E_3$. Their sections induce morphisms

$$\phi, \phi' : \bar{X} \to \mathbb{P}^2$$

blowing down triples of disjoint (-1)-curves. Let k'/k be a quadratic extension over which L and L' are $\mathrm{Gal}(\bar{k}'/k')$ invariant. Then $X_{k'}$ is a blow-up of a Brauer-Severi variety Y over k' at three conjugate points. The $d = 9$ case shows that Y (and hence $X_{k'}$) has lots of k'-rational points.

For the conic bundle case we apply the Tsen-Lang theorem twice. First, we show that $C(k) \neq \emptyset$ so $C \simeq \mathbb{P}^1$. Taking a generic $t \in \mathbb{P}^1(k)$ so that $X_t := f^{-1}(t)$ is a smooth conic, a second application gives $X_t(k) \neq \emptyset$. □

Exercises.

EXERCISE 3.5.1. Let X be a degree-one Del Pezzo surface over an arbitrary field k. Give a complete proof that $X(k) \neq \emptyset$, based on the sketch above. **Challenge:** When can you show that $|X(k)| > 1$?

4. Singular surfaces

In this section, we work over an algebraically closed field.

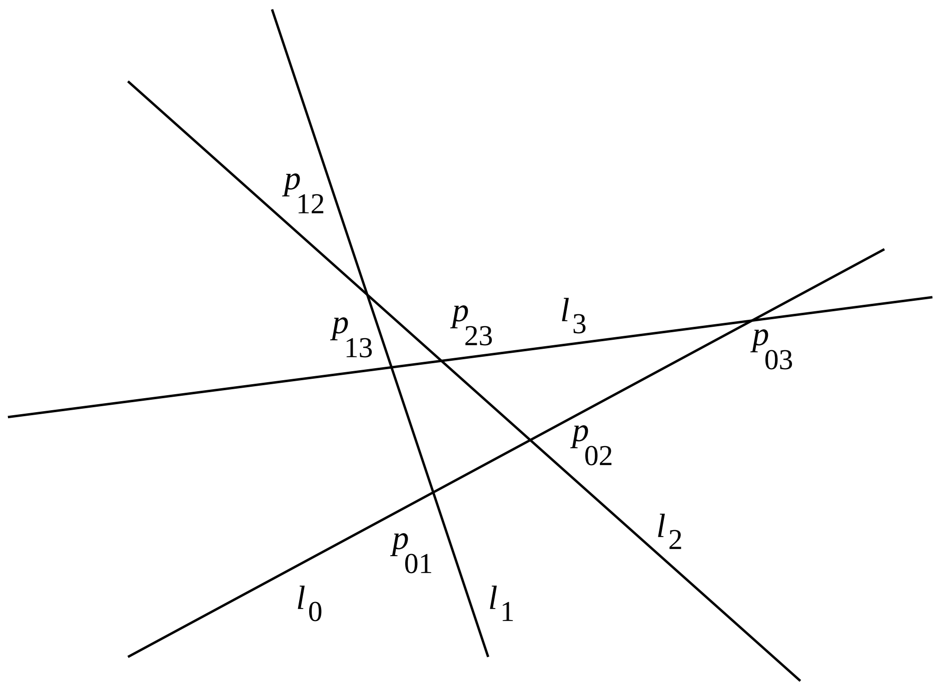

FIGURE 3. Four general lines in the plane

4.1. Cubic surfaces revisited: the Cayley cubic. In §1.1, we constructed smooth cubic surfaces by blowing up six points in general position on the plane. What happens when we relax this assumption?

Consider configurations of six points obtained as pairwise intersections of four general lines in the plane. Given four lines in general position, we can choose coordinates to put them in the standard form:

$$\ell_0 = \{x_0 = 0\},\ \ell_1 = \{x_1 = 0\},\ \ell_2 = \{x_2 = 0\},\ \ell_3 = \{x_0 + x_1 + x_2 = 0\}.$$

The intersection points are denoted $p_{ij} = \ell_i \cap \ell_j$ for $0 \leq i < j \leq 3$.

The points $p_{01}, \dots, p_{23}$ still impose independent conditions on homogeneous cubics in $x_0, \dots, x_3$, i.e.,

$$I_{p_{01},\dots,p_{23}} = \langle y_0, y_1, y_2, y_3 \rangle$$

where

$$\begin{array}{ll} y_0 = x_1x_2(x_0 + x_1 + x_2) & y_1 = x_0x_2(x_0 + x_1 + x_2) \\ y_2 = x_0x_1(x_0 + x_1 + x_2) & y_3 = -x_0x_1x_2 \end{array}.$$

These satisfy the relation

$$y_0y_1y_2 + y_1y_2y_3 + y_2y_3y_0 + y_3y_0y_1 = 0;$$

the resulting cubic surface $S \subset \mathbb{P}^3$ is called the *Cayley cubic surface* in honor of Arthur Cayley, who classified singular cubic surfaces [**Cay69**].

Here are some of its geometric properties:

- S has ordinary double points at

  $$s_0 = [1, 0, 0, 0],\ s_1 = [0, 1, 0, 0],\ s_2 = [0, 0, 1, 0],\ s_3 = [0, 0, 0, 1].$$

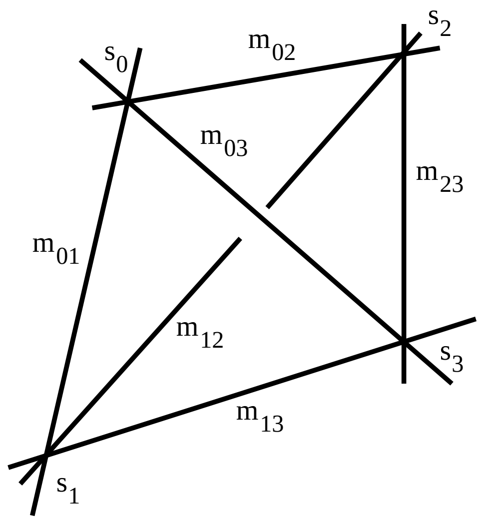

FIGURE 4. Some lines on the Cayley cubic

It is the unique cubic surface with this configuration of singularities, up to projective equivalence.

- S contains nine lines, i.e., the lines $m_{ij}, i, j = 0, \ldots, 3$ spanned by s_i and s_j, as well as the lines

$$\{y_0 + y_3 = y_1 + y_2 = 0\}, \quad \{y_0 + y_1 = y_2 + y_3 = 0\}, \quad \{y_0 + y_2 = y_1 + y_3 = 0\}.$$

- The birational map

$$[y_0, y_1, y_2, y_3] : \mathbb{P}^2 \overset{\sim}{\dashrightarrow} S$$

factors as

$$\begin{array}{ccc} & X & \\ {}^{\beta}\swarrow & & \searrow^{\sigma} \\ \mathbb{P}^2 & & S \end{array}$$

where β is the blow-up of $p_{01}, \ldots, p_{23}$ and σ is the blow-up of $s_0, \ldots, s_3$. The exceptional divisors of β are the proper transforms E_{ij} of the m_{ij}; the exceptional divisors of σ are the proper transforms ℓ_i' of the ℓ_i.

- Express

$$\mathrm{Pic}(X) = \mathbb{Z}L \oplus \mathbb{Z}E_{01} \oplus \cdots \oplus \mathbb{Z}E_{23}$$

where L is the pullback of the hyperplane class of $\mathbb{P}^2$ via β. The canonical class is

$$K_X = -3L + E_{01} + E_{02} + E_{03} + E_{12} + E_{13} + E_{23}$$

and the proper transforms of the lines are

$$\ell_0' = L - E_{01} - E_{02} - E_{03}, \; \ell_1' = L - E_{01} - E_{12} - E_{13}, \ldots$$

We have $K_X \cdot \ell_j' = 0$ and $(\ell_j')^2 = -2$ for each j, i.e., the exceptional divisors of the resolution σ are (-2)-curves.

4.2. Why consider singular cubic surfaces?

Reason 1: Reduction modulo primes

Let $X = \{F(y_0, y_1, y_2, y_3) = 0\} \subset \mathbb{P}^3$ be a smooth cubic surface defined over $\mathbb{Q}$; we may assume that $F \in \mathbb{Z}[y_0, y_1, y_2, y_3]$ and the greatest common divisor of the coefficients of F is 1. Consider the integral model

$$\pi : \mathcal{X} = \{F = 0\} \subset \mathbb{P}^3_{\mathbb{Z}} \to \mathrm{Spec}(\mathbb{Z}),$$

which is flat and projective over $\mathrm{Spec}(\mathbb{Z})$. For each prime p, we have $\mathcal{X}_p = \mathcal{X} \pmod p$, i.e., the fiber of $\mathcal{X}$ over $p \in \mathrm{Spec}(\mathbb{Z})$. If p divides the discriminant of F then $\mathcal{X}_p$ will have singularities. These singular fibers have a strong influence on the rational points of X.

Reason 1′: Degenerate fibers of families

This is the function-field analog of the previous situation. Let B be a complex curve and

$$\pi : \mathcal{X} \to B$$

a family of cubic surfaces, e.g., a pencil

$$\{sF(y_0, y_1, y_2, y_3) + tG(y_0, y_1, y_2, y_3) = 0\} \subset \mathbb{P}^3_{y_0,y_1,y_2,y_3} \times \mathbb{P}^1_{s,t}$$

with π being projection onto the second factor. At least some of the fibers $\mathcal{X}_b = \pi^{-1}(b), b \in B$ must be singular.

Reason 2: Counting rational points

Proving asymptotics for the number of rational points of bounded heights on *singular* cubic surfaces is often easier than the case of smooth cubic surfaces. Examples include toric cubic surfaces [**dlB98, Fou98, HBM99, Sal98**]

$$y_0^3 = y_1 y_2 y_3,$$

the Cayley cubic surface [**HB03**], the '$\mathbf{E}_6$ cubic surface' [**dlBBD07**]

$$y_1 y_2^2 + y_2 y_0^2 + y_3^3 = 0;$$

and a '$\mathbf{D}_4$ cubic surface' [**Bro06**]

$$y_1 y_2 y_3 = y_4 (y_1 + y_2 + y_3)^2.$$

4.3. What are 'good' singularities? Let S be a normal surface. A *resolution of singularities* $\sigma : X \to S$ is a birational proper morphism from a smooth surface. Abhyankar proved the existence of resolutions of surface singularities in arbitrary characteristic [**Abh56**]. A resolution $\sigma : X \to S$ is *minimal* if there exists no nontrivial factorization

$$X \xrightarrow{\phi} Y \to S$$

with Y smooth. This is equivalent to

- there are no (-1)-curves in the fibers of σ; or
- K_X is nef relative to σ.

A relative analog of Corollary 2.20 (see Remark 2.21) implies that minimal resolutions of singularities are unique, in the case of surfaces.

Recall that if $\phi : X \to Y$ is a birational morphism of smooth projective surfaces then (cf. Equation 1.1):

$$K_X = \phi^* K_Y + \sum_i m_i E_i, \quad m_i > 0.$$

The following definition represents a weakening of this condition:

DEFINITION 4.1. Suppose that S is a normal surface with a unique singularity p; assume that K_S is a $\mathbb{Q}$-Cartier divisor at p. (This is the case when S is a complete intersection in some neighborhood of p). Then $p \in S$ is a *canonical singularity* if, for each resolution of singularities $\sigma : X \to S$ we have

$$K_X = \sigma^* K_S + \sum_i m_i E_i, \quad m_i \geq 0,$$

where the E_i are exceptional divisors of σ.

Note here that *a priori* the $m_i \in \mathbb{Q}$; however, the classification of these singularities shows *a posteriori* that the $m_i \in \mathbb{Z}$.

PROPOSITION 4.2. *Suppose that (S, p) is a canonical singularity. A resolution of singularities $\sigma : X \to S$ is minimal if and only if each $m_i = 0$, i.e., $K_X = \sigma^* K_S$. In this case, each σ-exceptional curve is a (-2)-curve, i.e., a nonsingular rational curve E with $E^2 = -2$.*

PROOF. ($\Leftarrow$) Suppose that $m_i = 0$ for each i. Then $K_X \cdot E_i = 0$ for each σ-exceptional divisor. The Hodge index theorem implies that the σ-exceptional divisors have negative self-intersection, i.e., $E_i^2 < 0$. The adjunction formula implies that $E_i^2 = -2$ and E_i is a nonsingular curve of arithmetic genus zero.

($\Rightarrow$) Assume that σ is minimal, i.e., the fibers of σ contain no (-1)-curves. Suppose that $K_X \neq \sigma^* K_S$ so that some $m_i \neq 0$. It follows that $(\sum_i m_i E_i)^2 < 0$ and thus $(\sum_i m_i E_i) \cdot E_j < 0$ for some σ-exceptional curve E_j. Consequently $E_j^2 < 0$ and $K_X E_j < 0$, so E_j is a (-1)-curve by Proposition 1.6. □

Proposition 4.2 suggests the following variation on this definition

DEFINITION 4.3. A normal surface S has *Du Val singularities* if it admits a resolution $\sigma : X \to S$ such that $K_X \cdot E = 0$ for each σ-exceptional divisor E.

Patrick Du Val first classified surface singularities in terms of their discrepancies (or in his terminology, the 'conditions they impose on adjunction') in [**DV34**]. This definition is *a priori* more general than the class of canonical singularities: We do not insist that K_S is $\mathbb{Q}$-Cartier. However, we shall see later (Remark 4.7) that Du Val singularities are canonical.

Exercises.

EXERCISE 4.3.1. We give an example of a surface with 'bad' singularities. Suppose that $p_1, \ldots, p_4 \in \ell \subset \mathbb{P}^2$ are distinct points lying on a line ℓ. Consider

$$\beta : X := \mathrm{Bl}_{p_1,\ldots,p_4} \mathbb{P}^2 \to \mathbb{P}^2$$

and let $\tilde{\ell}$ denote the proper transform of ℓ, L the pullback of the hyperplane class via β, and $E_1, \ldots, E_4$ the exceptional divisors. Verify that

a. The divisor $4L - E_1 - E_2 - E_3 - E_4$ is basepoint-free and yields a morphism $\phi : X \to \mathbb{P}^{10}$.
b. If Y is the image of X under ϕ, show that $\phi : X \to Y$ is an isomorphism over $X \setminus \tilde{\ell}$ and contracts $\tilde{\ell}$ to a point $y \in Y$.
c. Show that Y is normal at y and the canonical class K_Y is $\mathbb{Q}$-Cartier. Compute the divisor $\phi^* K_Y$.
e. Show that $y \in Y$ is not a Du Val singularity.

4.4. Singular Del Pezzo surfaces.

Definition 4.4. A *singular Del Pezzo surface* is a projective surface S with Du Val singularities such that $-K_S$ is ample.

If $\sigma : X \to S$ is a minimal resolution of a singular Del Pezzo surface then $\sigma^* K_S = K_X$, i.e., $-K_X$ is semiample.

Here is one good source of singular Del Pezzo surfaces. Suppose X is a smooth projective surface with $-K_X$ nef and big. It has the following properties:

- $(-K_X)^2 > 0$;
- Any irreducible curve E with $K_X \cdot E = 0$ is a (-2)-curve.
- There are a finite number of (-2)-curves on X.

The first statement is a particular case of Corollary 2.4. The second is contained in the proof of Proposition 4.2. The third follows from the fact that $K_X^{\perp}$ is negative definite, and thus has a finite number of vectors of self-intersection -2.

Theorem 4.5. *Let X be a smooth projective surface with $-K_X$ nef and big. Then each nef divisor D on X is semiample.*

Corollary 4.6. *Let X be a smooth projective surface with $-K_X$ nef and big. Then $-K_X$ is semiample. In particular, there exists a birational morphism $\sigma : X \to S$ to a singular Del Pezzo surface with $\sigma^* K_S = K_X$.*

Proof. Remark 2.18 addresses this in the special case where D is not big, i.e., when $D^2 = 0$. Thus we may assume that $D^2 > 0$.

The Nakai criterion (Theorem 2.2) implies that D is ample unless $D \cdot E = 0$ for some irreducible curve $E \subset X$. The Hodge index theorem implies that each such curve satisfies $E^2 < 0$. Since $-K_X \cdot E \geq 0$, the only possibilities are (-1)-curves (see Proposition 1.6) or (-2)-curves (see Proposition 4.2). In either case $E \simeq \mathbb{P}^1$.

Suppose X admits (-1)-curves as above. We can apply the Castelnuovo contraction criterion (Theorem 1.12) to obtain a birational morphism $\beta : X \to Y$ such that Y admits a big and nef divisor M on Y with $\beta^* M = D$ and the only curves orthogonal to M are (-2)-curves. Furthermore, $-K_Y$ remains nef and big (cf. Corollary 2.8).

Let $E_1, \ldots, E_r$ denote the (-2)-curves orthogonal to M. We exhibit a birational morphism to a singular projective variety $\sigma : Y \to S$ contracting precisely these curves. Such a contraction exists for more general reasons [**Rei97**, 4.15] [**Art62**, 2.3] but we will sketch an argument in our situation.

We essentially copy the proof of the Castelnuovo Criterion. Let H be a very ample line bundle on Y such that each positive multiple nH has no higher cohomology. Write $d_i = H \cdot E_i$ for $i = 1, \ldots, r$. Since the intersection matrix of $\mathbb{Z}E_1 + \cdots + \mathbb{Z}E_r$ is negative definite, there exist positive integers n and $b_1, \ldots, b_r$ such that

$$nH \cdot E_i = -(b_1 E_1 + \cdots + b_r E_r) \cdot E_i$$

for each i. Let $B = b_1 E_1 + \cdots + b_r E_r$ so that $L := nH + B$ is orthogonal to each E_i.

The adjunction formula implies that each effective divisor A supported on $E_1 \cup \cdots \cup E_r$ has nonpositive arithmetic genus; a straightforward induction gives $H^1(\mathcal{O}_A) = 0$ as well. Here it is crucial that $K_Y \cdot E_i = 0$ for each i; it is not enough to assume that each component of the exceptional locus is rational. Thus we have

$\mathcal{O}_Y(L)|B \simeq \mathcal{O}_B$, i.e., an isomorphism of invertible sheaves, not just an equality of degrees. We obtain the exact sequence

$$0 \to \mathcal{O}_Y(nH) \to \mathcal{O}_Y(L) \to \mathcal{O}_B \to 0.$$

Our vanishing assumption show that

$$\Gamma(Y, \mathcal{O}_Y(L)) \twoheadrightarrow \Gamma(Y, \mathcal{O}_B),$$

i.e., for each point of B there is a section of $\mathcal{O}_Y(L)$ nonvanishing at that point. The sections of $\mathcal{O}_Y(L)$ induce an embedding away from $E_1 \cup \cdots \cup E_r$, so $\mathcal{O}_Y(L)$ is globally generated and induces a morphism $\sigma : Y \to S$ contracting precisely $E_1, \ldots, E_r$. In particular, $\mathcal{O}_Y(L)$ is the pullback of an ample line bundle on S via σ.

To complete the argument, we show there exists a Cartier divisor N on S such that σ^*N is a positive multiple of M. Repeating the previous argument for $M+mL$ with $m \gg 0$, we get the same contraction $\sigma : Y \to S$. Here the argument shows that $mL + M$ is the pullback of an ample line bundle from S. It follows that $M = \sigma^*N$ for some Cartier divisor N on X.

Finally, N is ample on S by the Nakai criterion, as we have contracted all the curves along which it is nonpositive. □

Remark 4.7. A variation on this argument shows that Du Val singularities are canonical. Suppose that $\sigma : Y \to S$ is a minimal resolution of Du Val singularities. The canonical class K_Y is nef relative to σ and thus globally generated relative to σ. We obtain a factorization

$$Y \to \operatorname{Proj}_S \left(\bigoplus_{n \geq 0} \sigma_* \mathcal{O}_Y(nK_Y) \right) \stackrel{\varpi}{\to} S.$$

Since ϖ is a bijective morphism of normal surfaces, it is an isomorphism. However, the canonical divisor of the intermediate surface is $\mathbb{Q}$-Cartier by construction.

Remark 4.8. Suppose the base field is algebraically closed of characteristic zero. There do exist smooth projective rational surfaces admitting nef divisors that are not semiample [**Zar62**, §2]. Thus the assumption that $-K_X$ be nef and big in Theorem 4.5 is necessary. (See Exercise 4.4.1 below and [**Laz04**, 2.3] for more discussion.)

We record one last consequence of Theorem 4.5, an extension of Corollary 2.13:

Proposition 4.9. *Let X be a smooth projective surface with $-K_X$ nef and big. Then $\overline{\mathrm{NE}}_1(X)$ is a finite rational polyhedral cone, generated by (-2)-curves and K_X-negative extremal rational curves.*

Proof. Apply Proposition 2.10 and Corollary 2.11: $\overline{\mathrm{NE}}_1(X)$ is generated by the nonnegative cone $\overline{\mathcal{C}}$, along with the (-1)-curves and (-2)-curves. The Hodge index theorem implies that K_X is negative on $\overline{\mathcal{C}} \setminus \{0\}$, so any extremal rays of $\overline{\mathrm{NE}}_1(X)$ arising from $\overline{\mathcal{C}}$ are necessarily K_X-negative.

The Cone Theorem 2.23 implies that the K_X-negative part of the effective cone is generated by curves L_i with $-K_X \cdot L_i \leq 3$. Theorem 4.5 gives that $-K_X$ is semiample and induces $\sigma : X \to S$. Thus there are at most a finite number of classes $[L_i]$ arising as K_X-negative extremal rays. Indeed, the curves in S with anticanonical degree ≤ 3 are parametrized by a scheme of finite type, as the curves in projective space of bounded degree are parameterized by a Hilbert scheme of finite type. We see in particular that X admits a finite number of (-1)-curves.

Clearly, there are a finite number of (-2)-curves, as these are all σ-exceptional. Thus $\overline{\mathrm{NE}}_1(X)$ admits a finite number of extremal rays, with the desired interpretations. □

Exercises.

EXERCISE 4.4.1. Assume the base field is of characteristic zero.

Let $C \subset \mathbb{P}^2$ denote a smooth cubic plane curve and H the hyperplane class on $\mathbb{P}^2$. Choose points $p_1, \dots, p_9 \in C$ such that the divisors $p_1 + \cdots + p_9$ and $H|C$ are linearly independent over $\mathbb{Q}$. Consider the blow-up

$$X := \mathrm{Bl}_{p_1,\dots,p_9}\mathbb{P}^2 \xrightarrow{\beta} \mathbb{P}^2$$

with exceptional curves $E_1, \dots, E_9$. Show that $D = -K_X = 3\beta^*H - E_1 - \cdots - E_9$ is nef but not semiample.

Now choose points $q_1, \dots, q_{12} \in C$ such that $q_1 + \cdots + q_{12}$ and $H|C$ are linearly independent. Consider the blow-up

$$Y := \mathrm{Bl}_{q_1,\dots,q_{12}}\mathbb{P}^2 \xrightarrow{\gamma} \mathbb{P}^2$$

with exceptional curves $F_1, \dots, F_{12}$. Show that $D' = 4\gamma^*H - F_1 - \cdots - F_{12}$ is nef but not semiample. Indeed, demonstrate that for each $n > 0$ the divisor nD' has the proper transform of C as a fixed component.

4.5. Classification of Du Val singularities. Suppose that $\sigma : X \to S$ is a minimal resolution of a Du Val surface singularity $p \in S$. Consider the intersection numbers of the irreducible components $E_1, \dots, E_r$ of $\sigma^{-1}(p)$, which we put into a symmetric matrix $(E_i \cdot E_j)_{i,j=1,\dots,r}$. This has the following properties:

- $(E_i \cdot E_j)$ is negative definite, by the Hodge index theorem;
- $E_i^2 = -2$ for each i, by Proposition 4.2;
- $E_i \cdot E_j = 0, 1$ for each $i \neq j$; indeed, if $E_i \cdot E_j > 1$ then $(E_i + E_j)^2 > 0$;
- we cannot express

$$\{E_1, \dots, E_r\} = \{E_{a_1}, \dots, E_{a_s}\} \cup \{E_{b_1}, \dots, E_{b_{r-s}}\}$$

with $E_{a_l} \cdot E_{b_m} = 0$ for each l, m; this is because $\sigma^{-1}(p)$ is connected.

Matrices of this type occur throughout mathematics, especially in the classification of the simple root systems via Dynkin diagrams/Cartan matrices in Lie theory. We cannot dwell too much on these interactions, except to refer the reader to some of the literature on this beautiful theory [**Bri71, Dur79, SB01**]. We list the possible matrices that can arise [**FH91**, 21.2]. First, we have the infinite series

$$\begin{matrix}\mathbf{A}_r \\ r \geq 1\end{matrix} \quad \begin{pmatrix} -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & \ddots & \ddots & \vdots \\ 0 & 1 & -2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 1 & -2 \end{pmatrix} \quad E_i \cdot E_i = \begin{cases} -2 & \text{if } i = j \\ 1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{D}_r \atop r \geq 4 \quad \begin{pmatrix} -2 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & -2 & 1 & 0 & \ddots & \ddots & \vdots \\ 1 & 1 & -2 & 1 & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \ddots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & \dots & \dots & 0 & 0 & 1 & -2 \end{pmatrix} \qquad E_i \cdot E_i = \begin{cases} -2 & \text{if } i = j \\ 1 & \text{if } |i-j| = 1, \\ & \quad i,j \geq 3 \\ & \text{or if } \{i,j\} \\ & = \{1,3\},\{2,3\} \\ 0 & \text{otherwise} \end{cases}$$

and then the exceptional lattices

$$\mathbf{E}_6 \quad \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 & 1 & -2 \end{pmatrix}$$

$$\mathbf{E}_7 \quad \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -2 \end{pmatrix}$$

$$\mathbf{E}_8 \quad \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & -2 \end{pmatrix}$$

Remarkably, in characteristic zero there is a *unique* singularity associated to each of these matrices.

PROPOSITION 4.10. *Assume that the base field is algebraically closed of characteristic zero. Then, up to analytic isomorphism, there is a unique Du Val surface singularity associated to each Cartan matrix enumerated above:*

$$\begin{array}{r|l} \mathbf{A}_r, r \geq 1 & z^2 = x^2 + y^{r+1} \\ \mathbf{D}_r, r \geq 4 & z^2 = y(x^2 + y^{r-2}) \\ \mathbf{E}_6 & z^2 = x^3 + y^4 \\ \mathbf{E}_7 & z^2 = y(x^3 + y^2) \\ \mathbf{E}_8 & z^2 = x^3 + y^5 \end{array}$$

For a modern proof of this, we refer the reader to [**KM98**, §4.2]. It turns out that these singularities are also related to the class of 'simple' hypersurface singularities, which can be independently classified [**AGZV85**]. There are a multitude of classical characterizations of Du Val singularities [**Dur79**].

Example 4.11. The Cayley cubic surface has four $\mathbf{A}_1$ singularities. The toric cubic surface

$$y_0^3 = y_1 y_2 y_3$$

has three $\mathbf{A}_2$ singularities at $[0,1,0,0]$,$[0,0,1,0]$, and $[0,0,0,1]$.

5. Cox rings and universal torsors

We work over an algebraically closed field k unless specified otherwise.

5.1. Universal torsors. Universal torsors are an important tool in higher-dimensional arithmetic geometry. They play a fundamental rôle in the modern theory of descent for rational varieties [**CTS87**]. They are also an important technique and conceptual tool for counting rational points of bounded height [**Sal98**].

Let X be a smooth projective variety. Assume that $\mathrm{Pic}(X)$ is a free abelian group of rank r, generated by the line bundles $L_1, \dots, L_r$ on X. Let $T_X = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{G}_m)$ denote the *Néron-Severi torus* of X, i.e., the torus with character group $\mathrm{Hom}(T_X, \mathbb{G}_m) = \mathrm{Pic}(X)$.

Definition 5.1. [**CTS87**] The *universal torsor* over X

$$\begin{array}{ccc} T_X & \rightarrow & U \\ & & \downarrow \\ & & X \end{array}$$

is a principal homogeneous space over X with structure group T_X with the following universal property: Given a line bundle L on X, if $\lambda_L : T_X \rightarrow \mathbb{G}_m = \mathrm{GL}_1$ denotes the corresponding character then the line bundle V_{λ_L} associated to U equals L. In other words, if U is given by a cocycle $\{\tau_{ij}\} \in H^1(X, T_X)$ then L is given by the cocycle $\{\lambda_L(\tau_{ij})\} \in H^1(X, \mathbb{G}_m)$.

Constructing U is straightforward in some sense: Choose $L_1, \dots, L_r$ freely generating $\mathrm{Pic}(X)$ and write

$$P_i = L_i^{-1} \setminus \mathbf{0}_X \subset L_i^{-1}$$

for the complement of the zero-section. This is a $\mathbb{G}_m$-principal bundle arising from L_i^{-1}. Then we can take

$$U = P_1 \times_X \cdots \times_X P_r$$

and T_X-action

$$\begin{array}{ccc} T_X \times U & \rightarrow & U \\ (t; s_1, \dots, s_r) & \mapsto & (\lambda_{-L_1}(t)s_1, \dots, \lambda_{-L_r}(t)s_r) \end{array}$$

where s_i is a local section of P_i and λ_L is the character associated with L.

However, for arithmetic applications it is important to have a more concrete presentation of the universal torsor.

Example 5.2. Consider the case $X = \mathbb{P}^n$. The standard quotient presentation

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$$

can be interpreted as an identification:

$$\begin{array}{ccccc} \mathcal{O}_{\mathbb{P}^n}(-1) \setminus \mathbf{0}_{\mathbb{P}^n} & & \stackrel{\sim}{\rightarrow} & & \mathbb{A}^{n+1} \setminus 0 \\ & \searrow & & \swarrow & \\ & & \mathbb{P}^n & & \end{array}$$

In other words, we regard $\mathcal{O}_{\mathbb{P}^n}(-1)$ as the 'universal line' over $\mathbb{P}^n$. Since $\mathcal{O}_{\mathbb{P}^n}(-1)$ generates $\mathrm{Pic}(\mathbb{P}^n)$, we have

$$U = \mathcal{O}_{\mathbb{P}^n}(-1) \setminus \mathbf{0}_{\mathbb{P}^n} = \mathbb{A}^{n+1} \setminus 0,$$

equivariant with respect to the action of $\mathbb{G}_m = T_{\mathbb{P}^n}$. Note that we can regard

$$\mathbb{A}^{n+1} = \mathrm{Spec}\left(\bigoplus_{N\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(N))\right).$$

More generally, the universal torsor

$$\begin{array}{ccc} T_{\mathbb{P}^m\times\mathbb{P}^n} & \rightarrow & U \\ & & \downarrow \\ & & \mathbb{P}^m \times \mathbb{P}^n \end{array}$$

can be identified with

$$\mathbb{A}^{m+n+2}_{x_0,\dots,x_m,y_0,\dots,y_n} \setminus (\{x_0 = \cdots = x_m = 0\} \cup \{y_0 = \cdots = y_n = 0\}).$$

Here the torus acts by the rule

$$(t_1, t_2) \cdot (x_0, \dots, x_m, y_0, \dots, y_n) = (t_1x_0, \dots, t_1x_m, t_2y_0, \dots, t_2y_n).$$

Decomposing the polynomial ring under this action, we can regard

$$\mathbb{A}^{m+n+2} = \mathrm{Spec}\left(\bigoplus_{N_1,N_2\in\mathbb{Z}} \Gamma(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^m\times\mathbb{P}^n}(N_1, N_2))\right).$$

Exercises.

EXERCISE 5.1.1. Realize the universal torsor over $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as an explicit open subset $U \subset \mathbb{A}^8$. Describe the action of T_X on U.

5.2. Universal torsors over nonclosed fields. We can only offer a brief summary here; we refer the reader to [**CTS87**] and [**Sko01**] for details and arithmetic applications.

Let k be a perfect field with absolute Galois group G. Suppose that X is defined over k and $\bar{X}$ satisfies the assumptions made in §5.1. The Galois action on $\mathrm{Pic}(\bar{X})$ allows us to define the torus T_X over k. Precisely, the action induces a representation on the character group

$$\varrho : G \to \mathrm{Aut}(\mathrm{Hom}(T_{\bar{X}}, \mathbb{G}_m)) = \mathrm{Aut}(\mathrm{Pic}(\bar{X})),$$

which gives the descent data for T_X. A *universal torsor* over X is a principal homogeneous space

$$\begin{array}{ccc} T_X & \rightarrow & U \\ & & \downarrow \\ & & X \end{array}$$

defined over k, such that the universal property is satisfied on passage to the algebraic closure.

Note our use of the indefinite article: Over a nonclosed field, a variety may have more than one universal torsor. Indeed, since any universal torsor U comes with a T_X-action over X, given a cocycle $\eta \in H^1_G(T_{\bar{X}})$ we can twist to obtain

$$\begin{array}{ccc} T_X & \rightarrow & U^\eta \\ & & \downarrow \\ & & X, \end{array}$$

another universal torsor over k. However, if the Galois action on $\mathrm{Pic}(\bar{X})$ is trivial then

$$H^1_G(T_{\bar{X}}) = H^1_G(\mathbb{G}_m^r) = 0$$

by Hilbert's Theorem 90. Here the universal torsor is unique whenever it exists.

On the other hand, there may be obstructions to descending a universal torsor over $\bar{X}$ to the field k. These reside in $H^2_G(T_{\bar{X}})$; indeed, the situation is analogous to the descent obstruction for line bundles discussed in Remark 3.3. Whenever $X(k) \neq \emptyset$ this obstruction vanishes [**CTS87**, 2.2.8], which makes universal torsors an important tool for deciding whether X has rational points.

5.3. Cox rings. Let X be a normal projective variety such that the Weil divisor class group is freely generated by $D_1, \ldots, D_r$.

Definition 5.3. The *Cox ring* of X is defined as

$$\mathrm{Cox}(X) = \bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r} \Gamma(X, \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r))$$

with multiplication

$$\Gamma(X, \mathcal{O}_X(m_1D_1 + \cdots + m_rD_r)) \times \Gamma(X, \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r)) \rightarrow \Gamma(X, \mathcal{O}_X((m_1+n_1)D_1 + \cdots + (m_r+n_r)D_r))$$

defined by $(s,t) \mapsto st$.

Example 5.4. We start with the eponymous example [**Cox95**]: Let X be a projective toric variety of dimension n

$$\mathbb{G}_m^n \times X \rightarrow X.$$

Let $D_1, \ldots, D_d$ denote the boundary divisors, i.e., the irreducible components of the complement of the dense open torus orbit. Let $s_i \in \Gamma(X, \mathcal{O}_X(D_i))$ denote the canonical section, i.e., the one associated with the inclusion

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D_i).$$

(Actually, s_i is canonical up to a nonzero scalar.) Recall that

- each effective divisor D on X can be expressed as a nonnegative linear combination

$$D \equiv n_1D_1 + \cdots + n_dD_d, \quad n_1, \ldots, n_d \geq 0;$$

- the canonical section s of $\mathcal{O}_X(D)$ admits a unique expression

$$s = f(s_1, \ldots, s_d)$$

  where f is a polynomial over k in d variables.

Thus the Cox ring of X is a polynomial ring

$$\mathrm{Cox}(X) \simeq k[s_1, \ldots, s_d]$$

with generators indexed by the boundary divisors.

We list some basic properties of the Cox ring of a *smooth* projective variety. We continue to assume that $D_1, \ldots, D_r$ are divisors freely generating $\mathrm{Pic}(X)$.

- $\mathrm{Cox}(X)$ is graded by $\mathrm{Pic}(X)$, i.e.,
$$\mathrm{Cox}(X) \simeq \bigoplus_{\mathcal{L} \in \mathrm{Pic}(X)} \mathrm{Cox}(X)_{\mathcal{L}}, \quad \mathrm{Cox}(X)_{\mathcal{L}} \simeq \Gamma(X, \mathcal{L}).$$
Indeed, for a unique choice of $(n_1, \dots, n_r) \in \mathbb{Z}^r$ we have an isomorphism
$$\mathcal{L} \simeq \mathcal{O}_X(n_1 D_1 + \cdots + n_r D_r).$$
- $\mathrm{Cox}(X)$ has a natural action by T_X via the rule
$$(t_1, \dots, t_r) \cdot s = t_1^{n_1} \cdots t_r^{n_r} s$$
when $s \in \Gamma(X, \mathcal{O}_X(n_1 D_1 + \cdots + n_r D_r))$.
- The nonzero graded pieces of $\mathrm{Cox}(X)$ are indexed by $\mathrm{NE}^1(X, \mathbb{Z})$. If $\mathrm{Cox}(X)$ is finitely generated then $\overline{\mathrm{NE}}^1(X)$ is a finitely generated rational polyhedral cone.

5.4. Two theorems. We start with a general result:

PROPOSITION 5.5. *Let X be a projective variety and $A_1, \dots, A_r$ semiample Cartier divisors on X. Then the ring*

$$\bigoplus_{n_1, \dots, n_r \geq 0} \Gamma(\mathcal{O}_X(n_1 A_1 + \cdots + n_r A_r)) \tag{5.1}$$

is finitely generated.

PROOF. (based on [**HK00**, 2.8], with suggestions from A. Várilly-Alvarado) It suffices to show that for some positive $N \in \mathbb{N}$ the ring
$$\bigoplus_{n_1, \dots, n_r \geq 0} \Gamma(\mathcal{O}_X(N(n_1 A_1 + \cdots + n_r A_r)))$$
is finitely generated. Indeed, the full ring is integral over this subring, so our result follows by finiteness of integral closure. Since $A_1, \dots, A_r$ are semiample, there exists an $N > 0$ such that $NA_1, \dots, NA_r$ are globally generated. Thus we may assume that $A_1, \dots, A_r$ are globally generated.

We first consider the special case $r = 1$. We obtain a morphism
$$\phi : X \to \mathbb{P}^m := \mathbb{P}(\Gamma(\mathcal{O}_X(A_1))^*),$$
with $\phi^* \mathcal{O}_{\mathbb{P}^m}(1) = \mathcal{O}_X(A_1)$. This admits a Stein factorization
$$X \xrightarrow{f} Y \xrightarrow{g} \mathbb{P}^m$$
with g finite and f having connected fibers, so in particular $f_* \mathcal{O}_X = \mathcal{O}_Y$. Furthermore, $g^* \mathcal{O}_{\mathbb{P}^m}(1)$ is ample on Y and thus
$$\bigoplus_{n \geq 0} \Gamma(Y, g^* \mathcal{O}_{\mathbb{P}^m}(n))$$
is finitely generated. The projection formula gives
$$g^* \Gamma(Y, g^* \mathcal{O}_{\mathbb{P}^m}(n)) \xrightarrow{\sim} \Gamma(X, \mathcal{O}_X(nA_1))$$
for each $n \in \mathbb{N}$, so
$$\bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nA_1))$$
is also finitely generated.

Now suppose r is arbitrary. Consider the vector bundle

$$V = A_1 \oplus \cdots \oplus A_r$$

and the associated projective bundle

$$\pi : \mathbb{P}(V^*) \to X.$$

We have the tautological quotient bundle

$$\pi^* V \twoheadrightarrow \mathcal{O}_{\mathbb{P}(V^*)}(1);$$

since V is globally generated (being the direct sum of globally generated line bundles), $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ is semiample. In particular, the ring

$$\bigoplus_{n \geq 0} \Gamma(\mathbb{P}(V^*), \mathcal{O}_{\mathbb{P}(V^*)}(n)) \tag{5.2}$$

is finitely generated.

The tautological quotient induces

$$\mathrm{Sym}^n \pi^* V \twoheadrightarrow \mathcal{O}_{\mathbb{P}(V^*)}(n),$$

and taking direct images via the projection formula we obtain

$$\mathrm{Sym}^n V \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathbb{P}(V^*)}(n)$$

and

$$\Gamma(X, \mathrm{Sym}^n V) = \Gamma(\mathbb{P}(V^*), \mathcal{O}_{\mathbb{P}(V^*)}(n)).$$

Since (5.2) is finitely generated, the algebra

$$\bigoplus_{n \geq 0} \Gamma(X, \mathrm{Sym}^n V)$$

is finitely generated as well. Using the decomposition

$$\mathrm{Sym}^n V = \bigoplus_{\substack{n_1 + \cdots + n_r = n \\ n_1, \ldots, n_r \geq 0}} \mathcal{O}_X(n_1 A_1 + \cdots + n_r A_r),$$

we conclude that (5.1) is finitely generated. □

REMARK 5.6 (due to A. Várilly-Alvarado). If A_1 and A_2 are ample then there exists an $N \in \mathbb{N}$ such that the multiplication maps

$$\Gamma(X, \mathcal{O}_X(Nm_1 A_1)) \otimes \Gamma(X, \mathcal{O}_X(Nm_2 A_2)) \to \Gamma(X, \mathcal{O}_X(N(m_1 A_1 + m_2 A_2)))$$

are surjective for each $m_1, m_2 \geq 0$. However, this fails for *semiample* divisors.

Let $h : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover branched over a smooth curve of bidegree $(2d, 2d)$; composing with the projections yield morphisms $g_i : X \to \mathbb{P}^1, i = 1, 2$, with connected fibers. Let f_1 and f_2 be the fibers of $\mathbb{P}^1 \times \mathbb{P}^1$; take A_1 and A_2 to be their preimages on X. Then we have

$$\Gamma(X, \mathcal{O}_X(mA_1)) = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$$

for each $m \geq 0$, i.e., sections of

$$\Gamma(X, \mathcal{O}_X(m_1 A_1)) \otimes \Gamma(X, \mathcal{O}_X(m_2 A_2))$$

are obtained via pullback from sections of $\Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m_1, m_2))$. Since $m_1 A_1 + m_2 A_2$ is very ample on X for suitable $m_1, m_2 \gg 0$, we conclude that

$$\Gamma(X, \mathcal{O}_X(m_1 A_1)) \otimes \Gamma(X, \mathcal{O}_X(m_2 A_2)) \to \Gamma(X, \mathcal{O}_X(m_1 A_1 + m_2 A_2))$$

cannot be surjective. Indeed, the decomposable sections cannot separate points in a fiber of h.

THEOREM 5.7. *Suppose X is a smooth projective variety with* Pic(X) *free of finite rank. Assume that* Cox(X) *is finitely generated. Then the universal torsor admits an embedding*

$$\iota : U \hookrightarrow \mathrm{Spec}(\mathrm{Cox}(X))$$

that is equivariant under the action of the Néron-Severi torus T_X.

PROOF. First, we construct the morphism ι. Again, let $D_1, \ldots, D_r$ denote divisors freely generating the divisor class group of X. The cone of effective divisors of X is finite rational polyhedral and strictly convex, so we can choose $D_1, \ldots, D_r$ such that each effective divisor on X can be written as a nonnegative linear combination of $D_1, \ldots, D_r$. (Of course, the D_i themselves need not be effective.)

Let $L_1, \ldots, L_r$ designate the line bundles associated to the invertible sheaves $\mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_r)$. Writing $P_i = L_i^{-1} \setminus \mathbf{0}_X$ we have

$$U = P_1 \times_X \cdots \times_X P_r \subset L_1^{-1} \times_X \cdots \times_X L_r^{-1}$$

which we interpret as the natural inclusion of

$$\mathrm{Spec}_X \left(\bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r} \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r) \right)$$

into

$$\mathrm{Spec}_X \left(\bigoplus_{n_1,\ldots,n_r\geq 0} \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r) \right).$$

For each $(n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0}$, we have

$$\Gamma(X, \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r)) \otimes \mathcal{O}_X \to \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r)$$

which induces

$$\mathrm{Spec}_X \left(\textstyle\bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r_{\geq 0}} \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r) \right) \longrightarrow \mathrm{Spec}_X \left(\textstyle\bigoplus_{(n_1,\ldots,n_r)\in\mathbb{Z}^r_{\geq 0}} \Gamma(\mathcal{O}_X(n_1D_1 + \cdots + n_rD_r)) \otimes \mathcal{O}_X \right).$$

Since each effective divisor is a nonnegative sum of the D_i, the target is isomorphic to $X \times \mathrm{Spec}(\mathrm{Cox}(X))$. Thus we get a morphism

$$\begin{array}{ccc} U & \rightarrow & X \times \mathrm{Spec}(\mathrm{Cox}(X)) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

and composing with the projection yields

$$\iota : U \to \mathrm{Spec}(\mathrm{Cox}(X)).$$

Our construction is clearly equivariant with respect to the actions of T_X.

We prove ι is an open embedding. First, observe that Spec(Cox(X)) is normal, i.e., Cox(X) is integrally closed in its fraction field. Since X is normal,

$$\bigoplus_{n_1,\ldots,n_r\geq 0} \mathcal{O}_X(n_1D_1 + \cdots + n_rD_r)$$

is a sheaf of integrally-closed domains, whose global sections form an integrally closed domain (cf, [**Har77**, Ex. 5.14(a)]). Furthermore, Cox(X) is even a UFD

[**EKW04**, Cor. 1.2]; this should not be surprising, as every effective divisor D on X naturally yields a principal divisor on $\mathrm{Spec}(\mathrm{Cox}(X))$, namely, the locus where the associated section $s \in \Gamma(X, \mathcal{O}_X(D)) \subset \mathrm{Cox}(X)$ vanishes.

We next exhibit a finitely-generated T_X-invariant subalgebra

$$R \subset \mathrm{Cox}(X)$$

such that the induced morphism

$$j : U \stackrel{\iota}{\to} \mathrm{Spec}(\mathrm{Cox}(X)) \to \mathrm{Spec}(R)$$

is an open embedding. Choose *ample* divisors $A_1, \dots, A_r$ freely generating $\mathrm{Pic}(X)$. (Since being ample is an open condition, we can certainly produce these.) For each ample A_i, we obtain an embedding

$$X \hookrightarrow \mathbb{P}(w_i)$$

into a weighted projective space, where the weights

$$w_i = (w_{i1}, \dots, w_{ij(i)})$$

index the degrees of a minimal set of homogeneous generators for the graded ring

$$x_{i1}, \dots, x_{ij(i)} \in \bigoplus_{N \geq 0} \Gamma(X, \mathcal{O}_X(NA_i)).$$

Take products to obtain

$$X \hookrightarrow \prod_{i=1}^{r} \mathbb{P}(w_i)$$

and let R denote the multihomogeneous coordinate ring of X, i.e., the quotient of the polynomial ring in the x_{ij} by the multihomogeneous polynomials cutting out X. We can then identify

$$U = \mathrm{Spec}(R) - \bigcup_{i=1}^{r} \{x_{i1} = \cdots = x_{ij(i)} = 0\}.$$

Thus we have a diagram

$$\begin{array}{ccccl} U & \stackrel{\iota}{\to} & V & := \mathrm{Spec}(\mathrm{Cox}(X)) \\ {\scriptstyle j}\downarrow & & \downarrow & \\ j(U) & \subset & W & := \mathrm{Spec}(R) \end{array}$$

with V normal and j an open embedding. Let $U' \subset V$ denote the pre-image of $j(U)$ in V. The induced morphism

$$\beta : U' \to j(U) \simeq U$$

is a birational morphism from a normal variety with a section, induced by $\iota \circ j^{-1}$. Any such morphism is an isomorphism. Indeed, the composed morphism

$$U' \stackrel{\beta}{\to} U \stackrel{\iota}{\to} U'$$

agrees with the identity on a dense subset of U', hence is the identity. Thus β and ι are inverses of each other. $\square$

THEOREM 5.8. *Let X be a smooth projective surface with $-K_X$ nef and big. Then* $\mathrm{Cox}(X)$ *is finitely generated.*

PROOF. Proposition 4.9 implies that $\overline{\mathrm{NE}}_1(X)$ is a finite rational polyhedral cone admitting a finite number of (-1)- and (-2)-curves. Thus the nef cone of X takes the form

$$\langle A_1, \ldots, A_r \rangle$$

where the A_i are nef divisors. Theorem 4.5 guarantees that each A_i is semiample. Consider the subring of the Cox ring

$$\mathrm{Cox}'(X) := \bigoplus_{D \in \langle A_1, \ldots, A_r \rangle} \Gamma(X, \mathcal{O}_X(D))$$

which is finitely generated by Proposition 5.5.

We next set up some notation, relying on the fact that $-K_X$ is semiample with associated contraction $\sigma : X \to S$ (Corollary 4.6). Let $E_1, \ldots, E_r$ denote the (-2)-curves on X, i.e., the curves contracted by σ. Let $F_1, \ldots, F_s$ denote the (-1)-curves on X. Choose generators $\eta_i \in \Gamma(\mathcal{O}_X(E_i))$ and $\xi_i \in \Gamma(\mathcal{O}_X(F_i))$, which are unique up to scalars. We regard these as elements of $\mathrm{Cox}(X)$.

LEMMA 5.9. Let D be an effective divisor on X. Express

$$D = M + F \tag{5.3}$$

where F is the fixed part and M is the moving part. Then the support of F consists of (-1)- and (-2)-curves.

PROOF. Suppose that the fixed part of F contains an irreducible component C that is not a (-1)- or (-2)-curve. It follows that $C^2 \geq 0$. Since C is effective, we have

$$h^2(\mathcal{O}_X(C)) = h^0(\mathcal{O}_X(K_X - C)) = 0.$$

Otherwise, $n(K_X - C)$ would be effective for each $n \geq 0$, which contradicts our assumption that $-K_X$ is big. The Hodge index theorem implies $-K_X \cdot C > 0$, so Riemann-Roch implies $h^0(\mathcal{O}_X(C)) > 1$, which means that C is not fixed. □

We interpret this via the Cox ring: Each homogeneous element $t \in \mathrm{Cox}(X)$ can be identified with an effective divisor $D = \{t = 0\}$. Expression (5.3) translates into $t = m \cdot f$, where $m \in \mathrm{Cox}'(X)$ and

$$f = \eta_1^{a_1} \cdots \eta_r^{a_r} \xi_1^{b_1} \cdots \xi_s^{b_s}, \quad a_1, \ldots, a_r, b_1, \ldots, b_s \in \mathbb{N}.$$

It follows then that

$$\mathrm{Cox}(X) = \mathrm{Cox}'(X)[\eta_1, \ldots, \eta_r, \xi_1, \ldots, \xi_s]$$

which completes our proof. □

REMARK 5.10. We make a few observations on the significance of Theorem 5.8 and recent generalizations.

- Hu and Keel [**HK00**] showed that smooth projective varieties with finitely generated Cox rings behave extremely well from the standpoint of birational geometry. Indeed, they designate such varieties *Mori Dream Spaces*.
- Shokurov [**Sho96**, §6] demonstrated how a robust version of the log minimal model program would imply that many classes of varieties have finitely generated Cox rings. For example, he established that log Fano threefolds over fields of characteristic zero have this property. These are a natural generalization of the singular Del Pezzo surfaces discussed here.

- As an application of their proof of the existence of minimal models for varieties of log general type (over fields of characteristic zero), Birkar, Cascini, Hacon, and M^{c}Kernan proved that log Fano varieties of arbitrary dimension have finitely generated Cox rings [**BCHM06**, 1.3.1].

Exercises. Suppose D is an effective divisor on a smooth projective surface X. Consider the graded ring

$$R(D) := \bigoplus_{m \geq 0} \Gamma(X, \mathcal{O}_X(mD)).$$

In the classic paper [**Zar62**], Zariski analyzed when this ring is finitely generated.

Exercise 5.4.1. Recall the notation of the second half of Exercise 4.4.1. Show that $R(D')$ is not finitely generated.

Exercise 5.4.2. Assume D admits a *Zariski decomposition* [**Zar62**, 7.7] [**Laz04**, 2.3.19], i.e.,

$$D = P + N \tag{5.4}$$

where P and N are $\mathbb{Q}$-divisors with the following properties:

- P is nef;
- N is effective with support
$$\mathrm{supp}(N) = \{C_i\}$$
generating a negative definite (or trivial) sublattice of the Néron-Severi group;
- $P \cdot C_i = 0$ for each $C_i \in \mathrm{supp}(N)$.

Deduce that

- for each $n \geq 0$ the map
$$\Gamma(X, \mathcal{O}_X(nD - \lceil nN \rceil)) \hookrightarrow \Gamma(\mathcal{O}_X(nD))$$
is an isomorphism;
- $\Gamma(X, \mathcal{O}_X(nP)) \simeq \Gamma(\mathcal{O}_X(nD))$ for $n \geq 0$ such that nN is integral.

If $-K_X$ is nef and big, deduce also that

- P is semiample;
- $\mathrm{supp}(N) \subset \{E_1, \dots, E_r, F_1, \dots, F_s\}$, the union of the (-1)- and (-2)-curves on X.

Hint: The second assertion is a corollary of the first. To prove this, note that any divisor A with

$$nD - \lceil nN \rceil \prec A \preceq nD$$

intersects some component in $\mathrm{supp}(N)$ negatively, and thus has that component in its fixed part.

Exercise 5.4.3. Let X be the Hirzebruch surface $\mathbb{F}_2$, Σ the class of a section at infinity, f the class of a fiber:

	Σ	f
Σ	2	1
f	1	0

This admits a unique (-2)-curve $E = \Sigma - 2f$.

- Show that $\mathrm{Cox}(X) \simeq k[\eta, f_0, f_\infty, t]$ where
$$\Gamma(\mathcal{O}_X(E)) = k\eta, \quad \Gamma(\mathcal{O}_X(f)) = kf_0 + kf_\infty,$$
and
$$\Gamma(\mathcal{O}_X(\Sigma)) = k\eta f_0^2 + k\eta f_0 f_\infty + k\eta f_\infty^2 + kt.$$
- Show that the Zariski decomposition of the divisor $D = \Sigma - f$ is
$$D = P + N, \quad P = \frac{1}{2}\Sigma, N = \frac{1}{2}E.$$
Verify that the fixed part of nD for $n \geq 0$ is $\lceil nN \rceil = \lceil n/2 \rceil E$.

5.5. More Cox rings of Del Pezzo surfaces. For blow-ups $\beta : X \to \mathbb{P}^2$, we write L for the pullback of the line class on $\mathbb{P}^2$ and $E_1, E_2, \ldots$ for the exceptional curves.

Example 5.11 (Degree Six Del Pezzo Surfaces). Let X be isomorphic to $\mathbb{P}^2$ blown up at three non-collinear points, which can be taken to be $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, and $p_3 = [0, 0, 1]$. This is a toric variety under the action of the diagonal torus. We have seen in §2.2 that $\overline{\mathrm{NE}}_1(X)$ is generated by the (-1)-curves:
$$\{E_1, E_2, E_3, E_{12}, E_{13}, E_{23}\}$$
where E_{ij} is the proper transform of the line joining p_i and p_j with class $L - E_i - E_j$. Here we have (cf. [**BP04**, 3.1]):
$$\mathrm{Cox}(X) = k[\eta_1, \eta_2, \eta_3, \eta_{12}, \eta_{13}, \eta_{23}].$$

Example 5.12 (Degree Five Del Pezzo Surfaces). This example is due to Skorobogatov [**Sko93**] (see also [**BP04**, 4.1]). Suppose that X is isomorphic to $\mathbb{P}^2$ blown up at four points in linear general position, which can be taken to be $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, $p_3 = [0, 0, 1]$, and $p_4 = [1, 1, 1]$. Let $E_i, i = 1, \ldots, 4$ denote the exceptional curves and E_{ij} the proper transforms of the lines joining p_i, with class $E_{ij} = L - E_i - E_j$. Skorobogatov shows there exist normalizations of the generators $\eta_{i5} \in \Gamma(\mathcal{O}_X(E_i))$ and $\eta_{ij} \in \Gamma(\mathcal{O}_X(E_{ij}))$ such that
$$\mathrm{Cox}(X) = k[\eta_{12}, \ldots, \eta_{45}] / \langle P_1, P_2, P_3, P_4, P_5 \rangle$$
where each P_i is a *Plücker relation*
$$P_i = \eta_{jk}\eta_{lm} - \eta_{jl}\eta_{km} + \eta_{jm}\eta_{kl}, \quad \{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}, j < k < l < m.$$
More geometrically, $\mathrm{Cox}(X)$ is the projective coordinate ring of the Grassmannian $\mathbb{G}(1, 4) \subset \mathbb{P}^9$.

Example 5.13 ($\mathbf{E}_6$ cubic surface). See [**HT04**, §3] for more details. Let $S \subset \mathbb{P}^3$ denote the (unique) cubic surface with a singularity of type $\mathbf{E}_6$
$$S = \{(w, x, y, z) : xy^2 + yw^2 + z^3 = 0\} \subset \mathbb{P}^3$$
and $\sigma : X \to S$ its minimal resolution of singularities. Let $E_1, \ldots, E_6$ denote the exceptional curves of σ and $\ell \subset X$ the proper transform of the unique line $\{y = z = 0\} \subset S$. The effective cone here is simplicially generated by (-1)- and (-2)-curves
$$\overline{\mathrm{NE}}_1(X) = \langle \ell, E_1, E_2, E_3, E_4, E_5, E_6 \rangle$$
but the corresponding elements $\xi_\ell, \xi_1, \ldots, \xi_6 \in \mathrm{Cox}(X)$ do not suffice to generate it. In this case, for a suitable ordering of the E_i we have
$$\mathrm{Cox}(X) \simeq k[\xi_1, ..., \xi_6, \xi_\ell, \tau_1, \tau_2, \tau_\ell] / \langle \tau_\ell \xi_\ell^3 \xi_4^2 \xi_5 + \tau_2^2 \xi_2 + \tau_1^3 \xi_1^2 \xi_3 \rangle.$$

We mention some other significant results:

- Batyrev and Popov [**BP04**] showed that the Cox ring of a Del Pezzo surface X of degree $d = 2, 3, 4, 5, 6$ is generated by sections associated with (-1)-curves on X. They show the relations (up to radical) are given by quadratic expressions analogous to the Plücker-type relations above. Furthermore, they conjectured that these quadratic relations actually generate the ideal of all relations.
- The Batyrev-Popov conjecture was proven for Del Pezzo surfaces of degree $d \geq 4$ and cubic surfaces without Eckardt points by Stillman, Testa, and Velasco [**STV07**]. Derenthal [**Der06a**] has also made significant contributions to our understanding of the relations in the Cox ring.
- Laface and Velasco [**LV07**] established the Batyrev-Popov conjecture when $d \geq 2$. Sturmfels and Xu [**SX08**] and Testa, Várilly-Alvarado, and Velasco [**TVAV08**] address Del Pezzo surfaces of degree one.
- For $d = 2, 3, 4, 5$ the affine variety $\mathrm{Spec}(\mathrm{Cox}(X))$ can be related to homogeneous spaces G/P, where G is a simply-connected algebraic group associated to the root system arising from $K_X^\perp \subset N^1(S, \mathbb{Z})$ (cf. §3.3.) Here P is the maximal parabolic subgroup associated to a representation of G naturally connected with the (-1)-curves on X. (This generalized the relation discussed between Grassmannians and Cox rings of degree-five Del Pezzos.) See [**SS07**] and [**Der07**] for details, as well as [**Pop01**] for the case of degree four.
- There are numerous examples of singular Del Pezzo surfaces (like the $\mathbf{E}_6$ cubic surface) whose Cox rings admit a single relation. These are classified in [**Der06b**].

Exercises.

EXERCISE 5.5.1. Let X be the blow-up of $\mathbb{P}^2$ at three *collinear* points. Compute generators and relations for $\mathrm{Cox}(X)$. *Hint:* You can find the answer in [**Has04**].

References

[Abh56] S. Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$*, Ann. of Math. (2) **63** (1956), 491–526. MR 0078017 (17,1134d)

[AGZV85] V. I. Arnol′d, S. M. Guseĭn-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. I*, Monographs in Mathematics, vol. 82, Birkhäuser Boston Inc., Boston, MA, 1985, The classification of critical points, caustics and wave fronts, Translated from the Russian by I. Porteous and M. Reynolds. MR 777682 (86f:58018)

[Art62] M. Artin, *Some numerical criteria for contractability of curves on algebraic surfaces*, Amer. J. Math. **84** (1962), 485–496. MR 0146182 (26 #3704)

[BCHM06] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, 2006, arXiv:math/0610203.

[Bea96] A. Beauville, *Complex algebraic surfaces*, second ed., London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996, Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. MR 1406314 (97e:14045)

[BP04] V. V. Batyrev and O. N. Popov, *The Cox ring of a del Pezzo surface*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), Progr. Math., vol. 226, Birkhäuser Boston, Boston, MA, 2004, pp. 85–103. MR 2029863 (2005h:14091)

[Bri71] E. Brieskorn, *Singular elements of semi-simple algebraic groups*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars, Paris, 1971, pp. 279–284. MR 0437798 (55 #10720)

[Bro06] T. D. Browning, *The density of rational points on a certain singular cubic surface*, J. Number Theory **119** (2006), no. 2, 242–283. MR 2250046 (2007d:14046)

[Cay69] A. Cayley, *A memoir on cubic surfaces*, Philosophical Transactions of the Royal Society of London **159** (1869), 231–326.

[CKM88] H. Clemens, J. Kollár, and S. Mori, *Higher-dimensional complex geometry*, Astérisque (1988), no. 166, 144 pp. (1989). MR 1004926 (90j:14046)

[CL97] T. Christof and A. Löbel, *Polyhedron representation transformation algorithm (PORTA)*, 1997, software available at `http://www.iwr.uni-heidelberg.de/groups/comopt/software/PORTA`.

[Cox95] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. MR 1299003 (95i:14046)

[CT87] J.-L. Colliot-Thélène, *Arithmétique des variétés rationnelles et problèmes birationnels*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986) (Providence, RI), Amer. Math. Soc., 1987, pp. 641–653. MR 934267 (89d:11051)

[CTS87] J.-L. Colliot-Thélène and J.-J. Sansuc, *La descente sur les variétés rationnelles. II*, Duke Math. J. **54** (1987), no. 2, 375–492. MR 899402 (89f:11082)

[Der06a] U. Derenthal, *On the Cox ring of Del Pezzo surfaces*, 2006, arXiv:math/0603111v1.

[Der06b] ———, *Singular Del Pezzo surfaces whose universal torsors are hypersurfaces*, 2006, arXiv:math.AG/0604194.

[Der07] ———, *Universal torsors of del Pezzo surfaces and homogeneous spaces*, Adv. Math. **213** (2007), no. 2, 849–864. MR 2332612 (2008i:14052)

[dlB98] R. de la Bretèche, *Sur le nombre de points de hauteur bornée d'une certaine surface cubique singulière*, Astérisque (1998), no. 251, 51–77, Nombre et répartition de points de hauteur bornée (Paris, 1996). MR 1679839 (2000b:11074)

[dlBBD07] R. de la Bretèche, T. D. Browning, and U. Derenthal, *On Manin's conjecture for a certain singular cubic surface*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 1, 1–50. MR 2332351 (2008e:11038)

[Dur79] A. H. Durfee, *Fifteen characterizations of rational double points and simple critical points*, Enseign. Math. (2) **25** (1979), no. 1-2, 131–163. MR 543555 (80m:14003)

[DV34] P. Du Val, *On isolated singularities of surfaces which do not affect the conditions of adjunction. I.*, Proc. Camb. Philos. Soc. **30** (1934), 453–459 (English).

[EKW04] E. J. Elizondo, K. Kurano, and K. Watanabe, *The total coordinate ring of a normal projective variety*, J. Algebra **276** (2004), no. 2, 625–637. MR 2058459 (2005b:14013)

[FH91] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics. MR 1153249 (93a:20069)

[Fou98] É. Fouvry, *Sur la hauteur des points d'une certaine surface cubique singulière*, Astérisque (1998), no. 251, 31–49, Nombre et répartition de points de hauteur bornée (Paris, 1996). MR 1679838 (2000b:11075)

[GHS03] T. Graber, J. Harris, and J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67 (electronic). MR 1937199 (2003m:14081)

[GJ00] E. Gawrilow and M. Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997) (G. Kalai and G.M. Ziegler, eds.), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, software available at `http://www.math.tu-berlin.de/polymake/`, pp. 43–73. MR 1785292 (2001f:52033)

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)

[Has04] B. Hassett, *Equations of universal torsors and Cox rings*, Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Summer Term 2004, Universitätsdrucke Göttingen, Göttingen, 2004, pp. 135–143. MR 2183138 (2007a:14046)

[HB03] D. R. Heath-Brown, *The density of rational points on Cayley's cubic surface*, Proceedings of the Session in Analytic Number Theory and Diophantine Equations (Bonn), Bonner Math. Schriften, vol. 360, Univ. Bonn, 2003, p. 33. MR 2075628 (2005d:14033)

[HBM99] D. R. Heath-Brown and B. Z. Moroz, *The density of rational points on the cubic surface* $X_0^3 = X_1X_2X_3$, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 3, 385–395. MR 1656797 (2000f:11080)

[HK00] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348, Dedicated to William Fulton on the occasion of his 60th birthday. MR 1786494 (2001i:14059)

[HT04] B. Hassett and Y. Tschinkel, *Universal torsors and Cox rings*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), Progr. Math., vol. 226, Birkhäuser Boston, Boston, MA, 2004, pp. 149–173. MR 2029868 (2005a:14049)

[Isk79] V. A. Iskovskih, *Minimal models of rational surfaces over arbitrary fields*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 19–43, 237, English translation: Math. USSR-Izv. 14 (1980), no. 1, 17–39. MR 525940 (80m:14021)

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959 (2000b:14018)

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 32, Springer-Verlag, Berlin, 1996. MR 1440180 (98c:14001)

[Lan59] S. Lang, *Abelian varieties*, Interscience Tracts in Pure and Applied Mathematics. No. 7, Interscience Publishers, Inc., New York, 1959. MR 0106225 (21 #4959)

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series. MR 2095471 (2005k:14001a)

[LV07] A. Laface and M. Velasco, *Picard-graded Betti numbers and the defining ideals of Cox rings*, 2007, arXiv:0707.3251.

[Man66] Yu. I. Manin, *Rational surfaces over perfect fields*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 30, 55–113, English translation: American Mathematical Society Translations. Series 2, Vol. 84 (1969): Twelve papers on algebra, algebraic geometry and topology. MR 0225780 (37 #1373)

[Man74] ———, *Cubic forms: algebra, geometry, arithmetic*, North-Holland Publishing Co., Amsterdam, 1974, Translated from Russian by M. Hazewinkel, North-Holland Mathematical Library, Vol. 4. MR 0460349 (57 #343)

[Mat02] K. Matsuki, *Introduction to the Mori program*, Universitext, Springer-Verlag, New York, 2002. MR 1875410 (2002m:14011)

[Mil80] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. MR 559531 (81j:14002)

[MT86] Yu. I. Manin and M. A. Tsfasman, *Rational varieties: algebra, geometry, arithmetic*, Uspekhi Mat. Nauk **41** (1986), no. 2(248), 43–94, English translation: Russian Math. Surveys 41 (1986), no. 2, 51–116. MR 842161 (87k:11065)

[Mum95] D. Mumford, *Algebraic geometry. I*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Complex projective varieties, Reprint of the 1976 edition. MR 1344216 (96d:14001)

[Pop01] O. N. Popov, *Del Pezzo surfaces and algebraic groups*, 2001, Diplomarbeit, Universität Tübingen.

[Rei97] M. Reid, *Chapters on algebraic surfaces*, Complex algebraic geometry (Park City, UT, 1993), IAS/Park City Math. Ser., vol. 3, Amer. Math. Soc., Providence, RI, 1997, pp. 3–159. MR 1442522 (98d:14049)

[Sal98] P. Salberger, *Tamagawa measures on universal torsors and points of bounded height on Fano varieties*, Astérisque (1998), no. 251, 91–258, Nombre et répartition de points de hauteur bornée (Paris, 1996). MR 1679841 (2000d:11091)

[SB01] N. I. Shepherd-Barron, *On simple groups and simple singularities*, Israel J. Math. **123** (2001), 179–188. MR 1835294 (2002c:14076)

[SD72] H. P. F. Swinnerton-Dyer, *Rational points on del Pezzo surfaces of degree* 5, Algebraic geometry, Oslo 1970 (Proc. Fifth Nordic Summer School in Math.), Wolters-Noordhoff, Groningen, 1972, pp. 287–290. MR 0376684 (51 #12859)

[Sho96] V. V. Shokurov, 3-*fold log models*, J. Math. Sci. **81** (1996), no. 3, 2667–2699, Algebraic geometry, 4. MR 1420223 (97i:14015)

[Sko93] A. N. Skorobogatov, *On a theorem of Enriques-Swinnerton-Dyer*, Ann. Fac. Sci. Toulouse Math. (6) **2** (1993), no. 3, 429–440. MR 1260765 (95b:14018)

[Sko01] ———, *Torsors and rational points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, Cambridge, 2001. MR 1845760 (2002d:14032)

[SS07] V. V. Serganova and A. N. Skorobogatov, *Del Pezzo surfaces and representation theory*, Algebra Number Theory **1** (2007), no. 4, 393–419. MR 2368955

[STV07] M. Stillman, D. Testa, and M. Velasco, *Gröbner bases, monomial group actions, and the Cox rings of del Pezzo surfaces*, J. Algebra **316** (2007), no. 2, 777–801. MR 2358614 (2008i:14054)

[SX08] B. Sturmfels and Z. Xu, *Sagbi bases of Cox-Nagata rings*, 2008, arXiv:0803.0892.

[TVAV08] D. Testa, A. Várilly-Alvarado, and M. Velasco, *Cox rings of degree one del Pezzo surfaces*, 2008, arXiv:0803.0353.

[Zar58a] O. Zariski, *On Castelnuovo's criterion of rationality* $p_a = P_2 = 0$ *of an algebraic surface*, Illinois J. Math. **2** (1958), 303–315. MR 0099990 (20 #6426)

[Zar58b] ———, *The problem of minimal models in the theory of algebraic surfaces*, Amer. J. Math. **80** (1958), 146–184. MR 0097404 (20 #3873)

[Zar62] ———, *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. of Math. (2) **76** (1962), 560–615. MR 0141668 (25 #5065)

Department of Mathematics, Rice University, Houston, Texas 77005, USA
E-mail address: hassett@math.rice.edu