Rational curves on K3 surfaces

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DRAFT VERSION
comments and corrections are welcome

Introduction

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1 Elements of the geometry of K3 surfaces

1.1 Definitions, key examples, and basic properties

Let k be a field.

Definition 1 A *K3 surface* is a smooth projective geometrically integral surface X/k such that the canonical class $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$.

A compact complex manifold X with these properties is also called a K3 surface.

Example 2 Equations of low degree K3 surfaces can be written quite explicitly:

1. Branched double covers: Assume that $\mathrm{char}(k) \neq 2$ and $G_6 \in k[x_0, x_1, x_2]$ is homogeneous of degree 6. The branched double cover of $\mathbb{P}^2$

$$X = \{[w, x_0, x_1, x_2] : w^2 = G_6(x_0, x_1, x_2)\}$$

is a K3 surface if it is smooth.

2. Quartic surfaces: For $F_4 \in k[x_0, x_1, x_2, x_3]$ homogeneous of degree four consider
$$X = \{F_4(x_0, x_1, x_2, x_3) = 0\} \subset \mathbb{P}^3.$$
Then X is a K3 surface if it is smooth.

3. Sextic surfaces: Consider $F_2, F_3 \in k[x_0, x_1, x_2, x_3, x_4]$ homogeneous of degrees two and three respectively defining a complete intersection surface
$$X = \{F_2 = F_3 = 0\} \subset \mathbb{P}^4.$$
Then X is a K3 surface if it is smooth.

4. Degree eight surfaces: Consider quadratic polynomials $P, Q, R \in k[x_0, \ldots, x_5]$ defining a complete intersection surface
$$X = \{P = Q = R = 0\} \subset \mathbb{P}^5.$$
Again, X is a K3 surface if it is smooth.

Observe that in each case the isomorphism classes of the resulting surfaces depend on 19 parameters. For instance, the Hilbert scheme of quartic surfaces in $\mathbb{P}^3$ can be interpreted as $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^3}(4))) \simeq \mathbb{P}^{34}$, and the projective linear group has dimension 15, so the associated quotient space is 19-dimensional.

From the definition, we can deduce some immediate consequences. Let Ω^1_X and Ω^2_X denote the sheaves of 1-forms and 2-forms on X and $T_X = (\Omega^1_X)^*$ the tangent bundle. Since $K_X = [\Omega^2_X]$ is trivial, there exists an everywhere non-vanishing section
$$\omega \in \Gamma(X, \Omega^2_X).$$
Contraction by ω induces an isomorphism of sheaves
$$\iota_\omega : T_X \xrightarrow{\sim} \Omega^1_X$$
and thus isomorphisms of cohomology groups
$$H^i(X, T_X) \xrightarrow{\sim} H^i(X, \Omega^1_X).$$
Serre duality for K3 surfaces takes the form
$$H^i(X, \mathcal{F}) \simeq H^{2-i}(X, \mathcal{F}^*)^*$$
so in particular
$$H^0(X, \Omega^1_X) \simeq H^2(X, T_X)^*$$
and
$$H^2(X, \mathcal{O}_X) \simeq \Gamma(X, \mathcal{O}_X)^* \simeq k^*.$$
The Noether formula
$$\chi(X, \mathcal{O}_X) = \frac{c_1(T_X)^2 + c_2(T_X)}{12}$$
therefore implies that
$$c_2(T_X) = 24.$$

1.2 Complex geometry

We briefly review the geometric properties of K3 surfaces over $k = \mathbb{C}$.

Computation of Hodge numbers and related cohomology Complex K3 surfaces are Kähler, even when they are not algebraic [Siu83]. Hodge theory [GH78] then gives additional information about the cohomology. We have the decompositions

$$\begin{aligned} H^1(X,\mathbb{C}) &= H^0(X,\Omega^1_X) \oplus H^1(X,\mathcal{O}_X) \\ H^2(X,\mathbb{C}) &= H^0(X,\Omega^2_X) \oplus H^1(X,\Omega^1_X) \oplus H^2(X,\mathcal{O}_X) \\ H^3(X,\mathbb{C}) &= H^1(X,\Omega^2_X) \oplus H^2(X,\Omega^1_X) \end{aligned}$$

where the outer summands are exchanged by complex conjugation. The first and third rows are flipped by Serre duality.

The symmetry under conjugation yields

$$\Gamma(X,\Omega^1_X) = 0,$$

so K3 surfaces admit no vector fields. Furthermore, using the Gauss-Bonnet theorem

$$\chi_{\mathrm{top}}(X) = c_2(T_X) = 24$$

we can compute

$$\dim H^1(X,\Omega^1_X) = 24 - 4 = 20.$$

We can summarize this information in the 'Hodge diamond':

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The fact that $H^1(X,\mathbb{C}) = 0$ implies that $H_1(X,\mathbb{Z})$ is a torsion abelian group; in fact, we have $H_1(X,\mathbb{Z}) = 0$. Otherwise there would exist a non-trivial finite covering $X' \to X$. The canonical class of X' remains trivial, hence X' is either an abelian surface or a K3 surface [Bea78, p. 126]. Since

$$\chi_{\mathrm{top}}(X') = \deg(X'/X)\chi_{\mathrm{top}}(X) = 24\deg(X'/X),$$

we derive a contradiction. Universal coefficient theorems imply then that $H^2(X,\mathbb{Z})$ and $H_2(X,\mathbb{Z})$ are torsion-free.

The Lefschetz Theorem on $(1,1)$-classes describes the Néron-Severi group of X:

$$\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^1(X,\Omega^1_X).$$

Thus we get a bound on its rank

$$\rho(X) := \mathrm{rank}(\mathrm{Pic}(X)) = \mathrm{rank}(H^1(X,\Omega^1_X) \cap H^2(X,\mathbb{Z})) \leq 20.$$

Another key application is to deformation spaces, in the sense of Kodaira and Spencer. Let $\mathrm{Def}(X)$ denote the deformations of X as a complex manifold. The tangent space

$$T_{[X]}\mathrm{Def}(X) \simeq H^1(X, T_X)$$

and the obstruction space is $H^2(X, T_X)$. However, since

$$H^2(X, T_X) \stackrel{\iota_\omega}{\to} H^2(X, \Omega^1_X) = 0$$

we may deduce:

Corollary 3 *If X is a K3 surface then the deformation space* $\mathrm{Def}(X)$ *is smooth of dimension* 20.

However, the general complex manifold arising as a deformation of X has no divisors or non-constant meromorphic functions. If h denotes a divisor, we can consider $\mathrm{Def}(X, h)$, i.e., deformations of X that preserve the divisor h. Its infinitesimal properties are obtained by analyzing cohomology the *Atiyah extension* [Ati57]

$$0 \to \mathcal{O}_X \to \mathcal{E} \to T_X \to 0$$

classified by

$$[h] \in H^1(X, \Omega^1_X) = \mathrm{Ext}^1(T_X, \mathcal{O}_X).$$

We have

$$T_{[X,h]}\mathrm{Def}(X, h) = H^1(X, \mathcal{E}) = \ker(H^1(X, T_X) \stackrel{\cap[h]}{\to} H^2(X, \mathcal{O}_X))$$

and using the contraction $\iota_\omega : T_X \to \Omega^1_X$ we may identify

$$T_{[X,h]}\mathrm{Def}(X, h) \simeq h^\perp \subset H^1(X, \Omega^1_X),$$

i.e., the orthogonal complement of h with respect to the intersection form. The obstruction space

$$H^2(X, \mathcal{E}) \simeq \mathrm{coker}(H^1(X, T_X) \stackrel{\cap[h]}{\longrightarrow} H^2(X, \mathcal{O}_X)) = 0,$$

because $H^2(X, T_X) = 0$.

A *polarized K3 surface* (X, h) consists of a K3 surface and an ample divisor h that is primitive in the Picard group. Its degree is the positive even integer $h \cdot h$, as described in Example 2.

Corollary 4 *If (X, h) is a polarized K3 surface then the deformation space* $\mathrm{Def}(X, h)$ *is smooth of dimension* 19.

Let $\mathcal{K}_g, g \geq 2$ denote the moduli space (stack) of complex polarized K3 surfaces of degree $2g - 2$; our local deformation-theoretic analysis shows this is smooth and connected of dimension 19.

In fact, the Hodge decomposition of a K3 surface completely determines its complex structure:

Theorem 5 (Torelli Theorem) *[PŠŠ71] [LP81] Suppose that X and Y are complex K3 surfaces and there exists an isomorphism*

$$\phi : H^2(X,\mathbb{Z}) \xrightarrow{\sim} H^2(Y,\mathbb{Z})$$

respecting the intersection form

$$\phi(\alpha).\phi(\beta) = \alpha.\beta$$

and the Hodge decomposition

$$(\phi \otimes \mathbb{C})(H^0(X,\Omega^2_X)) = H^0(Y,\Omega^2_Y).$$

Then there exists an isomorphism $X \simeq Y$.

The geometric properties of complex K3 surfaces are thus tightly coupled to its cohomology; most geometric information about X can be read off from $H^2(X)$.

Theorem 6 (Surjectivity of Torelli) *[LP81] [B+85] Let Λ denote the lattice isomorphic to the middle cohomology of a K3 surface under the intersection form. Each Hodge decomposition of $\Lambda \otimes \mathbb{C}$ arises as the complex cohomology of a (not-necessarily algebraic) K3 surface.*

The adjunction formula implies that any *divisor* D on a K3 surface has even self-intersection, i.e., for D effective we have $D \cdot D = 2p_a(D) - 2$. In fact, the intersection form on the full middle cohomology is even [LP81, B+85]. Standard results on the classification of even unimodular indefinite lattices allow us to explicitly compute

$$\Lambda \simeq U^{\oplus 3} \oplus (-E_8)^{\oplus 2},$$

where

$$U \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and E_8 is the positive definite lattice associated to the Lie group of the same name.

The surjectivity of the Torelli map then allows us to create K3 surfaces with very special geometric proprties 'out of thin air':

Example 7 Produce an example of a quartic K3 surface $X \subset \mathbb{P}^3$ with three disjoint lines L_1, L_2, L_3.

By the adjunction formula

$$L_i^2 + K_X.L_i = 2g(L_i) - 2$$

we know that $L_i^2 = -2$. Letting h denote the polarization class, the middle cohomology of the desired K3 surface would have a sublattice

$$M = \begin{array}{c|cccc} & h & L_1 & L_2 & L_3 \\ \hline h & 4 & 1 & 1 & 1 \\ L_1 & 1 & -2 & 0 & 0 \\ L_2 & 1 & 0 & -2 & 0 \\ L_3 & 1 & 0 & 0 & -2 \end{array}.$$

Using basic lattice theory, we can embed

$$M \hookrightarrow \Lambda.$$

Surjectivity of Torelli gives the existence of a K3 surface X with

$$\mathrm{Pic}(X) \supset M \supset \mathbb{Z}h;$$

we can even choose h to be a polarization on X. Global sections of $\mathcal{O}_X(h)$ give an embedding [SD74]

$$|h| : X \hookrightarrow \mathbb{P}^3$$

with image having the desired properties.

2 The Mori-Mukai argument

The following is attributed to Mumford, although it was known to Bogomolov around the same time:

Theorem 8 *[MM83] Every complex projective K3 surface contains at least one rational curve. Furthermore, suppose $(X, h) \in \mathcal{K}_g$ is very general, i.e., in the complement of a countable union of Zariski-closed proper subsets. Then X contains an infinite number of rational curves.*

Results on *density* of rational curves over the standard topology have recently been obtain by Chen and Lewis [CL10].

Idea: Let N be a positive integer. Exhibit a K3 surface $(X_0, h) \in \mathcal{K}_g$ and smooth rational curves $C_i \to X_0$, with $[C_1 \cup C_2] = Nh$ and $[C_i] \not\sim h$. Deform $C_1 \cup C_2$ to an irreducible rational curve in nearby fibers.

Kummer construction (for $N = 1$) We exhibit a K3 surface X_0 containing two smooth rational curves C_1 and C_2 meeting transversally at $g + 1$ points.

Let E_1 and E_2 be elliptic curves admitting an isogeny $E_1 \to E_2$ of degree $2g + 3$ with graph $\Gamma \subset E_1 \times E_2$, and $p \in E_2$ a 2-torsion point. The surface

$$(E_1 \times E_2)/\langle \pm 1 \rangle$$

has 16 simple singularities corresponding to the 2-torsion points; let X_0 denote its minimal resolution, the associated *Kummer surface*. Γ intersects $E_1 \times p$ transversally in $2g + 3$ points, one of which is 2-torsion in $E_1 \times E_2$. Take C_1 and C_2 to be the images of Γ and $E_1 \times p$ in X_0, smooth rational curves meeting transversally in $g + 1$ points. (The intersection of Γ and $E_1 \times p$ at the 2-torsion point does not give an intersection point in the Kummer surface.)

The sublattice of $\mathrm{Pic}(X_0)$ determined by C_1 and C_2 is:

	C_1	C_2
C_1	-2	$g+1$
C_2	$g+1$	-2

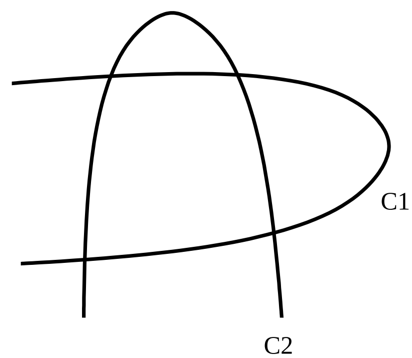

Figure 1: Two smooth rational curves in X_0

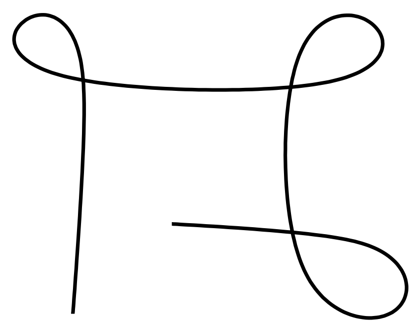

Figure 2: Deformation of $C_1 \cup C_2$ in nearby K3 surface

The divisor $h = C_1 + C_2$ is big and nef and has no higher cohomology (by Kawamata-Viehweg vanishing). It is also primitive: The divisor $(E_1 \times p) + \Gamma$ is primitive because the fibers of the projections onto E_1 and E_2 intersect it with degrees 2 and $2g + 3$, which are relatively prime; thus $C_1 + C_2$ is primitive as well. Deform (X_0, h) to a polarized $(X, h) \in \mathcal{K}_g$

$$\mathcal{X} \to B, \quad \dim(B) = 1,$$

with h ample and indecomposable in the effective monoid. Recall that an effective divisor h is *indecomposable* if we cannot write $h = D_1 + D_2$, for D_1 and D_2 nontrivial effective divisors.

We have $H^i(\mathcal{O}_{X_0}(h)) = 0, i > 0$ thus $C_1 \cup C_2$ is a specialization of curves in the generic fiber and $\mathrm{Def}(C_1 \cup C_2 \subset \mathcal{X})$ is smooth of dimension $g+1$. The locus in $\mathrm{Def}(C_1 \cup C_2 \subset \mathcal{X})$ parametrizing curves with at least ν nodes has dimension $\geq g + 1 - \nu$. When $\nu = g$ the corresponding curves are necessarily rational. The fibers of $\mathcal{X} \to B$ are not uniruled and thus contain a finite number of these curves, so the rational curves deform into nearby fibers.

Conclusion For $(X,h) \in \mathcal{K}_g$ generic, there exist rational curves in the linear series $|h|$. However, rational curves can only specialize to unions of rational curves (with multiplicities), thus *every* K3 surface in $\mathcal{K}_g$ contains at least one rational curve.

In fact, if D is any nonzero effective indecomposable divisor then D contains irreducible rational curves. Our argument above proves this when $D \cdot D > 0$. *Ad hoc* arguments give the remaining cases $D \cdot D = 0, -2$.

Remark 9 Yau-Zaslow [YZ96], Beauville [Bea99], Bryan-Leung [BL00], Xi Chen [Che99, Che02], etc. have beautiful *enumerative results* on the rational curves in $|h|$. Assume all the rational curves in $|h|$ are nodal and irreducible; Xi Chen showed this is the case for generic $(S,h) \in \mathcal{K}_g$. Then the number N_g of rational curves in $|h|$ is governed by the formula

$$\begin{aligned} \sum_{g\geq 0} N_g q^g &= \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{24}} \\ &= 1 + 24q + 324q^2 + 3200q^3 + \cdots \end{aligned}$$

Here N_0 counts the number of rational curves in a given (-2)-class and N_1 the number of singular fibers of a generic elliptic K3 surface.

Generalized construction (for arbitrary N) Let (X_0, h) be a polarized K3 surface of degree $2g-2$ with

$$\mathrm{Pic}(X_0)_{\mathbb{Q}} = \mathbb{Q}C_1 + \mathbb{Q}C_2, \quad \mathrm{Pic}(X_0) = \mathbb{Z}C_1 + \mathbb{Z}h,$$

where C_1 and C_2 smooth rational curves satisfying

	C_1	C_2
C_1	-2	$N^2(g-1)+2$
C_2	$N^2(g-1)+2$	-2

and

$$Nh = C_1 + C_2.$$

The existence of these can be deduced from surjectivity of Torelli, i.e., take a general lattice-polarized K3 as above.

Deform (X_0, h) to a polarized $(X,h) \in \mathcal{K}_g$ as above

$$\mathcal{X} \to B, \quad \dim(B) = 1.$$

$\mathrm{Def}(C_1 \cup C_2 \subset \mathcal{X})$ is smooth of dimension $N^2(g-1)+2$; the locus parametrizing curves with at least $N^2(g-1)+1$ nodes (i.e., the rational curves) has dimension ≥ 1. There are a finite number in each fiber, thus we obtain *irreducible* rational curves in $|Nh|$ for generic K3 surfaces in $\mathcal{K}_g$.

This argument proves that *very general* K3 surfaces admit irreducible rational curves in $|Nh|$ for each $N \in \mathbb{N}$. In particular, they have admit infinitely many rational curves. Conceivably, for special K3 surfaces these might coincide, i.e., so that the infinite number of curves *all* specialize to cycles

$$m_1 C_1 + \ldots + m_r C_r$$

supported in a *finite* collection of curves.

Remark 10 Lee-Leung [LL05] and Li-Wu [LW06, Wu07] have enumerated curves in $|2h|$, analyzing the contributions of reducible and non-reduced rational curves

A. Klemm, D. Maulik, R. Pandharipande, and E. Scheidegger [KMPS10] have recently shown that the BPS count of rational curves in $|Nh|$ (i.e., the number from Gromov-Witten theory, taking the multiple cover formula into account) depends only on the self intersection

$$(Nh) \cdot (Nh) = N^2(2g-2),$$

not on the divisibility N.

Despite the rapid growth of these numbers as $N \to \infty$, we do not know a Gromov-Witten proof that K3 surfaces necessarily admit infinitely many rational curves.

3 Questions on rational curves

3.1 Key conjectures

Let K be algebraically closed field of characteristic zero and X/K projective K3 surface. The following is well-known but hard to trace in the literature:

Question 11 (Main conjecture) There exist an infinite number of rational curves on X.

The following extreme version is more easily attributed:

Conjecture 12 (Bogomolov 1981) Let S be a K3 surface defined over number field F. Then each point $s \in S(F)$ is contained in a rational curve $C \subset S$ defined over $\bar{\mathbb{Q}}$.

While this seems out of reach, significant results of this flavor have been obtained for Kummer surfaces and related varieties over finite fields [BT05b, BT05a].

Conjecture 12 would imply that $\bar{S} = S_{\bar{\mathbb{Q}}}$ has an infinite number of rational curves, because $S(\bar{\mathbb{Q}})$ is Zariski dense in $\bar{S}$. Moreover, we can easily reduce the Main Conjecture to the case of number fields, using the following geometric result:

Proposition 13 *[Blo72, Ran95, Voi92] Let B be a smooth complex variety, $\pi : \mathcal{T} \to B$ a family of K3 surfaces, and $\mathcal{D}$ a divisor on $\mathcal{T}$. Then the set*

$$V := \{b \in B : \text{there exists a rational curve } C \subset \mathcal{T}_b = \pi^{-1}(b) \\ \text{with } [C] = \mathcal{D}_b\}.$$

is open. More precisely, any generic immersion

$$f_b : \mathbb{P}^1 \to \mathcal{T}_b, \quad f_{b*}[\mathbb{P}^1] = \mathcal{D}_b,$$

can be deformed to nearby fibers.

Thus rational curves deform provided their homology classes remain of type $(1,1)$. Note the use of Hodge theory!
Sketch: We indicate the key ideas behind Proposition 13, following [Blo72]; the key concept goes back to Kodaira and Spencer [KS59]: Suppose $Z \subset X$ is a Cartier divisor in a smooth complex projective variety of dimension n, with normal sheaf $\mathcal{N}_{Z/X}$. Adjunction induces

$$\Omega^n_X \to \omega_Z \otimes \mathcal{N}^*_{Z/X},$$

where ω_Z is the dualizing sheaf of Z. Taking cohomology

$$H^{n-2}(X, \Omega^n_X) \to H^{n-2}(Z, \omega_Z \otimes \mathcal{N}^*_{Z/X})$$

and applying Serre dualty we obtain the *semiregularity map*

$$r : H^1(Z, \mathcal{N}_{Z/X}) \to H^2(X, \mathcal{O}_X).$$

The deformation theory of the Hilbert scheme of X at $[Z]$ is governed by $\Gamma(Z, \mathcal{N}_{Z/X})$ (infinitesimal deformations) and $H^1(Z, \mathcal{N}_{Z/X})$ (obstructions). However, Hodge-theoretic arguments imply that all the obstructions that actually arise factor through the kernel of r. Furthermore, if Z is smoothly embedded in a K3 surface X then

$$H^1(Z, \mathcal{N}_{Z/X}) = H^1(Z, \omega_Z) \simeq \mathbb{C},$$

and the map r vanishes for deformations of X such that

$$[Z] \in H^2(X, \mathbb{Z}) \mapsto 0 \in H^2(X, \mathcal{O}_X),$$

i.e., deformations for which $[Z]$ remains algebraic.

Technical refinements of this argument allow one to relax the assumption that Z is smoothly embedded in X. We refer the reader to [Ran95] and [Voi92] for details.

Proof: Main Conjecture/$\bar{\mathbb{Q}}$ $\Rightarrow$ Main Conjecture/K

Suppose there exists a K3 surface T over K with a finite number of rational curves. We may assume that K is the function field of some variety $B/\bar{\mathbb{Q}}$. 'Spread out' to get some family $\mathcal{T} \to B$, and choose a point $b \in B(\bar{\mathbb{Q}})$ such that the fiber $\mathcal{T}_b$ has general Picard group

$$\mathrm{Pic}(\mathcal{T}_b) = \mathrm{Pic}(T_{\overline{K}}).$$

Since $\mathcal{T}_b$ has an infinite number of rational curves, the same holds for T.

3.2 Rational curves on special K3 surfaces

Bogomolov and Tschinkel [BT00] prove the following result: Let S be a complex projective K3 surface admitting either

1. a non-isotrivial elliptic fibration; or
2. an infinite group of automorphisms.

Then S admits an infinite number of rational curves.

The non-isotriviality assumption is much stronger than necessary; the argument of [BT00] actually goes through for all but the most degenerate elliptic K3 surfaces, which turn out to be either Kummer elliptic surfaces or to have Néron-Severi group of rank twenty. These can be dealt with in an *ad hoc* manner (see [BHT09]), thus *every* elliptic K3 surface admits infinitely many rational rational curves.

Here we focus on the case $|\mathrm{Aut}(S)| = \infty$; we sketch the existence of infinitely many rational curves in this context.

Consider the monoid of effective divisors on S. Each nonzero indecomposable element D contains rational curves by the Mori-Mukai argument (when $D \cdot D > 0$) or direct analysis (when $D \cdot D = 0, -2$). It suffices to show there must be infinitely many such elements. This is clear, because otherwise the image of

$$\mathrm{Aut}(S) \to \mathrm{Aut}(\mathrm{Pic}(S))$$

would be finite, so the kernel would have to be infinite, which is impossible.

Example 14 Let Λ be a rank-two even lattice of signature $(1, 1)$ that does not represent -2 or 0, and (S, f) polarized K3 surface with $\mathrm{Pic}(S) = \Lambda$.

The positive cone

$$\mathcal{C}_S := \{D \in \Lambda : D \cdot D > 0, D \cdot f > 0\}$$

equals the ample cone and is bounded by irrational lines. The existence of infinitely-many indecomposable effective divisors implies infinitely-many rational curves in S.

Remark 15 K3 surfaces with $\mathrm{Aut}(S)$ infinite or admitting an elliptic fibration have

$$\mathrm{rank}(\mathrm{Pic}(S)) \geq 2.$$

Thus these techniques do not apply to 'most' K3 surfaces. Indeed, I know no example before 2009 of a K3 surface $S/\bar{\mathbb{Q}}$ with $\mathrm{Pic}(S) = \mathbb{Z}$ admitting infinitely many rational curves. This is entirely consistent with the possibility that the Mori-Mukai argument might break down over a countable union of subvarieties in $\mathcal{K}_g$.

4 K3 surfaces in positive characteristic

4.1 What goes wrong in characteristic p?

In characteristic zero, K3 surfaces are never unirational (or even uniruled). Indeed, if there were a dominant map $\mathbb{P}^2 \dashrightarrow X$ then we could resolve indeterminacy to a morphism from a smooth projective rational surface $\phi : S \to X$.

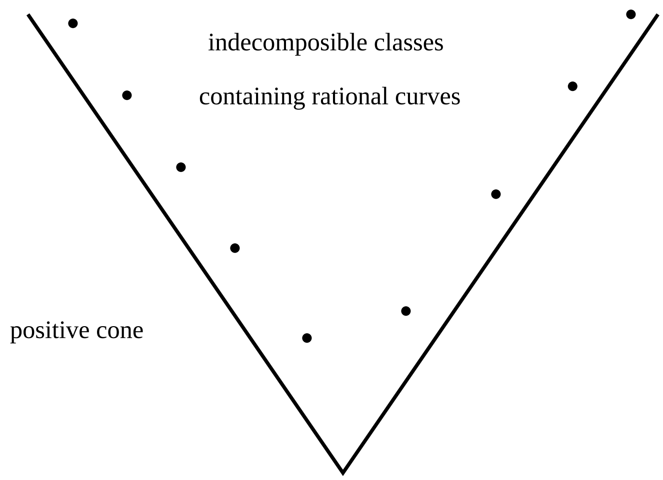

Figure 3: 'Typical' rank-two K3 surfaces have infinitely many curve classes containing rational curves

Since the derivative of ϕ is non-vanishing at the generic point, $\phi^*\omega$ would be a nonzero twoform on S.

In characteristic p the derivative of a map *can* vanish everywhere. This happens when the associated extension of function fields

$$\begin{array}{c} k(S) \\ | \\ k(X) \end{array}$$

has inseparability.

Example 16 Consider the Fermat hypersurface [Tat65, Shi74]

$$X = \{x_1^{q+1} - x_2^{q+1} = x_3^{q+1} - x_4^{q+1}\}$$

over a field of characteristic $p \neq 2$, where $q = p^e$. (Our main interest is the Fermat quartic K3 surface over a field of characteristic 3.) This is unirational.

Setting

$$x_1 = y_1 + y_2,\ x_2 = y_1 - y_2,\ x_3 = y_3 + y_4,\ x_4 = y_3 - y_4$$

we can rewrite our equation as

$$y_1 y_2 (y_1^{q-1} + y_2^{q-1}) = y_3 y_4 (y_3^{q-1} + y_4^{q-1}).$$

Dehomogenize by setting $y_4 = 1$ and write

$$y_2 = y_1 u, \quad y_3 = uv$$

so we obtain

$$y_1^{q+1}(1+u^{q-1}) = v(u^{q-1}v^{q-1}+1).$$

Take the *inseparable* field extension $t = y_1^{1/q}$ so we have

$$t^{(q+1)q}(1+u^{q-1}) = v(u^{q-1}v^{q-1}+1)$$

or

$$u^{q-1}(t^{q+1}-v)^q = v - t^{q(q+1)}.$$

Setting $s = u(t^{q+1}-v)$ we get

$$s^{q-1}(t^{q+1}-v) = v - t^{q(q+1)}$$

whence

$$v = t^{q+1}(s^{q-1}+t^{q^2-1})/(s^{q-1}+1).$$

Thus the function field $k(X)$ admits an extension equal to $k(t,s)$ and so there is a degree q dominant map

$$\mathbb{P}^2 \dashrightarrow X.$$

Example 17 The surface

$$X = \{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\}$$

is unirational over any field of characteristic p with $p \equiv 3 \pmod 4$.

Example 18 (Branched double covers in characteristic two) Let k be an algebraically closed field of characteristic two. Fix a generic homogeneous sextic polynomial

$$G_6 \in k[x_0, x_1, x_2]_6$$

and consider the branched double cover

$$X = \{w^2 = G_6(x_0, x_1, x_2)\}.$$

We regard this as a hypersurface in the weighted projective space with coordinates w, x_0, x_1, x_2.

The surface X is singular as presented. Indeed, passing to an affine open subset (say $z \neq 0$) we get the affine surface

$$w^2 = f(x,y), \quad f(x,y) = G_6(x,y,1),$$

which is singular at solutions to the equations

$$\partial f/\partial x = \partial f/\partial y = 0. \tag{4.1}$$

Taking the partial with respect to w automatically gives zero; and for each solution to (4.1) we can solve for w.

Expand

$$f(x,y) = a_0x^6 + a_1x^5y + a_2x^4y^2 + a_3x^3y^3 + a_4x^2y^4 + a_5xy^5 + a_6y^6 + \cdots$$

so that

$$\partial f/\partial x = a_1x^4y + a_3x^2y^3 + a_5y^5 + \cdots, \quad \partial f/\partial y = a_1x^5 + a_3x^3y^2 + a_5xy^4 + \cdots.$$

Thus our partials have four common zeros along the line at infinity; we therefore expect 21 solutions in our affine open subset. One can show, for generic choices of G_6, that the system (4.1) admits 21 distinct solutions, which correspond to ordinary double points on X.

Let $\widetilde{X} \to X$ denote the minimal desingularization; it is a K3 surface. Its Néron-Severi group has rank at least twenty-two, i.e., the 21 exceptional curves and the pull-back of the polarization from $\mathbb{P}^2$.

Finally, X is unirational, as the extension $k(X)/k(\mathbb{P}^2)$ can be embedded in the extension $k(\mathbb{P}^2)/k(\mathbb{P}^2)$ associated with the Frobenius morphism $\mathbb{P}^2 \to \mathbb{P}^2$. More concretely, we have

$$k(x,y) \subset k(x,y,\sqrt{f(x,y)}) \subset k(\sqrt{x},\sqrt{y})$$

because $\sqrt{f(x,y)} = f(\sqrt{x},\sqrt{y})$.

We refer the reader to [Shi04] for more discussion of this example.

Shioda has shown that the Néron-Severi groups of these kinds of surfaces behave quite strangely. Recall that the Néron-Severi group $\mathrm{NS}(X)$ of a smooth projective surface X over an algebraically closed field is the Picard group modulo 'algebraic equivalence': $D_1 \equiv D_2$ if there is a connected family of divisors containing D_1 and D_2.

Proposition 19 *Let X be a smooth projective unirational (or even uniruled) surface over a field of characteristic p. Suppose that X arises as the reduction mod p of a surface S defined over a field of characteristic zero. Then we have*

$$\rho(X) = \mathrm{rank}(\mathrm{NS}(X)) = \mathrm{rank}(H^2(S,\mathbb{Z})).$$

Thus our Fermat quartic surface

$$X = \{x_1^4 + x_2^4 + x_3^4 + x_4^4 = 0\}$$

has $\rho(X) = 22$!

K3 surfaces with $\rho = 22$ are said to be *supersingular* in the sense of Shioda [RS81, §5]. Artin has a different definition of supersingularity [RS81, §9, Prop. 2] [Art74], expressed in terms of the height of a K3 surface. This is computed from its formal Brauer group, which is associated to the system

$$\varprojlim_{A/k \text{ Artinian}} \mathrm{Br}(X \times_k A).$$

This is implied by Shioda's definition; the converse remains open. Artin [Art74, p. 552] and Shioda [Shi77, Ques. 11] have conjectured that supersingular K3 surfaces (in either sense) are all unirational.

4.2 What goes right in characteristic p?

Assume now that k is a field with $\mathrm{char}(k) = p$.

Theorem 20 *[RŠ78] [LN80] [Nyg79] K3 surfaces have no vector fields, i.e., if X is a K3 surface then $\Gamma(X, T_X) = 0$.*

Unfortunately, there appears to be no really simple proof of this important theorem. All the proofs I have seen start the same way: K3 surface with vector fields are unirational.

Given this, we can recover most of the deformation-theoretic results that make complex K3 surfaces so attractive:

Theorem 21 *[Del81] Suppose k is algebraically closed with $\mathrm{char}(k) = p$ and let X be a K3 surface defined over k. Then the formal deformation space $\mathrm{Def}(X)$ is smooth of dimension 20 over k. If h is a primitive polarization of X then $\mathrm{Def}(X, h)$ is of dimension 19 and arises from an algebraic scheme over k.*

The argument uses the Chern-class formalism described above and formal deformation theory of Schlessinger [Sch68]: For each Artinian local k-algebra A consider flat proper morphisms

$$\mathcal{X} \to \mathrm{Spec}\, A$$

with closed fiber $\mathcal{X}_0 = X$. The formal deformation space is obtained by taking the inverse limit of all such families over all Artinian k-algebras.

Even more remarkably, we can use the vanishing of vector fields to show that *every* K3 surface in characteristic p is obtained as the reduction mod p of a K3 surface in characteristic zero!

For each algebraically closed (or perfect) field k with $\mathrm{char}(k) = p$ let $W(k)$ denote the *Witt-vectors* with components in k. For example, if $k = \mathbb{F}_p$ then $W(k) = \mathbb{Z}_p$, the p-adic integers. When $k = \bar{\mathbb{F}}_p$ then $W(\bar{\mathbb{F}}_p)$ is the unique complete unramified extension of $\mathbb{Z}_p$ with algebraically closed residue class field. This can be obtained by adjoining all n-th roots of unity with $(n, p) = 1$.

Theorem 22 (Deligne's Lifting Theorem [Del81]) *Let (X, L) be a K3 surface over an algebraically closed field k of characteristic p. Consider the deformations spaces over the Witt-vectors*

$$\mathrm{Def}(X/W(k)), \mathrm{Def}((X, L)/W(k)) \to \mathrm{Spec}\, W(k)$$

i.e., the space associated to taking flat proper morphisms

$$\mathcal{X} \to \mathrm{Spec}\, A, \;\; \mathcal{X}_0 = X$$

where A is an Artinian module over $W(k)$. Then $\mathrm{Def}(X/W(k))$ is smooth over $\mathrm{Spec}\, W(k)$ and $\mathrm{Def}((X, L)/W(k))$ is flat over $\mathrm{Spec}\, W(k)$.

This uses the Schlessinger formalism for Artinian $W(k)$-algebras rather than k-algebras. For the finite field $k = \mathbb{F}_p$, this is the difference between considering schemes over $\operatorname{Spec} \mathbb{F}_p[t]/\langle t^n \rangle$ versus schemes over $\operatorname{Spec} \mathbb{Z}/\langle p^n \rangle$. In both contexts, the obstructions to lifting to order $n+1$ lie in a $\mathbb{F}_p$-vector space. Another key tool is crystalline cohomology, which allows us to imitate Hodge-theoretic techniques in characteristic p.

Corollary 23 *Suppose X is a K3 surface over an algebraically closed field k with $\operatorname{char}(k) = p$. Then there exists a finite extension T of $W(k)$ and a flat projective scheme*

$$\mathcal{X} \to \operatorname{Spec} T$$

such that X is isomorphic to the fiber over the closed point.

Remark 24 The question of whether we can lift X to a flat projective scheme over the Witt vectors

$$\mathcal{X} \to \operatorname{Spec} W(k)$$

is quite subtle, especially in characteristic two. We refer the interested reader to Ogus' work [Ogu79] for details.

Let S be the surface appearing as the generic fiber of $\mathcal{X} \to \operatorname{Spec} T$, which is defined over a field of characteristic zero. We know that

- S is smooth, because X is smooth;
- $K_S = 0$, because $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$ and canonical sheaf commutes with base extension;
- $H^1(S, \mathcal{O}_S) = 0$ by semicontinuity.

Thus S is a complex K3 surface and we can apply everything we know about its cohomology. Using the comparison theorem (relating complex and étale cohomology) and smooth basechange (relating the cohomology of the generic and special fibers) we find

$$H^2_{et}(X, \mu_{\ell^n}) = H^2(S, \mu_{\ell^n}) \simeq H^2(S, \mathbb{Z}/\ell^n\mathbb{Z}),$$

for each prime ℓ different from p and $n \in \mathbb{N}$. Furthermore, this is compatible with cup products.

Corollary 25 *Let X be a K3 surface over an algebraically closed field and ℓ a prime distinct from the characteristic. The middle ℓ-adic cohomology of X*

$$H^2(X, \mathbb{Z}_\ell(1)) = \varprojlim_{n \in \mathbb{N}} H^2_{et}(X, \mu_{\ell^n})$$

is given by the $\mathbb{Z}_\ell$-lattice

$$\Lambda \simeq U^{\oplus 3} \oplus (-E_8)^{\oplus 2}.$$

The odd-dimensional cohomologies of X vanish.

Remark 26 Each smooth projective surface X admits a $\mathbb{Z}$-valued intersection form on its Picard group. The induced nondegenerate form on the Néron-Severi group has signature $(1, 21)$ by the Hodge index theorem.

Suppose that k is algebraically closed and X/k is a unirational K3 surface with $\rho(X) = 22$, i.e., all the middle cohomology is algebraic. How can the signature of Λ be $(3, 19)$? The point is that

$$\mathrm{Pic}(X) \to H^2(X, \mathbb{Z}_\ell(1)) \simeq \Lambda \otimes \mathbb{Z}_\ell$$

as a lattice in an ℓ*-adic* quadratic form. It only makes sense to compare the ℓ-adic invariants, not the real invariants!

4.3 Frobenius and the Weil conjectures

Let X be a K3 surface defined over a finite field $\mathbb{F}_q$ and $\bar{X}$ the basechange to the algebraic closure $\bar{\mathbb{F}}_q$. Consider Frobenius $x \mapsto x^q$ which induces a morphism

$$\begin{array}{ccccc} X & & \stackrel{\mathrm{Fr}}{\to} & & X \\ & \searrow & & \swarrow & \\ & & \mathrm{Spec}\, \mathbb{F}_q & & \end{array}$$

whose fixed-points are precisely $X(\mathbb{F}_q)$. We get an induced action on the ℓ-adic cohomology groups

$$\mathrm{Fr}^* : H^i(\bar{X}, \mathbb{Z}_\ell) \to H^i(\bar{X}, \mathbb{Z}_\ell)$$

where ℓ is a prime not dividing q. We have the Lefschetz trace formula (due to Grothendieck!)

$$\begin{aligned} \# X(\mathbb{F}_q) &= \sum_{i=0}^4 (-1)^i \mathrm{tr}\mathrm{Fr}^* | H^i(\bar{X}, \mathbb{Q}_\ell) \\ &= 1 + \mathrm{tr}\mathrm{Fr}^* | H^2(\bar{X}, \mathbb{Q}_\ell) + q^2. \end{aligned}$$

Here we are using the vanishing of the odd-dimensional cohomology groups. The fundamental class $[X]$ and the point class give the 1 and q^2 contributions.

The Weil conjectures were proven for K3 surfaces before they were established in general:

Theorem 27 *[Del72] Let X be a K3 surface defined over a finite field with Frobenius endomorphism* Fr*. The characteristic polynomial*

$$p_{X,\mathrm{Fr}^*}(t) = \det(tI - \mathrm{Fr}^*) | H^2(\bar{X}, \mathbb{Q}_\ell)$$

is integral and its complex roots α satisfy $|\alpha| = q$.

The proof uses Clifford algebras: The middle integral cohomology of a polarized complex K3 surface (S, L) carries an integral quadratic form $\langle , \rangle$. Consider the orthogonal complement

$$V_{\mathbb{Z}} = L^{\perp} \subset H^2(S, \mathbb{Z})$$

which also inherits a quadratic form of signature $(2,19)$. Set $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$ and let $C(V, \langle,\rangle)$ denote the *Clifford algebra* of this quadratic form. This is a 2^{21}-dimensional associative $\mathbb{Q}$-algebra, admitting a linear injection $i : V \hookrightarrow C(V, \langle,\rangle)$, and determined by the following universal property: Given another $\mathbb{Q}$-algebra and a linear map $f : V \to A$ with $f(v)^2 = \langle v, v\rangle$, there exists a unique $\mathbb{Q}$-algebra homomorphism $g : C(Q, \langle,\rangle) \to A$ with $f = g \circ i$. If $e_1, \ldots, e_{21}$ is a basis for V then the Clifford algebra has basis $e_{j_1} \ldots e_{j_r}, j_1 < j_2 < \ldots < j_r$. We can decompose this into parts with even and odd degrees

$$C(V, \langle,\rangle) = C^+(V, \langle,\rangle) \oplus C^-(V, \langle,\rangle)$$

each of dimension 2^{20}.

Here is the first marvelous insight of Deligne:

> The Hodge decomposition on S
>
> $$H^2(S, \mathbb{C}) = H^0(S, \Omega^2_S) \oplus H^1(S, \Omega^1_S) \oplus H^2(S, \mathcal{O}_S)$$
>
> induces a Hodge decomposition on
>
> $$C^+(V, \langle,\rangle) \otimes \mathbb{C} = H^{01} \oplus H^{01}.$$
>
> Moreover, there is an isogeny-class of abelian varieties A of dimension 2^{19} such that
>
> $$H^1(A, \mathbb{C}) \simeq C^+(V, \langle,\rangle) \otimes \mathbb{C}$$
>
> as Hodge structures. Moreover, these abelian varieties come with a huge number of endomorphisms, e.g., the elements of the Clifford algebra.

This is the *Kuga-Satake construction* [KS67]. See [vG00] for a user-friendly introduction to these Hodge-theoretic techniques.

The second step is to do this over the whole moduli space of K3 surfaces in such a way that everything is defined over a number field. Suppose that $\mathcal{S} \to B$ is a single family (say over a 19-dimensional base) containing every complex K3 surface of degree $L \cdot L$. Then after finite basechange $B' \to B$ we want a family of 2^{19}-dimensional abelian varieties $\mathcal{A} \to B'$ such that, fiber-by-fiber, they are related to $\mathcal{S}' \times_B B' \to B'$ via the Kuga-Satake construction.

Remark 28 Note that we have no general *algebro-geometric* connection between S and A, and little explicit information about the field of definition of A. We only have a connection between their cohomologies. The Hodge conjecture predicts [vG00, 10.2] the existence of a correspondence

$$\begin{array}{ccc} Z & \to & A \times A \\ \downarrow & & \\ S & & \end{array}$$

inducing the cohomological connection. However, these are known to exist only in special cases, such as Kummer surfaces [Voi96].

Finally, the truly miraculous part: The universal construction relating $\mathcal{S} \times_B B'$ and $\mathcal{A}$ can be reduced mod p, so as to allow the Weil conjectures for $X = S \pmod p$ to be deduced from the Weil conjectures for the reductions of the fibers of $\mathcal{A} \to B$. □

4.4 On the action of Frobenius

Again, let X be a K3 surface over a finite field $\mathbb{F}_q$ with polarization L. What can we say *structurally* about the action of Fr^* on $H^2(\bar{X}, \mathbb{Q}_\ell)$?

First, it will make our analysis easier if we replace

$$H^2(\bar{X}, \mathbb{Z}_\ell) = \varprojlim_{n \in \mathbb{N}} H^2_{et}(\bar{X}, \mathbb{Z}/\ell^n\mathbb{Z})$$

with the twist

$$H^2(\bar{X}, \mathbb{Z}_\ell(1)) = \varprojlim_{n \in \mathbb{N}} H^2_{et}(\bar{X}, \mu_{\ell^n}).$$

The reason is that the Kummer sequence

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \stackrel{\times \ell^n}{\to} \mathbb{G}_m \to 0$$

gives connecting homomorphisms

$$\mathrm{Pic}(\bar{X}) = H^1(\bar{X}, \mathbb{G}_m) \to H^2_{et}(\bar{X}, \mu_{\ell^n})$$

inducing the cycle class map

$$\mathrm{Pic}(\bar{X}) \to H^2(\bar{X}, \mathbb{Z}_\ell(1))$$

and

$$\mathrm{Pic}(X) \to H^2(\bar{X}, \mathbb{Z}_\ell(1))^\Gamma$$

where

$$\Gamma = \langle \mathrm{Fr} \rangle = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q).$$

Since Frobenius acts via multiplication by q on μ_{ℓ^n}, the action of Fr^* on our original cohomology group and its twist differ by a factor of q.

Proposition 29 *If X is a K3 surface over $\mathbb{F}_q$ then*

$$\rho(X) \leq \dim\{\xi \in H^2(\bar{X}, \mathbb{Q}_\ell) : \mathrm{Fr}^*\xi = q\xi\},$$

i.e., the multiplicity of q as an eigenvalue of $p_{X,\mathrm{Fr}^*}(t)$.

Applying a similar analysis to the cup-product operation yields:

Proposition 30 Fr^* *respects the intersection form, i.e.,*

$$\langle \mathrm{Fr}^*\xi_1, \mathrm{Fr}^*\xi_2 \rangle = q^2 \langle \xi_1, \xi_2 \rangle .$$

If α is a root of $p_{X,\mathrm{Fr}^*}(t)$ then q^2/α is also a root.

Since $\langle , \rangle$ is nondegenerate, after passing to a field extension and changing coordinates we may assume the corresponding symmetric matrix is the identity. Let Φ denote the matrix of Fr^* in these coordinates, which satisfies

$$\Phi^t \Phi = q^2 I.$$

It follows that

$$\begin{aligned} p_{X,\mathrm{Fr}^*}(t) &= \det(tI - \Phi) \\ &= (-t)^{\deg(p)} \det(\Phi) \det(t^{-1}I - \Phi^{-1}) \\ &= (-t)^{\deg(p)} \det(\Phi) \det(t^{-1}I - q^{-2}\Phi^t) \\ &= (-t/q^2)^{\deg(p)} \det(\Phi) \det(q^2 t^{-1} I - \Phi^t) \\ &= (-t/q^2)^{\deg(p)} \det(\Phi) p_{X,\mathrm{Fr}^*}(q^2/t). \end{aligned}$$

Corollary 31 *Suppose X is a polarized K3 surface. Then the distinguished subspaces*

$$\{\xi \in H^2(\bar{X}, \mathbb{Q}_\ell) : \mathrm{Fr}^*\xi = \pm q\xi\}, \quad \{\xi \in H^2(\bar{X}, \mathbb{Q}_\ell(1)) : \mathrm{Fr}^*\xi = \pm \xi\}$$

are even dimensional.

4.5 Tate conjecture for K3 surfaces

The decomposition of the cohomology under Frobenius should strongly reflect the geometry:

Conjecture 32 Let X be a K3 surface over a finite field. Then Galois-invariant cycles come from divisors, i.e.,

$$\mathrm{Pic}(X) \otimes \mathbb{Q}_\ell \to H^2(\bar{X}, \mathbb{Q}_\ell(1))^\Gamma$$

is surjective.

This is a special case of the Tate conjecture [Tat65], a Galois-theoretic analog of the Lefschetz $(1,1)$ theorem on the Néron-Severi group.

For most K3 surfaces, the conjecture is known to be true: [NO85]

Theorem 33 *The Tate conjecture holds for K3 surfaces over finite fields of characteristic ≥ 5 that are not supersingular (in the sense of Artin).*

The following consequence is well-known to experts (and was ascribed to Swinnerton-Dyer in [Art74, p. 544]) but we do not know a ready reference:

Corollary 34 *The rank of the Néron-Severi group of a K3 surface over the algebraic closure of a finite field is always even, provided the characteristic is at least five and the surface is not supersingular, in the sense of Artin.*

Of course, the Artin-supersingular K3 surfaces are expected to have rank 22.

An especially nice special case of the Tate conjecture is K3 surfaces with elliptic fibrations [ASD73] $X \to \mathbb{P}^1$. Here the Tate conjecture is related to proving finiteness of the Tate-Shafarevich group of the associated Jacobian fibration $J(X) \to \mathbb{P}^1$.

Remark 35 For an analysis of the eigenvalues of Frobenius on the *non-algebraic* cohomology of a K3 surface, we refer the reader to [Zar93].

4.6 Reduction results

Let S be a projective K3 surface over a number field F with $\bar{S} = S_{\bar{\mathbb{Q}}}$. Let $\mathfrak{o}_F$ be the ring of integers with spectrum $B = \mathrm{Spec}\,(\mathfrak{o}_F)$ and $\pi : \mathcal{S} \to B$ a flat projective model for S. Fix $\mathfrak{p} \in B$ a prime of good reduction for S, i.e., $\mathcal{S}_{\mathfrak{p}} = \pi^{-1}(\mathfrak{p})$ is a smooth K3 surface over a *finite* field. Let k be a finite field with algebraic closure $\overline{k}$, $p = \mathrm{char}(k)$ and X/k K3 surface and $\bar{X} = X_{\bar{k}}$.

Consider the Frobenius endomorphism on $\bar{X}$ acting on ℓ-adic cohomology

$$\mathrm{Fr}^* : H^2(\bar{X}, \mathbb{Q}_\ell) \to H^2(\bar{X}, \mathbb{Q}_\ell);$$

X is *ordinary* if $p \nmid \mathrm{Trace}(\mathrm{Fr})$. This can also be expressed in terms of the formal Brauer group, i.e., it should have height one [RS81, §9]; thus ordinary K3 surfaces are not supersingular.

Joshi-Rajan [JR01] and Bogomolov-Zarhin [BZ09] have shown

$$\{\mathfrak{p} \in B : \mathcal{S}_{\mathfrak{p}} \text{ ordinary }\}$$

has positive Dirichlet density, even density one after a finite extension of the ground field.

5 Evaluating the Picard group in practice

In the analysis of complex K3 surfaces S, we used the Hodge decomposition

$$H^2(S, \mathbb{C}) = H^0(S, \Omega^2_S) \oplus H^1(S, \Omega^1_S) \oplus H^2(S, \mathcal{O}_S)$$

and the description

$$\mathrm{NS}(S) = H^2(S, \mathbb{Z}) \cap H^1(S, \Omega^1_S)$$

Our discussion may have left the impression that these objects are well-known, but much remains mysterious:

Problem 36 Let (S, h) be a polarized K3 surface over a number field k. Give an algorithm to compute

$$\rho(\bar{S}) = \mathrm{rank}(\mathrm{NS}(\bar{S})).$$

In particular, is there an effective test for deciding whether $\mathrm{Pic}(\bar{S}) = \mathbb{Z}h$?

There is one obvious constraint:

Proposition 37 *Suppose S is a K3 surface over $\mathbb{Q}$, p a prime, and assume that the reduction $X = S \pmod p$ is a smooth K3 surface. Then there is a restriction map*

$$\mathrm{Pic}(\bar{S}) \to \mathrm{Pic}(\bar{X})$$

compatible with the isomorphism on cohomology groups

$$H^2(\bar{S}, \mathbb{Z}_\ell(1)) \to H^2(\bar{X}, \mathbb{Z}_\ell(1))$$

arising from smooth base change.

Corollary 38 *If some reduction $X = S \pmod p$ has Néron-Severi group of rank $2m$ then*

$$\rho(\bar{S}) \leq 2m.$$

As we have seen, the Tate conjecture precludes using reduction mod p to prove that $\rho(\bar{S}) = 1$! Terasoma [Ter85], Ellenberg [Ell04], van Luijk [vL07], and Elsenhans-Jahnel [EJ08a, EJ08b, EJ09a, EJ09b] have demonstrated that we often can show this be reducing modulo multiple primes, and then comparing the various restrictions

$$\mathrm{Pic}(\bar{S}) \to \mathrm{Pic}(\bar{S} \pmod{p_i}).$$

Example 39 (van Luijk's example) This an a quartic K3 surface $S \subset \mathbb{P}^3$ over $\mathbb{Q}$ with

$$\mathrm{NS}(\bar{S} \pmod 2) = \begin{array}{c|cc} & h & C \\ \hline h & 4 & 2 \\ C & 2 & -2 \end{array}$$

and

$$\mathrm{NS}(\bar{S} \pmod 3) = \begin{array}{c|cc} & h & L \\ \hline h & 4 & 1 \\ L & 1 & -2 \end{array}.$$

In geometric terms, the reduction mod 2 contains a conic C and the reduction mod 3 contains a line L. The first lattice has discriminant -12 and the second lattice has discriminant -9, so these cannot both be specializations of rank-two sublattice of $\mathrm{NS}(\bar{S})$.

Thus the key questions is: How many primes must we check to determine the rank of $\mathrm{NS}(\bar{S})$? Can this be bounded in terms of arithmetic invariants of S?

There is a variant of this approach that uses just one prime [EJ09b]: Suppose we can compute $\mathrm{Pic}(\bar{S} \pmod p) \simeq \mathbb{Z}h \oplus \mathbb{Z}[C]$, for a suitable curve C. Often, if $[C]$ is in the image of the specialization

$$\mathrm{Pic}(\bar{S}) \to \mathrm{Pic}(\bar{S} \pmod p))$$

then there must exist a curve $C' \subset \bar{S}$ with $[C'] \mapsto [C]$ or even $C' \rightsquigarrow C$. However, there are algorithms for determining whether $\bar{S}$ contains a curve of prescribed degree. Barring such a curve, we may conclude that $\mathrm{Pic}(\bar{S}) = \mathbb{Z}h$.

6 Mori-Mukai in mixed characteristic

Our main goal is the following result

Theorem 40 (Bogomolov, H-, Tschinkel) *Let S be a K3 surface defined over a number field, with* $\mathrm{Pic}(\bar{S}) = \mathbb{Z}h$ *where* $h \cdot h = 2$. *Then* $\overline{S}$ *admits infinitely many rational curves.*

In other words, S is a double cover of $\mathbb{P}^2$ branched over a very general plane sextic curve. Here the associated involution greatly simplifies our analysis; this will be apparent in the part of our proof addressing multiplicities of components.

This result is an application of general techniques that should have a wider range of applications. The first aspect is lifting curves from characteristic p to characteristic zero:

Problem 41 Let S/F be a K3 surface defined over a number field. Designate $B = \mathrm{Spec}\,(\mathfrak{o}_F)$ and $\mathcal{S} \to B$ a flat projective model for S. Assume $\mathfrak{p} \in B$ is a prime such that $\mathcal{S}_{\mathfrak{p}}$ is a smooth K3 surface over $k = \mathbb{F}_q = \mathfrak{o}_F/\mathfrak{p}$.

Given distinct rational curves

$$C_1, \dots, C_r \subset \mathcal{S}_{\mathfrak{p}},$$

does their union lift to a rational curve in S?

Some *necessary conditions* should be apparent:

- The union $C_1 \cup \cdots \cup C_r$ should 'remain algebraic' i.e., there exists a $D \in \mathrm{Pic}(\bar{S})$ such that
$$D \mapsto C_1 + \cdots + C_r$$
under the specialization
$$\mathrm{Pic}(\bar{S}) \to \mathrm{Pic}(\mathcal{S}_{\mathfrak{p}}).$$
- The K3 surface $\mathcal{S}_{\mathfrak{p}}$ should not have 'too many' rational curves, e.g., it should not be uniruled.

These are reflected in the following result:

Proposition 42 *Let S/F be a K3 surface over a number field. Assume there exists a prime* $\mathfrak{p} \in \mathrm{Spec}\,(\mathfrak{o}_F)$ *satisfying*

1. *$\mathcal{S}_{\mathfrak{p}}$ is smooth and non-supersingular;*

2. *there exist distinct rational curves* $C_1, \ldots, C_r \subset \mathcal{S}_{\mathfrak{p}}$ *and an ample* $D \in \mathrm{Pic}(\bar{S})$ *such that*

$$D \mapsto C_1 + \cdots + C_r$$

under the specialization;

3. *no subset of the rational curves has this property, i.e., for each* $J \neq \emptyset \subsetneq \{1, \ldots, r\}$ *there exists no* $D' \in \mathrm{Pic}(\bar{S})$ *such that*

$$D' \mapsto \sum_{j \in J} C_j.$$

Then there exists a rational curve $C \to \bar{S}$ *such that*

$$C \rightsquigarrow C_1 \cup \cdots \cup C_r.$$

We sketch the proof, highlighting the main geometric ideas but referring to [BHT09] for some the deformation-theoretic details.

Choose a partial normalization

$$\phi_0 : T_0 \to C_1 \cup \cdots \cup C_r$$

where T_0 is a nodal connected projective curve of genus zero and ϕ_0 is birational onto its image. We interpret ϕ_0 as a stable map to $\mathcal{S}_{\mathfrak{p}}$. Let $\overline{\mathcal{M}}_0(\mathcal{S}_{\mathfrak{p}}, D)$ denote the stable map space containing ϕ_0. We make a few observations about this:

- there is a 'nice' open subset

$$\overline{\mathcal{M}}_0^{\circ}(\mathcal{S}_{\mathfrak{p}}, D) \subset \overline{\mathcal{M}}_0(\mathcal{S}_{\mathfrak{p}}, D)$$

  corresponding to maps birational onto their images; these admit no automorphisms, so this is in fact a space rather than an Artin stack;

- we only use this space for deformation-theoretic purposes, not as a compact moduli space/stack.

We relativize this construction: Let

$$\mathcal{Y} \to \mathrm{Def}(\mathcal{S}_{\mathfrak{p}}/W(\bar{k}))$$

denote the universal formal deformation space of $\mathcal{S}_{\mathfrak{p}}$ over the Witt vectors and

$$\mathcal{M} = \overline{\mathcal{M}}_0(\mathcal{Y}, D) \to \mathrm{Def}(\mathcal{S}_{\mathfrak{p}}/W(\bar{k}))$$

the universal formal stable map space. A deformation theoretic computation (see [BHT09] for details) shows this has relative dimension ≥ -1. However, at ϕ_0 the dimension of the fiber is at most zero-dimensional, as $\mathcal{S}_{\mathfrak{p}}$ is not uniruled. Thus $\mathcal{M}$ maps to a divisor in $\mathrm{Def}(\mathcal{S}_{\mathfrak{p}}/W(\bar{k}))$, namely, the formal deformation space of the *polarized* K3 surfaces $\mathrm{Def}((\mathcal{S}_{\mathfrak{p}}, D)/W(\bar{k}))$. The resulting

$$\overline{\mathcal{M}}_0(\mathcal{Y}, D) \to \mathrm{Def}((\mathcal{S}_{\mathfrak{p}}, D)/W(\bar{k}))$$

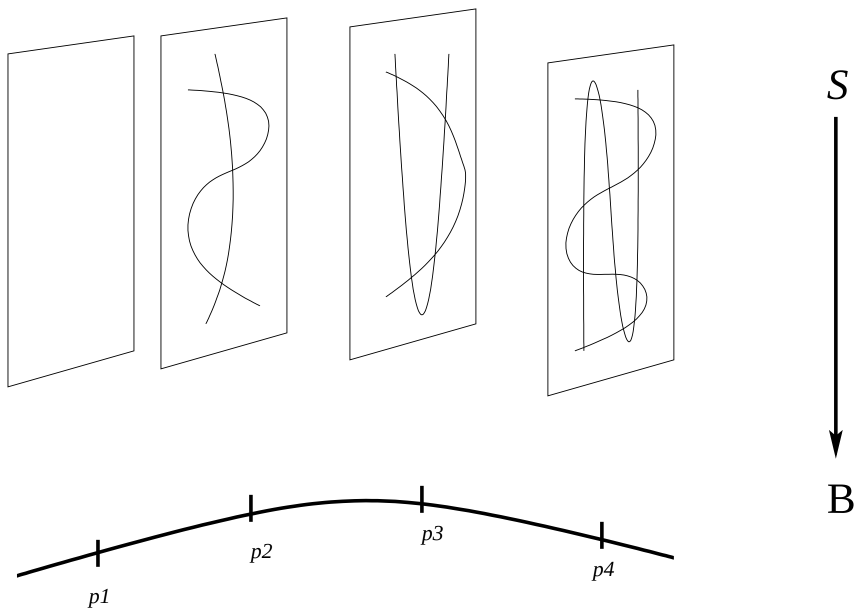

Figure 4: Reductions of a K3 surface mod $\mathfrak{p}$ have extra curve classes

is therefore algebraizable, of relative dimension zero near ϕ_0.

Hence we get a stable map

$$\phi_t : T_t \to \bar{S}$$

reducing to ϕ_0 over the prime $\mathfrak{p}$. However, we assume that no subset of the C_j arises as the specialization of a divisor from $\bar{S}$. It follows that $\phi_t(T_t)$ is irreducible; since ϕ_t remains birational onto its image, we deduce that $T_t \simeq \mathbb{P}^1$. Our desired rational curve is its image in $\bar{S}$. $\square$

We now assume that S is a K3 surface defined over a number field F with $\mathrm{Pic}(\bar{S}) = \mathbb{Z}h$, not necessarily of degree two. We collect some observations:

1. There exist infinitely many $\mathfrak{p}$ such that $\mathcal{S}_\mathfrak{p}$ is smooth and non-uniruled (see Prop. 19 and section 4.6).

2. For all these $\mathfrak{p}$, we have $\mathrm{rank}(\mathrm{Pic}(\mathcal{S}_\mathfrak{p})) \geq 2$ (see Cor. 34); in particular, h is in the interior of the effective cone of curves.

3. Fix $M \in \mathbb{N}$; for sufficiently large $\mathfrak{p}$ we have the following property: Any curve $C \subset \mathcal{S}_\mathfrak{p}$ with $C \cdot h \leq Mh \cdot h$ satisfies $[C] = mh$ for some $m \leq M$.

For the last assertion, let $\mu : B \to \mathcal{H}$ denote the classifying map to the Hilbert scheme of K3 surfaces corresponding to $\mathcal{S} \to B$. Note that the Hilbert scheme parametrizing curves of bounded degree is itself bounded; the same holds true for the subset of $\mathcal{G} \subset \mathcal{H}$ corresponding to K3 surfaces admitting non-complete intersection curves of bounded degree. Since $\mu(B) \not\subset \mathcal{G}$, there are finitely many primes at which the $\mu(B)$ intersects $\mathcal{G}$.

Fix N; our goal is to exhibit an irreducible curve $C \subset \bar{S}$ such that $[C] \in |N'h|$ for $N' \geq N$. Let $\mathfrak{p}$ be a large prime as described above, so that each non-complete intersection curve has degree greater than $Nh \cdot h$. Choose the smallest N' such that

$$N'h = \sum_{j=1}^{r} m_j C_j \tag{6.1}$$

for C_j indecomposable and not in $\mathbb{Z}h$. We have seen there exist irreducible rational curves in those classes. Furthermore, our assumption implies that $h \cdot C_j \geq N$, so we have $N' \geq N$. We would be done by Proposition 42 if

$$m_1 = m_2 = \cdots = m_r = 1$$

in Equation 6.1. How can we guarantee this?

Assume that (S, h) has degree two, thus is a double cover $S \to \mathbb{P}^2$; let $\iota : C \to C$ denote the covering involution. A straightforward computation shows that $\iota^*(h) = h$ and $\iota^*(D) = -D$ for $D \in h^\perp \subset H^2(S_{\mathbb{C}}, \mathbb{Z})$. In this situation, Equation 6.1 takes the special form

$$N'h = C_1 + C_2, C_2 = \iota^*(C_1),$$

which has the desired multiplicities.

Remark 43 What happens if we have non-trivial multiplicities? The argument for Proposition 42 still works under suitable geometric assumptions on $C_1, \ldots, C_r$ and their multiplicities.

Suppose that $C_1, C_2 \subset \mathcal{S}_{\mathfrak{p}}$ are smooth rational curves meeting transversally with intersection matrix:

	C_1	C_2
C_1	-2	3
C_2	3	-2

Suppose that $D \mapsto 2C_1 + C_2$ with $D \cdot D = 2$. Write $C_1 \cap C_2 = \{p, q, r\}$ and

$$T = \mathbb{P}^1 \cup_{p'} \mathbb{P}^1 \cup_{r'} \mathbb{P}^1 = C_1' \cup_{p'} C_2 \cup_{r''} C_1'',$$

i.e., a chain of three rational curves, the inner and outer curves C_1' and C_1'' identified with C_1 and the middle curve identified with C_2. Let ϕ_0 denote the morphism that restricts to the identity on each component. The associated stable map $\phi_0 \to \mathcal{S}_{\mathfrak{p}}$ still is unramified and lacks automorphisms, and can be utilized in the deformation-theoretic argument for Proposition 42.

Unfortunately, we lack general techniques to ensure that the new rational curves emerging mod p satisfy any transversality conditions.

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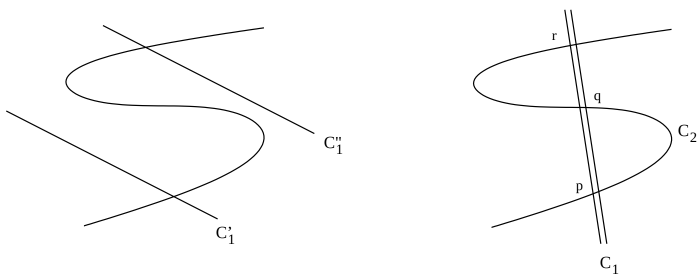

Figure 5: Multiplicities and the lifting argument

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