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Balanced line bundles and equivariant compactifications of homogeneous spaces

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Manin's conjecture predicts an asymptotic formula for the number of rational points of bounded height on a smooth projective variety X in terms of global geometric invariants of X. The strongest form of the conjecture implies certain inequalities among geometric invariants of X and of its subvarieties. We provide a general geometric framework explaining these phenomena, via the notion of balanced line bundles, and prove the required inequalities for a large class of equivariant compactifications of homogeneous spaces.

1 Introduction

Let X be a smooth projective variety over a number field. It is generally hoped that global geometric properties of X should be reflected in its arithmetic properties. For instance, assume that its anticanonical class $-K_X$ is ample. It has been conjectured that such X satisfy:

> Potential Density: there exists a finite extension F of the ground field such that $X(F)$ is Zariski dense (see [HT00a], [BT99] for first results in this direction and [Cam04], [Abr09] for a description of a general framework).

Supposing that X has dense rational points over F, we can ask for quantitative versions of density:

> Asymptotic Formulas: Let $\mathcal{L} = (L, \|\cdot\|)$ be an ample, adelically metrized, line bundle on X and $\mathsf{H}_{\mathcal{L}}$ the associated height (for definitions and more background see, e.g., [Tsc09, Section 4.8]). Then there exists a Zariski open $X^\circ \subset X$ such that

$$\#\{x \in X^\circ(F) \mid \mathsf{H}_{\mathcal{L}}(x) \leq \mathsf{B}\} \sim c(X, \mathcal{L})\mathsf{B}^{a(X,L)} \log(\mathsf{B})^{b(X,L)-1},$$

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as $\mathsf{B} \to \infty$. Here $a(X,L)$ and $b(X,L)$ are certain *geometric* constants introduced in this context in [FMT89] and [BM90] (and recalled in Section 2) and $c(X,\mathcal{L})$ is a Tamagawa-type number defined in [Pey95], [BT98b].

When $L = -K_X$ the main term of the asymptotic formula reads

$$\#\{x \in X^\circ(F) \mid \mathsf{H}_{-\mathcal{K}_X}(x) \leq \mathsf{B}\} \sim c(X, -\mathcal{K}_X)\mathsf{B}\log(\mathsf{B})^{\mathrm{rk}\,\mathrm{Pic}(X)-1}$$

as $\mathsf{B} \to \infty$, where $-\mathcal{K}_X$ is the metrized anticanonical bundle. For a survey addressing both aspects and containing extensive references, see [Tsc09].

The Asymptotic Formulas raise many formal questions. How do we choose $X^\circ \subset X$? Clearly, we want to exclude subvarieties $Y \subsetneq X$ contributing excessively to the number of rational points. For example, if X is a split cubic surface and $L = -K_X$ then lines on X contribute on the order of B^2 points of height $\leq \mathsf{B}$, more than the $\mathsf{B}\log(\mathsf{B})^6$ points expected from X°.

Furthermore, we should consider carefully whether to include subvarieties $Y \subsetneq X$ contributing rational points at the same rate as those from X°. For example, if $X \subset \mathbb{P}^5$ is a complete intersection of two quadrics then each line of X contributes on the order of B^2 points, the same as the conjectured total for X° (see Example 3.11). These lines are parametrized by an abelian surface. Including such subvarieties must have implications for the interpretation of the Tamagawa-type constant.

Returning to the case of general L, in order for the Asymptotic Formula to be internally consistent, all $Y \subsetneq X$ meeting X° must satisfy

$$(a(Y, L|_Y), b(Y, L|_Y)) \leq (a(X,L), b(X,L))$$

in the lexicographic order. Moreover, if the constant $c(X,\mathcal{L})$ is to be independent of the open set $X^\circ \subset X$ we must have

$$(a(Y, L|_Y)), b(Y, L|_Y)) < (a(X,L), b(X,L)). \tag{1}$$

However, there exist varieties of dimension ≥ 3 where these properties fail; these provide counterexamples to the Asymptotic Formulas [BT96b]. On the other hand, no counterexamples are known in the equivariant context, when X is an equivariant compactification of a linear algebraic group G or of a homogeneous space $H\backslash G$, and asymptotic formulas for the number of points of bounded height have been established for many classes of such compactifications (see [Tsc09]).

These arithmetic considerations motivate us to introduce and study the notion of *balanced line bundles*. A balanced line bundle is one for which general subvarieties Y satisfy the inequalities above, yielding compatibility with Manin's predictions. In this paper, we establish basic properties of balanced line bundles and investigate varieties that carry such line bundles.

Throughout the paper, we assume that the ground field is an algebraically closed field of characteristic zero. In Section 2, we introduce geometric invariants $a(X, L)$ and $b(X, L)$ and explore their basic propeties, i.e., questions of rationality and birational invariance. Then in Section 3, we define the notion of balanced line bundles and study some important varieties including flag varieties and some smooth Fano 3-folds. One important observation is

Proposition 1.1. *Let X be a generalized flag variety and L a big line bundle on X. Then L is balanced with respect to any smooth variety if and only if L is proportional to $-K_X$.* □

In Section 4, we study balanced line bundles on del Pezzo surfaces. We obtain the complete characterization of balanced line bundles which is described below:

Proposition 1.2. *Let X be a del Pezzo surface and L a big Cartier divisor on X. Then L is balanced if and only if the $\mathbb{Q}$-divisor $a(X, L)L + K_X$ is linearly equivalent to a rigid effective divisor on X.* □

Here we obtain an interesting equivalence between an analytic concept and a geometric concept. It claims that the rigidity of the adjoint divisor and the balanced property are equivalent. This observation has been made already in Proposition 1.1. In Section 5, we study the geometry of equivariant compactifications of homogeneous spaces and obtain the following result, which is a main result of this paper:

Theorem 1.3. *Let*

$$H \subset M \subset G$$

be connected linear algebraic groups. Let X be a smooth projective G-equivariant compactification of $H\backslash G$ and $Y \subset X$ the induced compactification of $H\backslash M$. Assume that the projection $G \to M\backslash G$ admits a rational section. Then $-K_X$ is balanced with respect to Y, i.e., inequality (1) *holds for $L = -K_X$.* □

A version of this geometric result, for $G = \mathbb{G}_a^n$, appeared in [CLT02, Section 7], where it was used to bound contributions from nontrivial characters to the Fourier expansion of height zeta functions and, ultimately, to prove asymptotic formulas for the number of rational points of bounded height (Manin's conjecture) for equivariant compactifications of $\mathbb{G}_a^n$. Another application can be found in [GTBT11], where this theorem plays an important role in an implementation of ideas from ergodic theory (mixing) in a proof of Manin's conjecture for equivariant compactifications of $G\backslash G^n$, where G is an absolutely simple linear algebraic group, acting diagonally on G^n. In Section 6 we investigate balanced line bundles on toric varieties in the context of the Minimal Model Program. The main result is

Proposition 1.4. *Let X be a smooth projective toric variety and L a big Cartier divisor. Then L is balanced with respect to all subtoric varieties if and only if the $\mathbb{Q}$-divisor $a(X, L)L + K_X$ is linearly equivalent to a rigid effective divisor on X.* □

Again, we observe the equivalence between the balanced property and the rigidity of the adjoint divisor.

2 Generalities

Definition 2.1. Let V be a finite dimensional vector space over $\mathbb{R}$. *A closed convex cone* $\Lambda \subset V$ is a closed subset which is closed under linear combinations with non-negative real coefficients. *An extremal face* $F \subset \Lambda$ is a closed convex subcone of Λ such that if $u, v \in \Lambda$ and $u + v \in F$ then $u, v \in F$. *A supporting function* is a linear functional $\sigma : V \to \mathbb{R}$ such that $\sigma \geq 0$ on Λ. A face of the form

$$F' = \{\sigma = 0\} \cap \Lambda$$

is called *a supported face*. A supported face is an extremal face, but the converse is not true, in general. The converse does hold when Λ is locally finitely generated in a neighborhood of F, i.e., there exist finitely many linear functionals

$$\lambda_i : V \to \mathbb{R}$$

such that $\lambda_i(v) > 0$ for $v \in F \setminus \{0\}$ and

$$\Lambda \cap \{v : \lambda_i(v) \geq 0 \text{ for any } i\},$$

is finitely generated. Note that when Λ is *strict*, i.e., does not contain a line, then $\{0\}$ is a supported face. □

We work over an algebraically closed field of characteristic zero. A variety is an integral separated scheme over this field. Let X be a smooth projective variety. We use

$$\Lambda_{\text{eff}}(X) \subset \mathrm{NS}(X, \mathbb{R}) \subset \mathrm{H}^2(X, \mathbb{R})$$

to denote the pseudo-effective cone, i.e., the closure of effective $\mathbb{Q}$-divisors on X in the real Néron-Severi group $\mathrm{NS}(X, \mathbb{R})$. Another common notation in the literature is $\overline{\mathrm{NE}}^1(X)$. Note that the pseudo-effective cone is strict and convex [BFJ09, Prop. 1.3]. Let $\Lambda^\circ_{\text{eff}}(X)$ denote the interior of the pseudo-effective cone; a divisor class D on X is *big* if $[D] \in \Lambda^\circ_{\text{eff}}(X)$. We denote the dual cone of the cone of pseudo-effective divisors by $\overline{\mathrm{NM}}_1(X)$. This is the closure of the cone generated by movable curves ([BDPP13].)

A *rigid effective* divisor is a reduced divisor $D \subset X$ such that

$$\mathrm{H}^0(\mathcal{O}_X(nD)) = 1 \quad \forall n \geq 1.$$

If D is rigid with irreducible components $D_1, \ldots, D_r$ then

$$\mathrm{H}^0(\mathcal{O}_X(n_1D_1 + \ldots + n_rD_r)) = 1 \quad \forall n_1, \ldots, n_r \geq 1$$

and

$$\operatorname{span}(D_1, \ldots, D_r) \cap \Lambda^{\circ}_{\text{eff}}(X) = \emptyset. \tag{2}$$

Definition 2.2. Assume that L is a big Cartier divisor on X. The *Fujita invariant* is defined by

$$\begin{aligned} a(X, L) &= \inf\{a \in \mathbb{R} : aL + K_X \sim \text{an effective } \mathbb{R}\text{-divisor }\} \\ &= \min\{a \in \mathbb{R} : a[L] + [K_X] \in \Lambda_{\text{eff}}(X)\}. \end{aligned}$$

□

Remark 2.3. Note that $a(X, L)$ only depends on the numerical class of L. □

Note that the Fujita invariant is positive if and only if K_X is not pseudo-effective. The invariant

$$\kappa\epsilon(X, L) = -a(X, L)$$

was introduced and studied by Fujita under the name *Kodaira energy* [Fuj97] (see also [Fuj87], [Fuj92], [Fuj96]). A similar invariant

$$\sigma(X, L) = \dim(X) + 1 - a(X, L)$$

appeared in [Som86] under the name *spectral value.*

Remark 2.4. A smooth projective variety X is uniruled if and only if K_X is not pseudo-effective [BDPP13], [Laz04b, Cor. 11.4.20]. □

The following result was conjectured by Fujita and proved by Batyrev for threefolds and [BCHM10, Cor. 1.1.7] in general. (See [DC12, Corollay 3.7 and Corollary 3.8] for recent generalizations.)

Theorem 2.5. *Let X be projective with Kawamata log terminal singularities such that K_X is not pseudo-effective, and L an ample Cartier divisor on X. Then $a(X, L)$ is rational.* □

However, this property can fail when L is big but not ample, and the following example going back to Cutkosky [Cut86] was suggested to us by Brian Lehmann:

Example 2.6. [Leh12, Example 4.9] Let Y be an abelian surface with Picard rank at least 3. The cone of nef divisors and the cone of pseudo-effective divisors coincide, and the boundary of these cones is circular. Let N be a Cartier divisor on Y such that $-N$ is ample and $X := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(N))$. Let $\pi : X \to Y$ denote the projection morphism and $S \subset X$ the section corresponding to the quotient map $\mathcal{O} \oplus \mathcal{O}(N) \to \mathcal{O}(N)$. Every divisor on X is linearly equivalent to $tS + \pi^*D$ where D is a divisor on Y. In particular, K_X is linearly equivalent to $-2S + \pi^*N$. The cone of pseudo-effective divisors $\Lambda_{\text{eff}}(X)$ is generated by S and $\pi^*\Lambda_{\text{eff}}(Y)$.

Consider a big $\mathbb{Q}$-divisor $L = tS + \pi^*D$, where $t > 0$ and D is a big $\mathbb{Q}$-divisor on Y. If t is sufficiently large, then $a(X,L)$ is characterized by the condition $a[D] + [N] \in \partial\Lambda_{\text{eff}}(Y)$. However, the boundary of $\Lambda_{\text{eff}}(Y)$ is circular, and $a(X,L) \notin \mathbb{Q}$, in general. □

From the point of view of Manin's conjecture, the global geometric invariants involved in its formulation should be functorial for birational transformations, and indeed this holds for the Fujita invariant:

Proposition 2.7. *Let $\beta : \tilde{X} \to X$ be a birational morphism of projective varieties, where $\tilde{X}$ is smooth, and X is normal and has only canonical singularities. Assume K_X is not pseudo-effective and L is a Cartier divisor which is big. Setting $\tilde{L} = \beta^* L$, we have*

$$a(X,L) = a(\tilde{X},\tilde{L}).$$

□

Proof. Since X has canonical singularities, we have

$$K_{\tilde{X}} = \beta^* K_X + \sum_i d_i E_i,$$

where the E_i are the irreducible exceptional divisors and the d_i are nonnegative rational numbers. It follows that for sufficiently divisible integers $m, n \geq 0$ we have

$$\begin{aligned}\Gamma(\mathcal{O}_X(mK_X + nL)) &= \Gamma(\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}} - \sum_i md_iE_i + n\tilde{L})) \\ &= \Gamma(\mathcal{O}_{\tilde{X}}(mK_{\tilde{X}} + n\tilde{L})),\end{aligned}$$

where the second equality reflects the fact that allowing poles in the exceptional locus does not increase the number of global sections. In particular, effective divisors supported in the exceptional locus of β are rigid. It follows from the assumption that no multiple of $K_{\tilde{X}}$ is effective and that $a(\tilde{X},\tilde{L}) \geq 0$. Definition 2.2 gives $a(\tilde{X},\tilde{L}) = a(X,L) > 0$. ■

Next, we discuss the second geometric invariant appearing in Manin's conjecture.

Definition 2.8. Let X be a projective variety with only $\mathbb{Q}$-factorial terminal singularities such that K_X is not pseudo-effective. Let L be a big Cartier divisor on X. Define

$$\begin{aligned} b(X,L) &= \text{the codimension of the minimal supported face of} \\ &\quad\ \Lambda_{\text{eff}}(X) \text{ containing the } \mathbb{R}\text{-divisor } a(X,L)L + K_X. \end{aligned}$$

□

Remark 2.9. Again $b(X,L)$ only depends on the numerical class of L □

This definition is relatively easy to grasp when $\Lambda_{\mathrm{eff}}(X)$ is finitely generated, which holds in a number of cases:

- A projective variety X is *log Fano* if there exists an effective $\mathbb{Q}$-divisor Δ on X such that (X, Δ) is divisorially log terminal (see [BCHM10, p. 424]) and $-(K_X + \Delta)$ is ample. If X is log Fano then the Cox ring of X is finitely generated [BCHM10, Cor. 1.3.2], so in particular, $\Lambda_{\mathrm{eff}}(X)$ is finite rational polyhedral and generated by effective divisors.
- Let X be a smooth projective variety that is toric or an equivariant compactification of the additive group $\mathbb{G}_a^n$. Then $\Lambda_{\mathrm{eff}}(X)$ is generated by boundary divisors, i.e., irreducible components of the complement of the open orbit, [HT99, Thm. 2.5], [BT95, Prop.1.2.11].

Let X be a smooth projective variety with $\Lambda_{\mathrm{eff}}(X)$ generated by a finite number of effective divisors and $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{NS}(X, \mathbb{Q})$. Since each irreducible rigid effective divisor on X is a generator of $\Lambda_{\mathrm{eff}}(X)$ (cf. (2)), we have

$$Z := \cup_{\text{rigid effective}} D$$

is a Zariski closed proper subset of X.

One of the reasons for adopting the terminology of supported faces in the definition of $b(X, L)$ is to simplify the verification of its birational invariance:

Proposition 2.10. *Let X be a normal $\mathbb{Q}$-factorial terminal projective variety such that K_X is not pseudo-effective and $\beta : \tilde{X} \to X$ a smooth resolution. Let L be a big Cartier divisor on X and put $\tilde{L} = \beta^* L$. Then*

$$b(X, L) = b(\tilde{X}, \tilde{L}).$$

□

Proof. Let F be the minimal supported face of $\Lambda_{\mathrm{eff}}(X)$ containing an $\mathbb{R}$-divisor $a(X, L)L + K_X$ and $\tilde{F}$ be the minimal supported face of $\Lambda_{\mathrm{eff}}(\tilde{X})$ containing $a(\tilde{X}, \tilde{L})\tilde{L} + K_{\tilde{X}}$. The vector spaces generated by F and $\tilde{F}$ will be denoted by V_F and $V_{\tilde{F}}$, respectively. There exists a nef cycle $\xi \in \overline{\mathrm{NM}}_1(\tilde{X})$ such that

$$\tilde{F} = \{\xi = 0\} \cap \Lambda_{\mathrm{eff}}(\tilde{X}).$$

Let $E_1, \ldots, E_n$ be irreducible components of the exceptional locus of β. The Negativity Lemma ([BCHM10, Lemma 3.6.2]) implies that $\mathrm{NS}(\tilde{X})$ is a direct sum of $\beta^*\mathrm{NS}(X)$ and $\mathbb{R}[E_i]$'s. Since X has only terminal singularities, it follows from Proposition 2.7 that

$$a(\tilde{X}, \tilde{L})\tilde{L} + K_{\tilde{X}} = \beta^*(a(X, L)L + K_X) + \sum_i d_i E_i,$$

where the d_i's are positive rational numbers. This implies that $\tilde{F}$ contains $\beta^*(a(X,L)L+K_X)$ and the E_i's. Thus $a(X,L)L+K_X$ is contained in an extremal face supported by a supporting function $\beta_*\xi$, i.e.,

$$\{\beta_*\xi=0\}\cap\Lambda_{\mathrm{eff}}(X),$$

so this supported face also contains F. We get a well-defined injection

$$\Phi : V_F \hookrightarrow V_{\tilde{F}}/(\sum_i \mathbb{R}E_i).$$

On the other hand, let $\eta \in \overline{\mathrm{NM}}_1(X)$ be a nef cycle supporting F. Consider a linear functional $\tilde{\eta} : \mathrm{NS}(\tilde{X}) \to \mathbb{R}$ defined by

$$\tilde{\eta} \equiv \eta \text{ on } \beta^*\mathrm{NS}(X) \text{ and } \tilde{\eta}\cdot E_i = 0 \text{ for any } i.$$

The projection from $\mathrm{NS}(\tilde{X})$ to $\beta^*\mathrm{NS}(X)$ maps pseudo-effective divisors to pseudo-effective divisors so that $\tilde{\eta} \in \overline{\mathrm{NM}}_1(\tilde{X})$. Moreover,

$$\tilde{\eta}\cdot(a(\tilde{X},\tilde{L})\tilde{L}+K_{\tilde{X}}) = \tilde{\eta}\cdot(\beta^*(a(X,L)L+K_X)+\sum_i d_iE_i) = 0,$$

so that $\{\tilde{\eta}=0\}\cap\Lambda_{\mathrm{eff}}(\tilde{X})$ contains $\tilde{F}$. It follows that Φ is bijective and our assertion is proved. ■

Definition 2.11. Let X be a uniruled projective variety with a big Cartier divisor L. We define

$$a(X,L)=a(\tilde{X},\beta^*L),\quad b(X,L)=b(\tilde{X},\beta^*L)$$

where $\beta : \tilde{X} \to X$ is some resolution of singularities. □

Note that $K_{\tilde{X}}$ is not pseudo-effective by Remark 2.4; Propositions 2.7 and 2.10 guarantee the invariants are independent of the choice of resolution.

For the anticanonical line bundle, invariants are computed as follows:

Example 2.12 (The anticanonical line bundle)**.** Let X be a projective variety with only $\mathbb{Q}$-factorial terminal singularities. As in the smooth case, the cone $\Lambda_{\mathrm{eff}}(X)$ of pseudo-effective divisors is strict. When the anticanonical class $-K_X$ is big, we have

$$a(X,-K_X)=1,\quad b(X,-K_X)=\mathrm{rk}\,\mathrm{NS}(X).$$

Let $\beta : \tilde{X} \to X$ be a smooth resolution and $E_1,\dots,E_n$ the irreducible components of the exceptional locus; we have

$$K_{\tilde{X}} = \beta^*K_X + \sum_i d_iE_i,$$

where $d_i \in \mathbb{Q}_{>0}$, for all i. Hence the minimal extremal face containing $-\beta^*K_X + K_{\tilde{X}}$ contains a simplicial cone $F := \oplus_i \mathbb{R}_{\geq 0}[E_i]$. The fact that F is a supported face follows from [Bou04, Theorem 3.19], which asserts that the pseudo-effective cone is locally polyhedral in this region, generated by the prime exceptional divisors. Hence we may compute

$$b(\tilde{X}, -\beta^*K_X) = \operatorname{rk} \mathrm{NS}(\tilde{X}, \mathbb{R}) - n = \operatorname{rk} \mathrm{NS}(X, \mathbb{R}) = b(X, -K_X).$$

□

Remark 2.13. There exist projective bundles over curves of arbitrary genus $g > 0$ with big anticanonical divisor [KMM92, 3.13]. When $g > 1$ these cannot have potentially dense rational points. □

Example 2.14. Example 2.12 can be generalized as follows: Let X be a smooth projective variety and $D = \sum_i e_i E_i$ an effective $\mathbb{R}$-divisor whose numerical dimension $\nu(D)$ is zero (see [Nak04, Section V.2] or [Leh13, Theorem 1.1] for definitions). The minimal extremal face containing D contains $F = \oplus_i \mathbb{R}_{\geq 0}[E_i]$, and we claim that F is a supported face of $\Lambda_{\mathrm{eff}}(X)$. First we prove that F is an extremal face. Let $u, v \in \Lambda_{\mathrm{eff}}(X)$ such that $u + v \in F$. For any pseudo-effective numerical class α, we denote the negative part and the positive part of the divisorial Zariski decomposition of α by $N_\sigma(\alpha)$ and $P_\sigma(\alpha) = \alpha - [N_\sigma(\alpha)] \in \Lambda_{\mathrm{eff}}(X)$ respectively (see [Nak04, Section III.1], also [Leh13, Section 3]). The assumption $\nu(D) = 0$ implies that

$$u + v \equiv N_\sigma(u + v) \leq N_\sigma(u) + N_\sigma(v).$$

This implies that $N_\sigma(u) \equiv u$ and $N_\sigma(v) \equiv v$ so that $u, v \in F$. Again by [Bou04, Theorem 3.19], the cone $\Lambda_{\mathrm{eff}}(X)$ is locally rational polyhedral in a neighborhood of F. Hence F is a supported face. □

Little is known about the geometric meaning of the invariant $b(X, L)$, in general. Here we consider situations relevant for our applications to equivariant compactifications of homogeneous spaces.

Definition 2.15. Let X be a $\mathbb{Q}$-factorial terminal and projective variety and D an $\mathbb{R}$-divisor in the boundary of $\Lambda_{\mathrm{eff}}(X)$. We say D is *locally rational polyhedral* if either $D \equiv 0$ numerically or there exist finitely many linear functionals

$$\lambda_i : \mathrm{NS}(X, \mathbb{Q}) \to \mathbb{Q}$$

such that $\lambda_i(D) > 0$ and

$$\Lambda_{\mathrm{eff}}(X) \cap \{v : \lambda_i(v) \geq 0 \text{ for any } i\},$$

is finite rational polyhedral and generated by effective $\mathbb{Q}$-divisors. In this case, the minimal extremal face F containing D is supported by a supporting function. □

Theorem 2.16. *Let X be a $\mathbb{Q}$-factorial terminal projective variety such that K_X is not pseudo-effective and L a big Cartier divisor on X. Suppose that $a(X, L)L + K_X$ has the form $c(A + K_X + \Delta)$, where A is an ample*

$\mathbb{R}$-divisor, (X, Δ) a Kawamata log terminal pair, and $c > 0$. Then $a(X, L)L + K_X$ is locally rational polyhedral and $a(X, L)$ is rational. □

Proof. When $D = a(X, L)L + K_X \equiv 0$, then $a(X, L)$ is rational. Suppose that $D \not\equiv 0$. The local finiteness of the pseudo-effective boundary is proved in [Leh12, Proposition 3.3] by using the finiteness of the ample models [BCHM10, Corollary 1.1.5]. Moreover, Lehmann proved that the pseudo-effective boundary is locally defined by movable curves. The generation by effective $\mathbb{Q}$-divisors follows from the non-vanishing theorem [BCHM10, Theorem D]. ■

In particular, if L is ample then $a(X, L)L + K_X$ is locally rational polyhedral. We have already seen in Example 2.6 that the local finiteness is no longer true if we only assume that L is big. However, there are certain cases where the local finiteness of $a(X, L)L + K_X$ still holds for any big Cartier divisor L:

Example 2.17 (Surfaces)**.** Let X be a smooth projective surface such that K_X is not pseudo-effective. Let L be a big Cartier divisor on X. If $D = a(X, L)L + K_X$ is numerically trivial, then $a(X, L)$ is rational. Suppose that D is a non-zero pseudo-effective divisor. We prove that D is locally rational polyhedral. We consider the Zariski decomposition of $D = P + N$, where P is a nef $\mathbb{R}$-divisor and N is the negative part of D. The boundary of $\Lambda_{\mathrm{eff}}(X)$ is locally rational polyhedral away from the nef cone (see [Bou04, Theorem 3.19] and [Bou04, Theorem 4.1]). Thus, if N is non-zero, then our assertion follows. Suppose that N is zero. Since $a(X, L)L \cdot P + K_X \cdot P = D \cdot P = 0$ and $L \cdot P > 0$, we have $K_X \cdot P < 0$. Therefore, $D = P$ is sitting in the K_X-negative part of the boundary of the cone of curves. Now our assertion follows from Mori's cone theorem. In particular, $a(X, L)$ is a rational number. □

In general, $a(X, L)L + K_X$ is not necessarily of the form of the adjoint divisor $c(A + K_X + \Delta)$, where A is an ample $\mathbb{R}$-divisor, (X, Δ) a Kawamata log terminal pair, and $c > 0$.

Example 2.18 (Equivariant compactifications of the additive groups)**.** Let X be a smooth projective equivariant compactification of the additive group $\mathbb{G}_a^n$. Then $\Lambda_{\mathrm{eff}}(X)$ is a simplicial cone generated by boundary components, by [HT99, Theorem 2.5]. However, this cannot be explained from Theorem 2.16. Indeed, consider the standard embedding of $\mathbb{G}_a^3$ into $\mathbb{P}^3$:

$$\mathbb{G}_a^3 \ni (x, y, z) \mapsto (x : y : z : 1) \in \mathbb{P}^3.$$

This is an equivariant compactification, and the group action fixes every point on the boundary divisor D, a hyperplane section. Let X be an equivariant blow up of 12 generic points on a smooth cubic curve in D. Write H for the pullback of the hyperplane class and $E_1, \ldots, E_{12}$ for the exceptional divisors. Consider

$$L = 4H - E_1 - \cdots - E_{12}.$$

Then L is big and nef, but not semi-ample (see [Laz04a, Section 2.3.A] for more details). In particular, the section ring of L is not finitely generated. On the other hand, consider

$$\Lambda_{\mathrm{adj}}(X) = \{\Gamma \in \Lambda_{\mathrm{eff}}(X) \mid \Gamma = c(A + K_X + \Delta)\},$$

where A is an ample $\mathbb{R}$-divisor, (X, Δ) a Kawamata log terminal pair, and c a positive number. Then $\Lambda_{\mathrm{adj}}(X)$ forms a convex cone. The existence of non-finitely generated divisors and [BCHM10, Corollary 1.1.9] imply that $\Lambda_{\mathrm{adj}}(X) \subsetneq \Lambda_{\mathrm{eff}}(X)$. □

It is natural to expect that the invariant $b(X, L)$ is related to the canonical fibration associated to $a(X, L)L + K_X$. A sample result in this direction is:

Proposition 2.19. *Let X be a smooth projective variety such that K_X is not pseudo-effective. Let L be a big Cartier divisor and assume that $D = a(X, L)L + K_X$ is locally rational polyhedral and semi-ample. Let $\pi : X \to Y$ be the semi-ample fibration of D. Then*

$$b(X, L) = \mathrm{rk}\, \mathrm{NS}(X) - \mathrm{rk}\, \mathrm{NS}_\pi(X),$$

where $\mathrm{NS}_\pi(X)$ is the lattice generated by π-vertical divisors, i.e., divisors $M \subset X$ such that $\pi(M) \subsetneq Y$ □

Proof. Let F be the minimal extremal face of $\Lambda_{\mathrm{eff}}(X)$ containing $D = a(X, L)L + K_X$ and V_F the vector space generated by F. We claim that $V_F = \mathrm{NS}_\pi(X)$. Let H be an ample $\mathbb{Q}$-divisor on Y such that $\pi^*H = D$. Let M be a π-vertical divisor on X. Then for sufficiently large m, there exists an effective Cartier divisor H' such that $mH \sim H'$ and the support of H' contains $\pi(M)$. Thus $mD = m\pi^*H \sim \pi^*H' \in F$ and the support of π^*H' contains M. We conclude that $M \in F$, and this proves that $\mathrm{NS}_\pi(X) \subset V_F$. Next, let X_y be a general fiber of π and $C \subset X_y$ a movable curve on X such that $[C]$ is in the interior of $\overline{\mathrm{NM}}_1(X_y)$. Then

$$F_C = \{[C] = 0\} \cap \Lambda_{\mathrm{eff}}(X),$$

is an extremal face containing D. The minimality implies $F \subset F_C$. On the other hand, the local rational finiteness of D implies that there exist effective $\mathbb{Q}$-divisors $D_1, \ldots, D_n \in F$ such that $D_1, \ldots, D_n$ form a basis of V_F. Since $D_1 \cdot C = \cdots = D_n \cdot C = 0$, the supports of D_i's are π-vertical. Hence it follows that $V_F \subset \mathrm{NS}_\pi(X)$. ■

Remark 2.20. When L is ample, it follows from [KMM87, Lemma 3.2.5] that the codimension of the minimal extremal face of nef cone containing D is equal to the relative Picard rank $\rho(X/Y)$. □

In Section 6, we explore this further in the case of toric varieties.

3 Balanced line bundles

Definition 3.1. Let X be a uniruled projective variety, L a big line bundle on X, and $Y \subsetneq X$ an irreducible uniruled subvariety. L is *weakly balanced with respect to* Y if

- $L|_Y$ is big;
- $a(Y, L|_Y) \leq a(X, L)$;
- if $a(Y, L|_Y) = a(X, L)$ then $b(Y, L|_Y) \leq b(X, L)$.

It is *balanced with respect to* Y if it is weakly balanced and one of the two inequalities is strict.

L is *weakly balanced* (resp. *balanced*) *on* X if there exists a Zariski closed subset $Z \subsetneq X$ such that L is weakly balanced (resp. balanced) with respect to every Y not contained in Z. The subset Z will be called *exceptional*. □

Remark 3.2. The restriction to uniruled subvarieties is quite natural: If Y is a smooth projective variety that is *not* uniruled and $Y \to X$ is a morphism such that $L|_Y$ is big, then $a(Y, L|_Y) \leq 0 < a(X, L)$ (see Remark 2.4). □

Remark 3.3. Any big Cartier divisor can be expressed as a sum of an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E. In particular, $L|_Y$ is big for any $Y \not\subseteq \mathrm{Supp(E)}$. □

We first explore these properties for projective homogeneous spaces:

Proposition 3.4. *Let G be a connected semi-simple algebraic group, $P \subset G$ a parabolic subgroup and $X = P\backslash G$ the associated generalized flag variety. Let L be a big Cartier divisor on X. We have:*

- *if L is not proportional to $-K_X$ then L is not balanced but is weakly balanced with respect to smooth subvarieties $Y \subset X$;*
- *if L is proportional to $-K_X$ then L is balanced with respect to smooth subvarieties.*

□

Proof. For generalized flag varieties, the nef cone and the pseudo-effective cone coincide so that L is ample. Moreover, the nef cone of a flag variety is finitely generated by semi-ample line bundles. Also note that since every rationally connected smooth proper variety is simply connected, all parabolic subgroups are connected.

Assume that L is not proportional to the anticanonical bundle, i.e., $D = a(X, L)L + K_X$ is a non-zero effective $\mathbb{Q}$-divisor. Let $\pi : X \to X'$ be the semi-ample fibration of D. Then X' is also a G-variety so that there exists a parabolic subgroup $P' \supset P$ such that $X' = P'\backslash G$ and π is the natural projection map. We have the following exact sequence:

$$0 \to \mathrm{Pic}(P'\backslash G)_{\mathbb{Q}} \to \mathrm{Pic}(P\backslash G)_{\mathbb{Q}} \to \mathrm{Pic}(P\backslash P')_{\mathbb{Q}} \to 0.$$

Indeed, the surjectivity follows from [KKV89, Proposition 3.2(i)]. Note that the picard group of a linear algebraic group is necessary finite (see [KKLV89, Proposition 4.5]). Then the exactness of other parts follows from

[KMM87, Lemma 3.2.5]. Let W be a fiber of π. Then $a(X,L) = a(W, L|_W)$ since $K_X|_W = K_W$ and $D|_W = 0$. The exact sequence and Remark 2.20 imply that

$$b(X,L) = \rho(X/X') = \operatorname{rk}\operatorname{Pic}(W) = b(W, L|_W).$$

Thus L is not balanced with respect to any fiber of π.

Let L be an arbitrary ample Cartier divisor and $Y \subset X$ a smooth subvariety. Let $\mathfrak{g}$ be the Lie algebra of G. For any $\partial \in \mathfrak{g}$, we can construct a global vector field ∂^X on X such that for any open set $U \subset X$ and any $f \in \mathcal{O}_X(U)$,

$$\partial^X(f)(x) = \partial_g f(x \cdot g)|_{g=1}.$$

It follows that the normal bundle $\mathcal{N}_{Y/X}$ is globally generated and its determinant is as well.

The restriction

$$aL|_Y + K_Y = (aL + K_X)|_Y + \det(\mathcal{N}_{Y/X})$$

with $\det(\mathcal{N}_{Y/X})$ globally generated hence contained in $\Lambda_{\mathrm{eff}}(Y)$. Thus we have $a(Y, L|_Y) \leq a(X,L)$. Suppose equality holds; our goal then is to prove that

$$b(Y, L|_Y) \leq b(X,L),$$

and the strict inequality holds when L is proportional to $-K_X$. Let $D = aL + K_X$ which is semi-ample.

First we assume that $\det(\mathcal{N}_{Y/X})$ and $D|_Y$ are trivial so that $\mathcal{N}_{Y/X}$ is the trivial vector bundle of rank $r = \operatorname{codim}(Y,X)$. The above construction of vector fields defines a surjective map:

$$\varphi : \mathfrak{g} \to \mathrm{H}^0(Y, \mathcal{N}_{Y/X}).$$

We may assume that $e = P \in Y$ so that the Lie algebra $\mathfrak{p}$ of P is contained in the kernel of φ. Consider the Hilbert scheme $\mathrm{Hilb}(X)$ and note that $\mathrm{H}^0(Y, \mathcal{N}_{Y/X})$ is naturally isomorphic to the Zariski tangent space of $\mathrm{Hilb}(X)$ at $[Y]$. Consider the morphism:

$$\pi : G \ni g \mapsto [Y \cdot g] \in \mathrm{Hilb}(X).$$

Since Y is Fano, $\mathrm{H}^1(Y, \mathcal{N}_{Y/X}) = 0$, $\mathrm{Hilb}(X)$ is smooth at $[Y]$, and

$$\dim_{[Y]}(\mathrm{Hilb}(X)) = r.$$

Moreover, since φ is surjective, π is a smooth morphism and $\pi(G)$ is a smooth open subscheme in $\mathrm{Hilb}(X)$. Let

$\mathcal{H}$ be the connected component of $\mathrm{Hilb}(X)$ containing $[Y]$ and $P' = \mathrm{Stab}(Y)$. Since the kernel of φ contains $\mathfrak{p}$, we have $P \subset P'$. This implies that $\pi(G) = P'\backslash G$ is open and closed so that

$$\mathcal{H} = \pi(G) = P'\backslash G.$$

In particular, $\dim(G) - \dim(P') = r$, so the kernel of φ is exactly equal to the Lie algebra $\mathfrak{p}'$ of P'. Consider the universal family $\mathcal{U} \subset X \times \mathcal{H}$ on $\mathcal{H}$. It follows that G acts on $\mathcal{U}$ transitively, and we conclude that $\mathcal{U} = P\backslash G$ and $Y = P\backslash P'$. Since $D|_Y$ is trivial, $b(Y, L|_Y) = \mathrm{rk}\,\mathrm{Pic}(P\backslash P')$. The exact sequence, which we discussed before, and [KMM87, Lemma 3.2.5] indicate that $\mathrm{rk}\,\mathrm{Pic}(P\backslash P') = \rho(X/\mathcal{H})$, the relative Picard rank of π. It follows from Remark 2.20 and the triviality of $D|_Y$ that

$$\rho(X/\mathcal{H}) \leq b(X, L).$$

When L is proportional to $-K_X$, $b(X, L) = \mathrm{rk}\,\mathrm{NS}(X)$ so that the strict inequality holds.

In the general case, we still know that $\mathcal{N}_{Y/X}$ and its determinant are globally generated. Consider the semi-ample fibration $Y \to W$ associated to $aL|_Y + K_Y = D|_Y + \det(\mathcal{N}_{Y/X})$ with generic fiber Y_w, which is smooth. Note that $\mathcal{N}_{Y_w/X}$ is trivial, as $\det(\mathcal{N}_{Y/X})|_{Y_w}$ and $\mathcal{N}_{Y_w/Y}$ are both trivial. $D|_Y$ is also trivial. The above construction shows that Y_w is the fiber of a fibration $\rho : X \to B$ with $W \subset B$; Y is the pullback of W. Theorem 2.16 and Proposition 2.19 imply that

$$b(Y, L|_Y) = \mathrm{rk}\,\mathrm{NS}(Y) - \mathrm{rk}\,\mathrm{NS}_\rho(Y).$$

The restriction map

$$\Phi : \mathrm{NS}(Y)/\mathrm{NS}_\rho(Y) \to \mathrm{NS}(Y_w),$$

is injective; this follows from [KMM87, Lemma 3.2.5]. Note that ρ is an isotrivial family. Hence we have

$$b(Y, L|_Y) \leq b(Y_w, L|_{Y_w}) \leq b(X, L).$$

If L is proportional to $-K_X$, then the last inequality is a strict inequality. ■

We can also analyze the balanced condition with respect to hypersurfaces:

Proposition 3.5. *Let X be a smooth Fano variety of Picard rank one and $Y \subset X$ an irreducible smooth effective divisor. Then $-K_X$ is balanced with respect to Y.* □

Proof. For smooth divisors $Y \subset X$ the claim follows from adjunction formula:

$$-K_X|_Y + K_Y \in \Lambda_{\mathrm{eff}}(Y)^\circ,$$

because Y is an ample divisor on X. Thus we obtain

$$a(Y, -K_X|_Y) < a(X, -K_X) = 1.$$

■

However, this may fail when Y is singular:

Example 3.6 (Mukai-Umemura 3-folds, [MU83])**.** Consider the standard action of SL_2 on $V = \mathbb{C}x \oplus \mathbb{C}y$. Let $R_{12} = \mathrm{Sym}^{12}(V)$ be a space of homogeneous polynomials of degree 12 in two variables and $f \in R$ a form with distinct roots. Let X be the Zariski closure of the SL_2-orbit $\mathrm{SL}_2 \cdot [f] \subset \mathbb{P}(R_{12})$. Then X is a smooth Fano 3-fold of index 1 with $\mathrm{Pic}(X) = \mathbb{Z}$, for some special f. The complement of the open orbit $\mathrm{SL}_2 \cdot [f]$ is an irreducible divisor

$$D = \overline{\mathrm{SL}_2 \cdot [x^{11}y]} = \mathrm{SL}_2 \cdot [x^{11}y] \cup \mathrm{SL}_2 \cdot [x^{12}],$$

a hyperplane section on $\mathbb{P}(R_{12})$ whose class generates $\mathrm{Pic}(X)$. Furthermore, D is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ by a linear series of bidegree $(11, 1)$, which is injective, an open immersion outside of the diagonal, but not along the diagonal. In particular, D is singular along the diagonal.

Let $\beta : \tilde{D} \to D$ be the normalization of D which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then $-\beta^* K_X|_{\tilde{D}}$ is a line bundle of bidegree $(11, 1)$ so that

$$a(D, -K_X|_D) = a(\tilde{D}, -\beta^* K_X|_{\tilde{D}}) = 2 > 1 = a(X, -K_X).$$

Thus Proposition 3.5 does not hold for D. □

Remark 3.7. The authors do not know whether Proposition 3.5 holds for singular surfaces in $\mathbb{P}^3$. It is quite interesting to see whether the balanced property holds for very singular rational surfaces in $\mathbb{P}^3$. □

For Fano varieties of index one, one might hope to use the Fujita invariant $a(X, L)$ to identify the exceptional locus $X \setminus X^\circ$. However, this is quite non-trivial even in the following situation, considered in [Deb03] (see also [LT10] and [Beh06]):

Conjecture 3.8 (Debarre - de Jong conjecture)**.** Let $X \subset \mathbb{P}^n$ be a Fano hypersurface of degree $d \leq n$. Then the dimension of the variety of lines is $2n - d - 3$. In particular, when $d = n$, for any line C, we have

$$a(C, -K_X|_C) = 2 > 1 = a(X, -K_X).$$

The conjecture predicts that the dimension of the variety of lines is $n - 3$ so that lines will not sweep out X. □

Some of simplest examples of Fano threefolds fail to be balanced.

Example 3.9. Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold, which is Fano of index 2. The Picard group of X is generated by the hyperplane class L. By Proposition 3.5, $-K_X$ is balanced with respect to every smooth divisor on X. Let $Y \subset X$ be a line. Note that $2L$ restricts to the anticanonical class on X and Y, and $b(Y, L|_Y) = b(X, L) = 1$. Thus $-K_X$ is weakly balanced, but not balanced, with respect to Y. Since the family of lines dominates X, $-K_X$ is not balanced on X. □

Remark 3.10. However, assume that X is defined over a number field. The family of lines dominating X are surfaces of general type, which embeded into their Albanese varieties. By Faltings' theorem, lines defined over a fixed number field lie on a proper subvariety and cannot dominate X. □

One can rephrase the above example in the following way: the anticanonical line bundle of a smooth cubic threefold is not geometrically balanced, but arithmetically balanced. Next example shows that it is also possible that the anticanonical line bundle is not even arithmetically balanced.

Example 3.11. Let $X \subset \mathbb{P}^5$ denote a smooth complete intersection of two quadrics. The anticanonical class $-K_X = 2L$ where L is the hyperplane class, which generates the Picard group. The variety A parametrizing lines $Y \subset X$ is an abelian surface [GH78, p. 779]. Four lines pass through a generic point $x \in X$ [GH78, p. 781], so these lines dominate X. We have $a(X, L) = a(Y, L|_Y) = 2$ and $b(X, L) = b(Y, L|_Y) = 1$ so X is not balanced.

Suppose that X is defined over a number field F with $X(F)$ Zariski dense; fix a metrization $\mathcal{L}$ of L. Manin's formalism predicts the existence of an open set $X^\circ \subset X$ such that

$$\#\{x \in X^\circ(F) : \mathsf{H}_{\mathcal{L}}(x) \leq \mathsf{B}\} \sim c\mathsf{B}^2.$$

However, each line $Y \subset X$ defined over F contributes

$$\#\{x \in Y(F) : \mathsf{H}_{\mathcal{L}}(x) \leq \mathsf{B}\} \sim c'(Y, \mathcal{L})\mathsf{B}^2,$$

where $c'(Y, \mathcal{L})$ is a Tamagawa type number associated to Y and $\mathcal{L}$. Moreover, after replacing F by a suitable finite extension these lines are Zariski dense in X, because rational points on abelian surfaces are potentially dense (see [HT00b, §3], for instance). □

The weakly balanced property may fail too:

Example 3.12. [BT96b] Let f, g be general cubic forms on $\mathbb{P}^3$ and

$$X := \{sf + tg = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^3,$$

the Fano threefold obtained by blowing up the base locus of the pencil. The projection onto the first factor exhibits a cubic surface fibration

$$\pi : X \to \mathbb{P}^1,$$

so that $-K_X$ restricts to $-K_Y$, for every smooth fiber Y of π. Thus

$$a(Y, -K_Y) = a(X, -K_X) = 1.$$

Furthermore, the Néron-Severi rank of a smooth fiber of π is 7. On the other hand, by the Lefschetz theorem, we have $\mathrm{rk}\,\mathrm{NS}(X) = 2$ and

$$7 = b(Y, -K_Y) > b(X, -K_X) = 2,$$

i.e., $-K_X$ is not weakly balanced on X.

Let Z be the union of singular fibers of π, exceptional curves in general smooth fibers Y, and the exceptional locus of the blow up to $\mathbb{P}^3$. Note that $-K_X$ is balanced with respect to every rational curve on X which is not contained in Z. □

4 del Pezzo surfaces

Let X be a smooth projective surface with ample $-K_X$, i.e., a del Pezzo surface. Their deformation types are determined by the degree of the canonical class $d := (K_X, K_X)$. Basic examples are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$; more examples are obtained by blowing up $9 - d$ general points on $\mathbb{P}^2$. We have

- $\mathrm{rk}\,\mathrm{NS}(X) = 10 - d$;
- for $1 \leq d \leq 7$ the cone $\Lambda_{\mathrm{eff}}(X)$ is generated by classes of exceptional curves, i.e., smooth rational curves of self-intersection -1.

Let L be a big Cartier divisor on X. When is it balanced? The only subvarieties of X on which we need to test the values of a and b are rational curves $C \subset X$, and $b(C, L|_C) = 1$.

It is easy to characterize curves breaking the balanced condition for the Fujita invariant. Let Z be the union of exceptional curves, if $d > 1$. Here, exceptional means in traditional sense. When $d = 1$, let Z be the union of exceptional curves plus singular rational curves in $|-K_X|$.

Lemma 4.1. *Let X be a del Pezzo surface of degree d, C an irreducible rational curve with $(C, C) \neq -1$, and L a big Cartier divisor on X. Then*

$$a(C, L|_C) \leq a(X, L), \tag{3}$$

i.e., L is weakly balanced on X outside of Z. □

Proof. If C' is an irreducible rational curve with $(-K_X, C') = 1$ then $C' \subseteq Z$. Indeed, if $(C', C') < 0$, then $(C', C') = -1$, by adjunction, and C' is exceptional. On the other hand, if C' and $-K_X$ are linearly independent, the Hodge index theorem implies that $d(C', C') - 1 < 0$, i.e., $(C', C') = -1$ or 0. The second case is impossible since $(K_X, C') + (C', C')$ must be even. If C' and $-K_X$ are linearly dependent, then $d(C', C') - 1 = 0$ so that $d = 1$ and C' is a singular rational curve in $|-K_X|$.

Let $C'' \subset X$ be a rational curve which is not in Z. After rescaling, we may assume that $a(X, L) = 1$, in particular, we do not assume that L is an integral divisor. Writing $L + K_X \sim D$, where D is an effective $\mathbb{Q}$-divisor, and computing the intersection with C'' we obtain

$$(L, C'') = (-K_X, C'') + (D, C'') \geq (-K_X, C'').$$

Since C'' is not in Z, $(-K_X, C'') \geq 2$, i.e., $(L, C'') \geq 2$. It follows that

$$(L, C'') + \deg(K_{\tilde{C}''}) = (L, C'') - 2 \geq 0,$$

where $\tilde{C}''$ is the normalization of C'', i.e., $a(C'', L|_{C''}) \leq 1$, as claimed. ■

We proceed with a characterization of $b(X, L)$. Consider the Zariski decomposition

$$a(X, L)L + K_X = P + E,$$

where P is a nef $\mathbb{Q}$-divisor and $E = \sum_{i=1}^n e_i E_i$, $e_i \in \mathbb{Q}_{>0}$, $(E_i, E_j) < 0$. We have $(P, E) = 0$. Since X is a Mori dream space, P is semi-ample and defines a semi-ample fibration

$$\pi : X \to B.$$

We have two cases:

Case 1. B is a point. Then

$$a(X, L)L + K_X = \sum_{i=1}^n e_i E_i$$

is rigid, and the classes E_i are linearly independent in $\mathrm{NS}(X)$. In particular, $\oplus_i \mathbb{R}_{\geq 0} E_i$ is an extremal face of $\Lambda_{\mathrm{eff}}(X)$, and in fact the minimal extremal face containing $a(X, L)L + K_X$. It follows that

$$b(X, L) = \mathrm{rk}\, \mathrm{NS}(X) - n.$$

Case 2. B is a smooth rational curve. Then the minimal extremal face containing

$$a(X, L)L + K_X = P + \sum_{i=1}^n e_i E_i$$

is given by

$$\mathrm{NS}_\pi(X) \cap \Lambda_{\mathrm{eff}}(X) = \{P = 0\} \cap \Lambda_{\mathrm{eff}}(X),$$

where $\mathrm{NS}_\pi(X) \subset \mathrm{NS}(X)$ is the subspace generated by vertical divisors, i.e., divisors $D \subset X$ not dominating B. It follows that

$$b(X, L) = \operatorname{rk} \mathrm{NS}(X) - \operatorname{rk} \mathrm{NS}_\pi(X) = 1.$$

Proposition 4.2. *Let X be a del Pezzo surface and L a big Cartier divisor on X. Then L is balanced if and only if $a(X, L)L + K_X \sim D$, where D is a rigid effective divisor.* □

Proof. Assume that $a(X, L) = 1$. In *Case 1*, we must have

$$L + K_X = D = \sum_{i=1}^{n} e_i E_i, \quad e_i > 0,$$

with E_i disjoint exceptional curves. Assume that L is not balanced so that $b(X, L) = 1$. Let $\pi : X \to \mathbb{P}^2$ be the blowdown of $E_1, \ldots, E_n$ and h a hyperplane class on $\mathbb{P}^2$. Then

$$L = -K_X + D = 3\pi^* h + \sum_{i=1}^{n} (e_i - 1)E_i.$$

Let C be an irreducible rational curve which is not in Z. If C does not meet any of the E_i then

$$(L, C) = (3\pi^* h, C) \geq 3 > 2.$$

If C meets at least one of the E_i then

$$(L, C) = (-K_X, C) + (D, C) > 2$$

since the first summand is ≥ 2. It follows that $a(C, L_C) < 1$, i.e., L is balanced, contradicting our assumption.

In *Case 2*, we have

$$L + K_X = D = P + \sum_{i=1}^{n} e_i E_i, \quad e_i \geq 0,$$

where P is nef and E_i are disjoint exceptional divisors. Let $\pi : X \to \mathbb{P}^1$ be the fibration induced by the semi-ample line bundle P. The general fiber F of π is a conic and

$$\operatorname{rk} \mathrm{NS}(X) - \operatorname{rk} \mathrm{NS}_\pi(X) = 1.$$

We have $(F, F) = 0$, $(-K_X, F) = 2$, and the class of F is proportional to P. Hence, for any such F,

$$a(F, L|_F) = a(X, L), \quad b(F, L|_F) = b(X, L) = 1.$$

Thus L is not balanced. ■

5 Equivariant geometry

Let G be a connected linear algebraic group, $H \subset G$ a closed subgroup, and X a projective equivariant compactification of $X^\circ := H\backslash G$, a quasi-projective variety [Bor91, Ch. II]. Applying equivariant resolution of singularities we may assume that X is smooth and the boundary

$$\cup_{\alpha\in\mathcal{A}} D_\alpha = X \setminus X^\circ$$

is a divisor with normal crossings with irreducible components D_α. If H is a parabolic subgroup of a semi-simple group G, then there is no boundary, i.e., $\mathcal{A}$ is empty, and $H\backslash G$ is a generalized flag variety which was discussed in Section 3. Throughout, we will assume that $\mathcal{A}$ is not empty.

Let $\mathfrak{X}(G)^*$ be the group of algebraic characters of G and

$$\mathfrak{X}(G,H)^* = \{\, \chi : G \to \mathbb{G}_m \,|\, \chi(hg) = \chi(g), \quad \forall h \in H \,\}$$

the subgroup of characters whose restrictions to H are trivial. Let $\mathrm{Pic}^G(X)$ be the group of isomorphism classes of G-linearized line bundles on X and $\mathrm{Pic}(X)$ the Picard group of X. For $L \in \mathrm{Pic}^G(X)$, the subgroup $H \subset G$ acts linearly on the fiber L_x at $x = H \in H\backslash G$. This defines a homomorphism

$$\mathrm{Pic}^G(X) \to \mathfrak{X}(H)^*$$

to characters of H. Let $\mathrm{Pic}^{(G,H)}(X)$ be the kernel of this map. We will identify line bundles and divisors with their classes in $\mathrm{Pic}(X)$.

Proposition 5.1. *Let G be a connected linear algebraic group and H a closed subgroup of G. Let X be a smooth projective equivariant compactification of $X^\circ := H\backslash G$ with a boundary $\cup_{\alpha\in\mathcal{A}} D_\alpha$. Then*

1. *we have an exact sequence*

$$0 \to \mathfrak{X}(G,H)^* \to \oplus_{\alpha\in\mathcal{A}} \mathbb{Z} D_\alpha \to \mathrm{Pic}(X) \to \mathrm{Pic}(X^\circ) \to 0;$$

2. *we have an exact sequence*

$$0 \to \mathfrak{X}(G,H)^*_{\mathbb{Q}} \to \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}};$$

and the last homomorphism is surjective when

$$\mathfrak{C}(G,H) := \mathrm{Coker}(\mathfrak{X}(G)^* \to \mathfrak{X}(H)^*)$$

or equivalently, $\mathrm{Pic}(X^\circ)$*, is finite.*

3. *we have a canonical injective homomorphism*

$$\Phi : \oplus_{\alpha\in\mathcal{A}}\mathbb{Q}D_\alpha \hookrightarrow \mathrm{Pic}^{(G,H)}(X)_{\mathbb{Q}};$$

which is an isomorphism when $\mathfrak{C}(G,H)$ *is finite.*

□

Proof. The first statement is easy. The second assertion follows from [MFK94, Corollary 1.6] and [KKV89, Proposition 3.2(i)].

For the last assertion: Corollary 1.6 of [MFK94] implies that some multiple of D_α is G-linearizable. After replacing D_α by a multiple of D_α, we may assume that G acts on the finite-dimensional vector space $\mathrm{H}^0(X,\mathcal{O}_X(D_\alpha))$, via this G-linearization. Let s_α be the section corresponding to D_α. Then $s_\alpha\in\mathrm{H}^0(X,\mathcal{O}_X(D_\alpha))$ is an eigenvector of the action by G. After multiplying by a character of G, if necessary, we may assume that s_α is fixed by the action of G. We let $\Phi(D_\alpha)$ be this G-linearization.

Suppose that $\Phi(\sum_\alpha d_\alpha D_\alpha)=\mathcal{O}_X$, with trivial G-linearization, where $d_\alpha\in\mathbb{Z}$. Then there exists a rational function f such that

$$\mathrm{div}(f)=\sum_\alpha d_\alpha D_\alpha.$$

We may assume that f is a character of G whose restriction to H is trivial. By the definition of Φ, the function f must be fixed by the G-linearization. This implies that $f\equiv 1$. When $\mathfrak{C}(G,H)$ is finite, the surjectivity of Φ follows from (1) and (2). ■

From now on we consider the following situation: let $H\subset M\subset G$ be connected linear algebraic groups. Typical examples arise when G is a unipotent group or a product of absolutely simple groups and H and M are arbitrary subgroups such that $H\backslash M$ is connected. Let X be a smooth projective G-equivariant compactification of $H\backslash G$, and Y the induced compactification of $H\backslash M$.

Lemma 5.2. *Let* $\pi : X\to X'$ *be a* G*-equivariant morphism onto a projective equivariant compactification of* $M\backslash G$*. Assume that the projection* $G\to M\backslash G$ *admits a rational section. Then*

- $\pi(D_\alpha)=X'$ *if and only if* $D_\alpha\cap Y\neq\emptyset$*;*
- *if* $D_\alpha\cap Y\neq\emptyset$ *then* $D_\alpha\cap Y$ *is irreducible;*
- *if* $D_\alpha\cap Y\neq\emptyset$ *and* $D_{\alpha'}\cap Y\neq\emptyset$*, for* $\alpha\neq\alpha'$ *then* $D_\alpha\cap Y\neq D_{\alpha'}\cap Y$*.*

□

Proof. We have the diagram

$$\begin{array}{ccccc} H\backslash G & \subset & X & \supset & Y \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M\backslash G & \subset & X' & \supset & M \cdot e = \text{point} \end{array}$$

The first claim is evident. To prove the second assertion, choose a rational section $\sigma : M\backslash G \dashrightarrow G$ of the projection $G \to M\backslash G$. We may assume that a rational section is well-defined at a point $M \in M\backslash G$. Consider the diagram

$$\begin{array}{ccc} D_\alpha & \supset & D_\alpha^\circ \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X' & \supset & M\backslash G \end{array}$$

where $D_\alpha^\circ = D_\alpha \cap \pi^{-1}(M\backslash G)$. We define a rational map

$$\begin{array}{rcl} \Psi : D_\alpha^\circ & \dashrightarrow & D_\alpha \cap Y \\ x & \mapsto & x \cdot (\sigma \circ \pi(x))^{-1}. \end{array}$$

Since Ψ is dominant, $D_\alpha \cap Y$ is irreducible. Since the G-orbit of $D_\alpha \cap Y$ is D_α°, the third claim follows. ■

Remark 5.3. When M is a connected solvable group, then G is birationally isomorphic to $M \times (M\backslash G)$ so that the projection $G \to M\backslash G$ has a rational section. See [Bor91, Corollary 15.8]. □

Theorem 5.4. *Let*

$$H \subset M \subset G$$

be connected linear algebraic groups. Let X be a smooth projective G-equivariant compactification of $H\backslash G$ and $Y \subset X$ the induced compactification of $H\backslash M$. Let L be a big Cartier divisor on X. Assume that

- *the projection $G \to M\backslash G$ admits a rational section;*
- *and $a(X, L)L + K_X$ is linearly equivalent to a rigid effective $\mathbb{Q}$-divisor D.*

Furthermore, assume that either

1. *$\Lambda_{\mathrm{eff}}(X)$ is finitely generated by effective divisors; or*
2. *there exists a birational contraction map $f : X \dashrightarrow Z$ contracting D, where Z is a normal projective variety.*

Then L is balanced with respect to Y. □

Proof. Let X' be any smooth projective equivariant compactification of $M\backslash G$. We consider a G-rational map $\pi : X \dashrightarrow X'$ mapping

$$\pi : G \ni g \mapsto Mg \in M\backslash G.$$

After applying a G-equivariant resolution of the indeterminacy of the projection π if necessary, we may assume that π is a surjective morphism and X is a smooth equivariant compactification of $H\backslash G$ with a boundary divisor $\cup_\alpha D_\alpha$. Note that Y is a general fiber of π so that Y is smooth. Write the rigid effective $\mathbb{Q}$-divisor $D = a(X, L)L + K_X$ by

$$D = a(X, L)L + K_X = \sum_{i=1}^{n} e_i E_i,$$

where E_i's are irreducible components of $a(X, L)L + K_X$ and $e_i \in \mathbb{Q}_{>0}$. Our goal is to show that

$$(a(Y, L|_Y), b(Y, L|_Y)) < (a(X, L), b(X, L)).$$

Since the E_i's are rigid effective divisors, they are boundary components. This implies that

$$a(X, L)L|_Y + K_Y = (a(X, L)L + K_X)|_Y = \sum_{i=1}^{n} e_i E_i|_Y \in \Lambda_{\mathrm{eff}}(Y).$$

It follows that

$$a(Y, L|_Y) \leq a(X, L).$$

Assume that $a(Y, L|_Y) = a(X, L) =: a$. Let F be the minimal supported face of $\Lambda_{\mathrm{eff}}(X)$ containing $D = aL + K_X = \sum e_i E_i$ and V_F a vector subspace generated by F. Either condition (1) or (2) guarantees that F is generated by E_i's so that

$$b(X, L) = \operatorname{rk} \mathrm{NS}(X) - n.$$

(See Proposition 2.10 and Example 2.12.) Let F' be the minimal supported face of $\Lambda_{\mathrm{eff}}(Y)$ containing

$$D|_Y = aL|_Y + K_Y = \sum e_i E_i|_Y.$$

Let V' be a vector subspace generated by all components of $E_i \cap Y$. Since F' contains all components of $E_i \cap Y$, we have $b(Y, L|_Y) \leq \operatorname{codim}(V')$. Consider the restriction map:

$$\Phi : \mathrm{NS}(X)/V_F \to \mathrm{NS}(Y)/V'.$$

It follows from [KKV89, Proposition 3.2(i)], Lemma 5.2, and the exact sequence (1) in Proposition 5.1 that Φ is surjective. On the other hand, $\pi^*\mathrm{NS}(X')$ is contained in the kernel of Φ, so Φ has the nontrivial kernel. We conclude that

$$b(Y, L|_Y) \leq \operatorname{codim}(V') < b(X, L).$$

■

Remark 5.5. Conditions (1) or (2) can be replaced by the condition: the numerical dimension $\nu(D)$ is zero (see [Leh13] for definitions). Note that (2) implies that the numerical dimension $\nu(D)$ is zero. $\square$

Corollary 5.6. *Let $H \subset M \subset G$ be connected linear algebraic groups and X a smooth projective equivariant compactification of $H\backslash G$. Let $Y \subset X$ be the induced compactification of $H\backslash M$. Assume that the projection $G \to M\backslash G$ admits a rational section. Then $-K_X$ is balanced with respect to Y.* $\square$

Note that $-K_X$ is necessarily big by [FZ13, Thm. 1.2].

The existence of rational sections is important, and the second statement in Lemma 5.2 is not true in general:

Example 5.7. Consider the standard action of PGL_3 on $\mathbb{P}^2$. Let $\mathbb{P}^5$ be the space of conics and consider

$$X^\circ = \{T = (C, [p_1, p_2, p_3]) \in \mathbb{P}^5 \times \mathrm{Hilb}^{[3]}(\mathbb{P}^2) \mid T \text{ satisfies } (*)\},$$

$$(*) : C \text{ is smooth, } p_i\text{'s are distinct, and } p_i \in C.$$

Let X be the Zariski closure of X°, it is the Hilbert scheme of conics with zero dimensional subschemes of length 3, and is a smooth equivariant compactification of a homogeneous space $S_3\backslash\mathrm{PGL}_3$. Consider a $\mathbb{P}^2$-fibration $f : X \to \mathrm{Hilb}^{[3]}(\mathbb{P}^2)$, the fiber over a general point $Z \in \mathrm{Hilb}^{[3]}(\mathbb{P}^2)$ is a $\mathbb{P}^2$, parametrizing conics passing through Z. On $f^{-1}(Z)$, degenerate conics correspond to two lines passing through Z; these form three boundary components l_i on $\mathbb{P}^2$. However, general points on these components are on the same PGL_3-orbit, so there exists an irreducible boundary divisor $D \subset X$ such that $D \cap f^{-1}(Z) = l_1 \cup l_2 \cup l_3$. In other words, there is a non-trivial monodromy action on l_i's. However, the monodromy action on the Picard group is trivial, and the balanced property still holds with respect these fibers. $\square$

6 Toric varieties

Manin's conjecture for toric varieties was settled by Batyrev and Tschinkel via harmonic analysis on the associated adele groups in [BT98a] and [BT96a]. Implicitly, [BT96a] established a version of the balanced property. Here, we will use MMP to determine balanced line bundles. We expect that these techniques would also be applicable to some non-equivariant varieties. We refer to [FS04] for details concerning toric Mori theory, though most of properties we use hold formally for Mori dream spaces.

We start by recalling basic facts regarding toric Mori theory, which also hold for all Mori dream spaces (see [HK00, Section 1]):

Proposition 6.1 (D-Minimal Model Program)**.** *Let X be a $\mathbb{Q}$-factorial projective toric variety and D a $\mathbb{Q}$-divisor. Then the minimal model program with respect to D runs, i.e.,*

1. *for any extremal ray R of $\mathrm{NE}_1(X)$, there exists the contraction morphism φ_R;*

2. *for any small contraction φ_R of a D-negative extremal ray R, the D-flip $\psi : X \dashrightarrow X^+$ exists;*
3. *any sequence of D-flips terminates in finite steps;*
4. *and every nef line bundle is semi-ample.*

□

Proof. See Theorem 4.5, Theorem 4.8, Theorem 4.9, and Proposition 4.6 in [FS04]. ■

Proposition 6.2 (Zariski decomposition)**.** *Let X be a $\mathbb{Q}$-factorial projective toric variety and D a $\mathbb{Q}$-effective divisor. Applying D-MMP we obtain a birational contraction map $f : X \dashrightarrow X'$, with nef proper transform D' of D. Consider a common resolution:*

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\mu"'] \arrow[d, "\nu"] \\ X \arrow[r, dashed, "f"'] & X' \end{tikzcd}$$

Then

1. *$\mu^* D = \nu^* D' + E$, where E is a ν-exceptional effective $\mathbb{Q}$-divisor;*
2. *the support of E contains all divisors contracted by f;*
3. *if $g : \tilde{X} \to Y$ is the semi-ample fibration associated to $\nu^* D'$, then for any ν-exceptional effective Cartier divisor E', the natural map*
$$\mathcal{O}_Y \to g_* \mathcal{O}(E'),$$
is an isomorphism.

□

Proof. The assertions (1) and (2) follow from the Negativity lemma (see [FS04, Lemma 4.10]). Also see [FS04, Theorem 5.4]. ■

The invariant $b(X, L)$ can be characterized in terms of Zariski decomposition of $a(X, L)L + K_X$:

Proposition 6.3. *Let X be a $\mathbb{Q}$-factorial projective toric variety and D an effective $\mathbb{Q}$-divisor on X. Suppose that*

1. *$D = P + N$, where P is a nef divisor and $N \geq 0$;*
2. *let $g : X \to Y$ be the semi-ample fibration associated to P. For any effective Cartier divisor E which is supported by* Supp(N), *the natural map*
$$\mathcal{O}_Y \to g_* \mathcal{O}(E),$$
is an isomorphism.

Then the minimal extremal face of $\Lambda_{\mathrm{eff}}(X)$ containing D is generated by vertical divisors of g and components of N. □

Proof. When D is big, the assertion is trivial. We may assume that $\dim(Y) < \dim(X)$. Let F be the minimal extremal face of $\Lambda_{\mathrm{eff}}(X)$ containing D. Since F is extremal, it follows that F contains all vertical divisors of g and components of N.

On the other hand, our assumption implies that for general fiber X_y, $N|_{X_y}$ is a rigid divisor on X_y (see [Leh14, Theorem 6.1]), and its irreducible components generate an extremal face F' of $\Lambda_{\mathrm{eff}}(X_y)$. Let $\alpha \in \overline{\mathrm{NM}}_1(X_y)$ be a nef cycle supporting F' and $F_\alpha := \{\alpha = 0\} \cap \Lambda_{\mathrm{eff}}(X)$. Since X_y is a general fiber, $\alpha \in \overline{\mathrm{NM}}_1(X)$ so that F_α is an extremal face (see [Pet12, Theorem 6.8]). Since $D \cdot \alpha = 0$ and F is minimal we have $F \subset F_\alpha$. Let $D' \in F$ be an effective $\mathbb{Q}$-divisor. D' is linear equivalent to a torus-invariant effective $\mathbb{Q}$-divisor, so we may assume that D' is a boundary divisor. Let $T \subset Y$ be the big torus of Y. Then $g^{-1}(T) \cong X_y \times T$. Since $D' \cdot \alpha = 0$, D' is a sum of vertical divisors of g and components of N; our assertion follows. ■

Remark 6.4. Note that since the cone of pseudo-effective divisors is finitely generated, every extremal face is a supported face. □

Proposition 6.5. *Let X be a projective toric variety and Y an equivariant compactification of a subtorus of codimension one (possibly singular). Let L be a big Cartier divisor on X. Then L is weakly balanced with respect to Y.* □

Proof. Let M be the class of $\mathcal{O}_X(Y)$. Applying an equivariant embedded resolution of singularities, if necessary, we may assume that X and Y are smooth or at least $\mathbb{Q}$-factorial terminal. Due to a group action of a torus, Y is not rigid, so that

$$a(X,L)L|_Y + K_Y = (a(X,L)L + K_X)|_Y + M|_Y \in \Lambda_{\mathrm{eff}}(Y).$$

Note that $a(X,L)L + K_X$ is an effective $\mathbb{Q}$-divisor on X. Thus we have

$$a(Y, L|_Y) \leq a(X,L).$$

Suppose that $a(Y, L|_Y) = a(X,L) =: a$. Let $D = aL + K_X + Y$ and consider the Zariski decomposition of D:

$$\begin{tikzcd} & \tilde{X} \supset \tilde{Y} \arrow[dl, "\mu"'] \arrow[d, "\nu"] \\ D \subset X \arrow[r, dashed, "f"'] & X' \supset D', \end{tikzcd}$$

where D' is the strict transform of D, which is nef, and $\tilde{Y}$ is the strict transform of Y. We may assume that both $\tilde{X}$ and $\tilde{Y}$ are smooth. Let F be the minimal extremal face of $\Lambda_{\mathrm{eff}}(\tilde{X})$ containing

$$a\mu^*L + K_{\tilde{X}} + \tilde{Y}.$$

Since $a\mu^*L + K_{\tilde{X}} \in F$, it follows that $\operatorname{codim}(F) \leq b(X, L)$. Since X has only terminal singularities, we have

$$a\mu^*L + K_{\tilde{X}} + \tilde{Y} = a\mu^*L + \mu^*K_X + \sum_i d_i E_i + \tilde{Y},$$

where d_i's are positive integers and E_i's are μ-exceptional divisors. It follows that F is the minimal extremal face containing μ^*D and all μ-exceptional divisors. Let $g : \tilde{X} \to B$ be the semi-ample fibration associated to ν^*D'. Note that $\dim(B) < \dim(\tilde{X})$ since D is not big. Proposition 6.3 implies that F is generated by all vertical divisors of g and all ν-exceptional divisors. We denote the vector space, generated by F, by V_F.

Let F' be the minimal extremal face of $\Lambda_{\mathrm{eff}}(\tilde{Y})$ containing

$$a\mu^*L|_{\tilde{Y}} + K_{\tilde{Y}} = (a\mu^*L + K_{\tilde{X}} + \tilde{M})|_{\tilde{Y}},$$

where $\tilde{M}$ be the class of $\mathcal{O}_{\tilde{X}}(\tilde{Y})$. Then F' is also the minimal extremal face containing $\mu^*D|_{\tilde{Y}}$ and all components of $(E_i \cap \tilde{Y})$'s so that F' is the minimal extremal face containing $\nu^*D'|_{\tilde{Y}}$ and all components of $(G_j \cap \tilde{Y})$'s, where G_j's are all ν-exceptional divisors. In particular, F' contains all vertical divisors of $g|_{\tilde{Y}} : \tilde{Y} \to H = g(\tilde{Y})$. Since a multiple of ν^*D' admits a section vanishing along $\tilde{Y}$, H is a Weil divisor of B, which is a subtoric variety. Let $V' \subset \mathrm{NS}(\tilde{Y})$ be a vector space generated by vertical divisors of $g|_{\tilde{Y}}$ and components of $(G_j \cap \tilde{Y})$'s. Then $b(Y, L) \leq \operatorname{codim}(V')$. Consider the following restriction map:

$$\Phi : \mathrm{NS}(\tilde{X})/V_F \to \mathrm{NS}(\tilde{Y})/V'.$$

We claim that Φ is surjective. Let N be an irreducible component of the boundary divisor of $\tilde{Y}$ which dominates H. There exists an irreducible component N' of the boundary divisor of $\tilde{X}$ such that N' contains N. Since N meets with a general fiber of $g|_{\tilde{Y}}$, N' also meets with a general fiber of g. This implies that N' dominates B. As in the proof of Lemma 5.2,

$$N' \cap \tilde{Y} = mN + (\text{vertical divisors of } g|_{\tilde{Y}}).$$

Our claim follows from this. Hence

$$b(Y, L) \leq \operatorname{codim}(V') \leq \operatorname{codim}(V_F) \leq b(X, L).$$

■

Proposition 6.6. *Let X be a $\mathbb{Q}$-factorial terminal projective toric variety and L a big Cartier divisor on X. Suppose that the positive part of Zariski decomposition of $D := a(X, L)L + K_X$ is nontrivial. Then L is not balanced.* □

Proof. After blowing up, if necessary, we may assume that D itself admits a Zariski decomposition

$$D = P + N,$$

where P is a nef $\mathbb{Q}$-divisor and $N \geq 0$ is the negative part. Let $g : X \to Y$ be the semi-ample fibration associated to P. We consider a general fiber X_y of g. Since $a(X,L)L|_{X_y} + K_{X_y} = N|_{X_y}$ is a rigid effective divisor, we conclude that $a(X,L) = a(X_y, L|_{X_y})$. Let $V \subset \mathrm{NS}(X)$ be the vector space generated by vertical divisors of g and components of N and $V' \subset \mathrm{NS}(X_y)$ the vector space generated by components of $N|_{X_y}$. The restriction map

$$\Phi : \mathrm{NS}(X)/V \to \mathrm{NS}(X_y)/V',$$

is surjective, by Lemma 5.2. On the other hand, let T be the big torus of Y. Then the preimage $g^{-1}(T)$ of T is a product of T and a general fiber X_y. It follows that Φ is injective. Thus we have

$$b(X,L) = b(X_y, L|_{X_y}).$$

Hence L is not balanced on X. ■

An alternative proof of Theorem 5.4 for toric varieties is provided below:

Proposition 6.7. *Let X be a $\mathbb{Q}$-factorial terminal projective toric variety, L a big Cartier divisor on X, and Y an equivariant compactification of a subtorus of codimension one (possibly singular). Suppose that the positive part of the Zariski decomposition of $a(X,L)L + K_X$ is trivial. Then L is balanced with respect to Y.* □

Proof. We follow the notations in the proof of Proposition 6.5. We only need to explain why $b(Y, L|_Y) < b(X,L)$, when $a(Y, L|_Y) = a(X,L)$. Since $a\mu^*L + K_{\tilde{X}}$ is rigid, it follows that

$$\mathrm{codim}(V_F) < b(X,L).$$

Thus our assertion follows. ■

Corollary 6.8. *Let X be a $\mathbb{Q}$-factorial terminal projective toric variety. A big Cartier divisor L is balanced with respect to all toric subvarieties if and only if $a(X,L)L + K_X$ is rigid.* □

In general, it is possible that for a codimension one subtoric variety $Y \subset X$, the rank of $\mathrm{NS}(Y)$ exceeds the rank of $\mathrm{NS}(X)$. However, the balanced property still can be verified:

Example 6.9. Consider the standard action of $\mathbb{G}_m^3 = \{(t_0, t_1, t_2)\}$ on $\mathbb{P}^3$ by

$$(t_0, t_1, t_2) \cdot (x_0 : x_1 : x_2 : x_3) \mapsto (t_0 x_0 : t_1 x_1 : t_2 x_2 : x_3).$$

Consider the subtorus

$$M = \{(t_0, t_1, (t_0t_1)^{-1})\} \subset \mathbb{G}_m^3,$$

and let S be the equivariant compactification of M defined by

$$x_0x_1x_2 = x_3^3.$$

This is a singular cubic surface with three isolated singularities of type A_2. We denote them by $p_1, p_2, p_3 \in \mathbb{P}^3$. Since they are fixed under the action of $\mathbb{G}_m^3$ on $\mathbb{P}^3$, the blow-up $B := \mathrm{Bl}_{p_1,p_2,p_3}(\mathbb{P}^3)$ is an equivariant compactification of $\mathbb{G}_m^3$. Moreover, the closure $\tilde{S}$ of M in B is the minimal desingularization of S and the class of $\tilde{S}$ in $\mathrm{Pic}(B)$ is ample. Put $X := B \times \mathbb{P}^1$ and $Y := \tilde{S} \times \mathbb{P}^1$. We have a diagram

$$\begin{tikzcd} X \arrow[d, "\pi"'] & Y \arrow[l, hook'] \arrow[d] \\ B & \tilde{S}, \arrow[l, hook'] \end{tikzcd}$$

Then Y is a nef divisor, and we have

$$\mathrm{rk}\,\mathrm{NS}(Y) = 8 > \mathrm{rk}\,\mathrm{NS}(X) = 5.$$

However, the anticanonical class $-K_X$ is still balanced with respect to Y since

$$a(Y, -K_X|_Y) = a(X, -K_X) = 1$$
$$b(Y, -K_X|_Y) = 1 < b(X, -K_X) = 5.$$

This shows that, in general, we cannot expect to control the subgroup of $\mathrm{NS}(X)$ generated by vertical divisors. In the proof of Proposition 6.5, we were able to control the *quotient* by this subgroup. □

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References

[Abr09] D. Abramovich. Birational geometry for number theorists. In *Arithmetic geometry*, volume 8 of *Clay Math. Proc.*, pages 335–373. Amer. Math. Soc., Providence, RI, 2009.

[BCHM10] C. Birkar, P. Cascini, Chr. D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Paun, and Thomas Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. *J. Algebraic Geom.*, 22(2):201–248, 2013.

[Beh06] Roya Beheshti. Lines on projective hypersurfaces. *J. Reine Angew. Math.*, 592:1–21, 2006.

[BFJ09] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Differentiability of volumes of divisors and a problem of Teissier. *J. Algebraic Geom.*, 18(2):279–308, 2009.

[BM90] V. V. Batyrev and Yu. I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.*, 286(1-3):27–43, 1990.

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[Bou04] Sébastien Boucksom. Divisorial Zariski decompositions on compact complex manifolds. *Ann. Sci. École Norm. Sup. (4)*, 37(1):45–76, 2004.

[BT95] Victor V. Batyrev and Yuri Tschinkel. Rational points of bounded height on compactifications of anisotropic tori. *Internat. Math. Res. Notices*, (12):591–635, 1995.

[BT96a] V. Batyrev and Yu. Tschinkel. Height zeta functions of toric varieties. *J. Math. Sci.*, 82(1):3220–3239, 1996. Algebraic geometry, 5.

[BT96b] V. V. Batyrev and Y. Tschinkel. Rational points on some Fano cubic bundles. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(1):41–46, 1996.

[BT98a] V. V. Batyrev and Y. Tschinkel. Manin's conjecture for toric varieties. *J. Algebraic Geom.*, 7(1):15–53, 1998.

[BT98b] V. V. Batyrev and Y. Tschinkel. Tamagawa numbers of polarized algebraic varieties. *Astérisque*, (251):299–340, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).

[BT99] F. A. Bogomolov and Yu. Tschinkel. On the density of rational points on elliptic fibrations. *J. Reine Angew. Math.*, 511:87–93, 1999.

[Cam04] F. Campana. Orbifolds, special varieties and classification theory. *Ann. Inst. Fourier (Grenoble)*, 54(3):499–630, 2004.

[CLT02] A. Chambert-Loir and Y. Tschinkel. On the distribution of points of bounded height on equivariant compactifications of vector groups. *Invent. Math.*, 148(2):421–452, 2002.

[Cut86] Steven D. Cutkosky. Zariski decomposition of divisors on algebraic varieties. *Duke Math. J.*, 53(1):149–156, 1986.

[DC12] Gabriele Di Cerbo. On Fujita's log spectrum conjecture, 2012. `arXiv:1210.5234`.

[Deb03] Olivier Debarre. Lines on smooth hypersurfaces, 2003. `www.math.ens.fr/~debarre/Lines_hypersurfaces.pdf`.

[FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel. Rational points of bounded height on Fano varieties. *Invent. Math.*, 95(2):421–435, 1989.

[FS04] O. Fujino and H. Sato. Introduction to the toric Mori theory. *Michigan Math. J.*, 52(3):649–665, 2004.

[Fuj87] T. Fujita. On polarized manifolds whose adjoint bundles are not semipositive. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 167–178. North-Holland, Amsterdam, 1987.

[Fuj92] T. Fujita. On Kodaira energy and adjoint reduction of polarized manifolds. *Manuscripta Math.*, 76(1):59–84, 1992.

[Fuj96] T. Fujita. On Kodaira energy of polarized log varieties. *J. Math. Soc. Japan*, 48(1):1–12, 1996.

[Fuj97] T. Fujita. On Kodaira energy and adjoint reduction of polarized threefolds. *Manuscripta Math.*, 94(2):211–229, 1997.

[FZ13] Baohua Fu and De-Qi Zhang. A characterization of compact complex tori via automorphism groups. *Math. Ann.*, 2013.

[GH78] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[GTBT11] A. Gorodnik, R. Takloo-Bighash, and Y. Tschinkel. Multiple mixing for adele groups and rational points, 2011. `arXiv:1112.5743`.

[HK00] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[HT99] B. Hassett and Y. Tschinkel. Geometry of equivariant compactifications of $\mathbf{G}_a^n$. *Internat. Math. Res. Notices*, (22):1211–1230, 1999.

[HT00a] J. Harris and Y. Tschinkel. Rational points on quartics. *Duke Math. J.*, 104(3):477–500, 2000.

[HT00b] Brendan Hassett and Yuri Tschinkel. Abelian fibrations and rational points on symmetric products. *Internat. J. Math.*, 11(9):1163–1176, 2000.

[KKLV89] Friedrich Knop, Hanspeter Kraft, Domingo Luna, and Thierry Vust. Local properties of algebraic group actions. In *Algebraische Transformationsgruppen und Invariantentheorie*, volume 13 of *DMV Sem.*, pages 63–75. Birkhäuser, Basel, 1989.

[KKV89] Friedrich Knop, Hanspeter Kraft, and Thierry Vust. The Picard group of a G-variety. In *Algebraische Transformationsgruppen und Invariantentheorie*, volume 13 of *DMV Sem.*, pages 77–87. Birkhäuser, Basel, 1989.

[KMM87] Yujiro Kawamata, Katsumi Matsuda, and Kenji Matsuki. Introduction to the minimal model problem. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 283–360. North-Holland, Amsterdam, 1987.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rationally connected varieties. *J. Algebraic Geom.*, 1(3):429–448, 1992.

[Laz04a] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Laz04b] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Leh12] Brian Lehmann. A cone theorem for nef curves. *J. Algebraic Geom.*, 21(3):473–493, 2012.

[Leh13] Brian Lehmann. Comparing numerical dimensions. *Algebra Number Theory*, 7(5):1065–1100, 2013.

[Leh14] Brian Lehmann. On Eckl's pseudo-effective reduction map. *Trans. Amer. Math. Soc.*, 366(3):1525–1549, 2014.

[LT10] J. M. Landsberg and Orsola Tommasi. On the Debarre-de Jong and Beheshti-Starr conjectures on hypersurfaces with too many lines. *Michigan Math. J.*, 59(3):573–588, 2010.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)].* Springer-Verlag, Berlin, third edition, 1994.

[MU83] Shigeru Mukai and Hiroshi Umemura. Minimal rational threefolds. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 490–518. Springer, Berlin, 1983.

[Nak04] Noboru Nakayama. *Zariski-decomposition and abundance*, volume 14 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2004.

[Pet12] Thomas Peternell. Varieties with generically nef tangent bundles. *J. Eur. Math. Soc. (JEMS)*, 14(2):571–603, 2012.

[Pey95] E. Peyre. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.*, 79(1):101–218, 1995.

[Som86] A. J. Sommese. On the adjunction theoretic structure of projective varieties. In *Complex analysis and algebraic geometry (Göttingen, 1985)*, volume 1194 of *Lecture Notes in Math.*, pages 175–213. Springer, Berlin, 1986.

[Tsc09] Y. Tschinkel. Algebraic varieties with many rational points. In *Arithmetic geometry*, volume 8 of *Clay Math. Proc.*, pages 243–334. Amer. Math. Soc., Providence, RI, 2009.