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EQUATIONS OF UNIVERSAL TORSORS AND COX RINGS

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Abstract. We discuss several constructions of universal torsors over rational surfaces.

1. Universal torsors and Cox rings

1.1. Motivating Example. All fields are supposed to be of characteristic 0.

Let X/K be a quintic Del Pezzo surface over a number field K. We have $\overline{X} = X_{\overline{K}} = \mathrm{Bl}_{P_1,P_2,P_3,P_4}\,\mathbb{P}^2$, i.e. geometrically, X is the blow-up of $\mathbb{P}^2$ in four points in general position. Without loss of generality, we may assume that

$$P_1 = [1,0,0], P_2 = [0,1,0], P_3 = [0,0,1], P_4 = [1,1,1].$$

***Theorem 1* (Enriques, Swinnerton-Dyer).** *Even in the non-split case,*

$$X(K) \neq \emptyset.$$

Proof. See [**Sko93**]. □

Notes by Ulrich Derenthal, June 18th and 19th, 2004.

Since there is a unique projectivity taking arbitrary generic points P_1, P_2, P_3, P_4 (i.e., distinct and no three of them collinear) to $[1,0,0], \ldots, [1,1,1]$ as above, the geometry behind this over $\overline{K}$ is (where $P_5 \in \overline{X}$ is the point we want to describe):

$$\begin{aligned}\overline{X} &= \mathrm{SL}_3 \backslash\backslash \{(P_1, \ldots, P_5) \in \mathbb{P}^2\} \\ &= \mathrm{SL}_3 \backslash\backslash \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix} //\mathbb{G}_m^5 .\end{aligned}$$

Consider the Grassmannian of 3-dimensional subspaces of 5-dimensional space $\mathrm{Gr}(3,5)$. Since such a subspace is described by a basis which is unique only up to an action of GL_3, we have $\mathrm{GL}_3 \backslash\backslash M(3 \times 5) \cong \mathrm{Gr}(3,5)$, where we interpret the three rows of a 3×5 matrix as a basis. This implies that $\mathrm{SL}_3 \backslash\backslash M(3 \times 5)$ is the cone over this Grassmannian.

Therefore, $\overline{X} \cong \mathrm{Cone}(\mathrm{Gr}(3,5))//\mathbb{G}_m^5$. Here, $\mathrm{Gr}(3,5)$ is embedded into $\mathbb{P}^9$ by the Plücker embedding.

The "miracle" is that this generalizes to non-closed fields.

Remark 2. The permutation group S_5 of the five points acts on the situation, and actually $\mathrm{Aut}(\overline{X}) = S_5$.

Descent data for X is given by representations $\rho : \mathrm{Gal}(\overline{K}/K) \to S_5$. Let T_ρ be the nonsplit form of $\mathbb{G}_m^5$ corresponding to ρ. In fact, X is $\mathrm{Cone}(\mathrm{Gr}(3,5))//T_\rho$.

1.2. Universal torsors. Let X be a smooth projective variety over $\overline{K}$. Assume $\mathrm{Pic}(X)$ is free of rank r. Let T_X be the Néron-Severi torus, i.e., its character group is $\chi^*(T_X) = \mathrm{Pic}(X)$.

Definition 3. A *universal torsor* $\mathscr{U}$ is a T_X-principal homogeneous space

$$\begin{array}{ccc} T_x & \longrightarrow & \mathscr{U} \\ & & \downarrow \\ & & X \end{array}$$

so that given an element $\lambda \in \chi^*(T_X)$ (i.e., $\lambda : T_X \to \mathbb{G}_m$), then $\mathscr{M}_\lambda^\times \cong \mathscr{L}_\lambda - \{0\text{-section}\}$ as $\mathbb{G}_m$-bundles over X. Here, $\mathscr{L}_\lambda \in \mathrm{Pic}(X)$ is the line bundle associated to λ by $\chi^*(T_X) \cong \mathrm{Pic}(X)$, and $\mathscr{M}_\lambda$ is the associated bundle to the principal bundle $\mathscr{U} \to X$ induced by the representation λ.

Example 4. 1. Let $X = \mathbb{P}^n$. Then $\mathscr{U} = \mathbb{A}^{n+1} - \{0\}$ is the corresponding universal torsor with the torus acting diagonally. We have $\mathscr{U}/\mathbb{G}_m \cong X = \mathbb{P}^n$.

2. Let X be the quintic Del Pezzo surface as above with the action of T_X on Cone(Gr(3,5)). Then the universal torsor $\mathscr{U}$ is the open subset of Cone(Gr(3,5)) on which T_X acts freely.

The abstract approach to universal torsors is as follows: Choose a minimal set $\mathscr{L}_1, \dots, \mathscr{L}_r$ generating $\mathrm{Pic}(X)$ over $\mathbb{Z}$. Denote $\mathscr{L}_j - \{0\text{-section}\}$ by $\mathscr{L}_j^\times$. Let $\mathscr{U} = \mathscr{L}_1^\times \times \cdots \times \mathscr{L}_r^\times$. Then $T_X \to \mathscr{U} \to X$ is a T_X-principal bundle defining the universal torsor.

However, this abstract definition is not very useful, e.g., for number theoretic applications.

Remark 5. Over non-closed fields, we may not be able to descend the universal torsor $\mathscr{U}$.

For example, consider a non-split conic X. It is geometrically isomorphic to $\mathbb{P}^1$, but it has no line bundle isomorphic to $\mathscr{O}_{\mathbb{P}^1}(1)$ over the ground field. It only has line bundles of even degree, so there cannot exist a universal torsor over the ground field.

1.3. Total coordinate rings / Cox rings.

Definition 6. Let X be a projective variety with properties as above. Let $\mathscr{L}_1, \dots, \mathscr{L}_r$ be a basis of $\mathrm{Pic}(X)$. Then the Cox ring of X is defined as

$$\mathrm{Cox}(X) = \bigoplus_{(v_1,\dots,v_r)\in\mathbb{Z}^r} \Gamma(X, \mathscr{L}_1^{v_1} \otimes \cdots \otimes \mathscr{L}_r^{v_r}).$$

Properties of $\mathrm{Cox}(X)$ are:

1. It is graded by $\mathrm{Pic}(X)$: for $\lambda \in \chi^*(T_X) \cong \mathrm{Pic}(X)$, the part of degree λ is given by $\mathrm{Cox}(X)_\lambda = \Gamma(X, \mathscr{L}_\lambda)$.
2. The torus T_X acts naturally on $\mathrm{Cox}(X)$: For $t \in T_X$, $s \in \mathrm{Cox}(X)_\lambda$, this action is given by $t(s) := \lambda(t) \cdot s$.
3. $\mathrm{Cox}(X)$ is independent of the choice of generators $\mathscr{L}_i$ of the Picard group. Given two sets of generators $\mathscr{L}_i$ and $\mathscr{M}_j$, the induced isomorphism of rings is canonical only up to the action of the torus T_X. The reason is that the isomorphism depens on a choice of isomorphisms

$$L_j \cong \mathscr{M}_1^{n_1} \otimes \cdots \otimes \mathscr{M}_r^{n_r}, j \in \{1, \dots, r\}.$$

However, such an isomorphism is not canonical: $\mathscr{L}_j$ has automorphisms given by scalar multiplication. For details, see [**HT04**].

The existence of non-trivial automorphisms makes the descent of universal torsors an interesting question.

4. The graded pieces of $\mathrm{Cox}(X)$ which are non-zero correspond to effective divisors on X.

The Cox ring does not need to be finitely generated:

***Example 7* (Mukai).** Let $X = \mathrm{Bl}_{P_1,\ldots,P_n} \mathbb{P}^{r-1}$ be the blowup of projective space in n points in general position. If $\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} \geq 1$, then $\mathrm{Cox}(X)$ is not finitely generated (i.e., for $\mathbb{P}^2$: $n \geq 9$; for $\mathbb{P}^3$: $n \geq 8$). Details can be found in [**Muk01**].

However, it is finitely generated if one of the following conditions is true:

1. The cone of effective divisors $\mathrm{NE}(X)$ is generated by a finite collection of semi-ample line bundles (e.g., $X = G/P$ where P is parabolic subgroup of an algebraic group G).
2. X is (log) Fano of dimension ≤ 3.
3. X is toric. In this case, for $X - \mathbb{G}_m^{\dim X} = \bigcup_{j=1}^{N} D_j$ where the D_j are subvarieties of codimension 1, and $s_j \in \Gamma(\mathscr{O}_X(D_j))$ is non-zero, then $\mathrm{Cox}(X) \cong K[s_1, \ldots, s_N]$.

1.4. Relations between universal torsors and Cox rings. From now on, assume that $\mathrm{Cox}(X)$ is finitely generated. Let $\mathscr{V} = \mathrm{Spec}(\mathrm{Cox}(X))$. It is affine with T_X-action $T_X \times \mathscr{V} \to \mathscr{V}$. Fix an open subset $\mathscr{U}$ on which T_X acts freely. The basic fact is that $\mathscr{U}$ is a T_X-principal bundle over X:

$$\begin{array}{ccc} T_X & \longrightarrow & \mathscr{U} \\ & & \downarrow \\ & & X \end{array}$$

and $\mathscr{U}$ is a universal torsor.

The punchline is that this way, the universal torsor $\mathscr{U}$ is naturally a quasi-affine variety. Therefore, giving equations for $\mathscr{U}$ is equivalent to giving generators and relations for $\mathrm{Cox}(X)$. This can be done by algebro-geometric methods, which may be seen as an improvement to the existing number theoretic method to calculate universal torsors.

To sketch a proof of these results, observe that X is naturally a Geometric Invariant Theory quotient $(\mathscr{V}//T_X)_\lambda$ (by Keel–Hu, [**HK00**]) after specifying a linearization $\lambda \in \chi^*(T_X)$ so that $\mathscr{L}_\lambda$ is an ample line bundle on X.

Note that we need to mix affine invariant theory and the usual projective Geometric Invariant Theory to interpret $(\mathscr{V}//T_X)_\lambda$: First take the affine quotient under the action of $\ker(\lambda)$, which gives an affine variety. Then take Proj using the grading coming from the character λ.

Then $\mathrm{Proj}(\bigoplus_{n\geq 0}\mathrm{Cox}(X)_{n\lambda}) = (\mathscr{V}//T_X)_\lambda$ by Geometric Invariant Theory. The left hand side is $\mathrm{Proj}(\bigoplus_{n\geq 0}\Gamma(X,\mathscr{L}_\lambda^{\otimes n}))$, which is just X since $\mathscr{L}_\lambda$ is ample.

A second observation is that given $\lambda \in \chi^*(T_X)$, i.e., $\lambda : T_X \to \mathbb{G}_m$, the associated bundle induces $\mathscr{L}_\lambda^{-1}$. Therefore, it suffices to check the claim for ample λ.

We have an inclusion $\bigoplus_{n\geq 0}\mathrm{Cox}(X)_{n\lambda} \to \mathrm{Cox}(X)$ which induces a dominant map $\mathscr{V} \to \mathrm{Cone}(X \subset \mathbb{P}^N, \mathscr{L}_\lambda)$. Therefore, we have

$$\begin{array}{ccc} \mathscr{V} & \longrightarrow & \mathrm{Cone}(X\subset \mathbb{P}^N) \\ \uparrow & & \uparrow \\ \mathscr{U} & \longrightarrow & (\mathrm{Cone}(X\subset\mathbb{P}^N)-\{0\}) \cong (\mathscr{L}_\lambda^{-1})^\times \end{array}$$

The point is: One gets hold of the universal torsor by embedding it into the affine variety $\mathrm{Spec}(\mathrm{Cox}(X))$.

2. Equations of universal torsors

From now on, let X be a smooth projective variety over on algebraically closed field K of characteristic 0 with $\mathrm{Pic}(X) \cong \mathbb{Z}^r$ whose Cox ring is finitely generated. Therefore, the cone of effective divisors $\mathrm{NE}(X)$ is finitely generated.

2.1. The method of Colliot-Thélène and Sansuc. This approach to the calculation of Cox rings can be found in [**CTS87**].

On X, choose effective divisors $D_1, \dots, D_N$ generating $\mathrm{Pic}(X)$. Let $W = X\backslash(D_1 \cup \cdots \cup D_N$. Since removing these generators kills the Picard group, $\mathrm{Pic}(W) = 0$.

We have an exact sequence

$$0 \to K[W]^*/K^* \to \bigoplus_{j=1}^{N} \mathbb{Z}D_j \to \mathrm{Pic}(X) \to 0$$

where $K[W]^*/K^*$ describes the linear equivalences among $\{D_1, \dots, D_N\}$.

Dualizing this sequence by applying $\mathrm{Hom}(\cdot, \mathbb{G}_m)$, we obtain

$$1 \to T_X \to \mathbb{G}_m^N \xrightarrow{q} R_W \to 1.$$

Remark 8. A morphism $\varphi : Z \to R_W$ gives a T_X-torsor:

$$\begin{array}{ccccc} T_X \longrightarrow \mathbb{G}_m^N \times_{R_W} Z & & \supset & & q^{-1}(\varphi(z)) \\ \downarrow & & & & \downarrow \\ Z & & \ni & & z \end{array}$$

The strategy is to construct a T_X-torsor $\mathscr{U}_W$ over W which extend to a universal torsor over X. This strategy works well in many cases, but not in general.

The morphism $\varphi : W \to R_W$ is constructed by constructing a splitting σ to the quotient

$$K[W]^* \underset{\longrightarrow}{\overset{\sigma}{\longleftarrow}} K[W]^*/K^* :$$

Note that σ induces a K-algebra homomorphism

$$K[R_W] = K[t_1, t_1^{-1}, \ldots, t_{N-r}, t_{N-r}^{-1}] \to K[W], \quad t_j \mapsto \sigma(t_j),$$

where the t_j form a basis for $\chi^*(R_W)$ and $r = \mathrm{Rank}(\mathrm{Pic}(X))$. Since R_W is affine, such a homomorphism corresponds to a K-morphism $W \to R_W$, which defines φ.

The key fact is that the morphism φ extracted from σ gives a torsor $T_X \to \mathscr{U}_W \to W$ on W admitting an extension to a universal torsor $T_X \to \mathscr{U} \to X$ over X.

$$\begin{array}{ccccc} T_X \longrightarrow \mathscr{U}_W & & \rightsquigarrow & & T_X \longrightarrow \mathscr{U} \\ \downarrow & & & & \downarrow \\ W & & & & X \end{array}$$

An explicit method for constructing such an extension is not known. Only the existence is proven in **[CTS87]**.

***Remark 9* (Batyrev).** Given a point $P \in W$, we get a natural splitting $\sigma_P : K[W]^*/K^* \to K[W]^*$: for every element of $K[W]^*/K^*$, choose a representing f satisfying $f(P) = 1$.

2.2. The example of the quintic Del Pezzo surface. Let $X = \mathrm{Bl}_{P_1,\ldots,P_4}\mathbb{P}^2$ be again the blow-up of $\mathbb{P}^2$ in

$$P_1 = [1,0,0], P_2 = [0,1,0], P_3 = [0,0,1], P_4 = [1,1,1].$$

We will see how to obtain the Plücker equations defining the universal torsor by this method.

Consider the exceptional divisors E_i and the transforms l_{ij} of the lines through P_i and P_j ($i \neq j \in \{1, \dots, 4\}$). Choose coordinates $[x, y, z]$ and let $u = \frac{x}{z}$, $v = \frac{y}{z}$.

Consider

$$\begin{aligned} \operatorname{div}(u = x/z) &= l_{23} + E_3 - l_{12} - E_1 \\ \operatorname{div}(v = y/z) &= l_{13} + E_3 - l_{12} - E_2 \\ \operatorname{div}(u - 1) &= l_{24} + E_4 - l_{12} - E_1 \\ \operatorname{div}(v - 1) &= l_{14} + E_4 - l_{12} - E_2 \\ \operatorname{div}(u - v) &= l_{34} + E_3 + E_4 - l_{12} - E_1 - E_2 \end{aligned}$$

Next, we normalize these functions by constructing a section σ_P from a chosen point, say $P = [3, 2, 1]$. This gives a morphism $\varphi : W \to R_W$ as above.

Consider the sections λ_{ij} corresponding to l_{ij} and η_i to E_i. Using the normalization, we obtain:

$$\frac{u}{3} = \frac{\lambda_{23}\eta_3}{\lambda_{12}\eta_1}, \quad \frac{v}{2} = \frac{\lambda_{13}\eta_3}{\lambda_{12}\eta_2}, \quad \frac{u-1}{2} = \frac{\lambda_{24}\eta_4}{\lambda_{12}\eta_1}, \quad v-1 = \frac{\lambda_{14}\eta_4}{\lambda_{12}\eta_2}, \quad u-v = \frac{\lambda_{34}\eta_3\eta_4}{\lambda_{12}\eta_1\eta_2}.$$

Then the relations between the sections $u, v, u-1, v-1, u-v$ give relations between the sections λ_{ij}, η_i:

$$\begin{aligned} 3\frac{u}{3} - 2\frac{v}{2} = u - v \quad &\rightsquigarrow \quad -(3\lambda_{23})\eta_2 + (2\lambda_{13})\eta_1 + \lambda_{34}\eta_4 = 0 \\ 2\frac{v}{2} = (v-1) + 1 \quad &\rightsquigarrow \quad \lambda_{14}\eta_4 - (2\lambda_{13})\eta_3 + \lambda_{12}\eta_2 = 0 \\ 2\frac{u-1}{2} - (v-1) = u - v \quad &\rightsquigarrow \quad \lambda_{34}\eta_3 - (2\lambda_{24})\eta_4 + \lambda_{14}\eta_1 = 0 \\ 3\frac{u}{3} = 2\frac{u-1}{2} + 1 \quad &\rightsquigarrow \quad (2\lambda_{24})\eta_4 - (3\lambda_{23})\eta_3 + \lambda_{12}\eta_1 = 0 \\ -(u-v) + v(u-1) - (v-1)u = 0 \quad &\rightsquigarrow \quad \lambda_{12}\lambda_{34} - (2\lambda_{13})(2\lambda_{24}) + (3\lambda_{23})\lambda_{14} = 0 \end{aligned}$$

Replacing $3\lambda_{23}, 2\lambda_{13}, 2\lambda_{24}$ by new variables exactly gives the Plücker relations.

2.3. The Cox ring approach. Consider a different example:

$$X = \mathrm{Bl}_{P_1,P_2,P_3}\mathbb{P}^2 \text{ where } P_1 = [1,0,0], P_2 = [1,1,0], P_3 = [0,1,0],$$

i.e., X is the blow-up of $\mathbb{P}^2$ in three points lying on a line. Let l_{123} be the transform of this line.

Basic facts on X are:

1. $\mathrm{NE}(X) = \langle l_{123}, E_1, E_2, E_3 \rangle$ is a simplicial cone, i.e., there are no relations between its generators. Therefore, the previous method does not work.

We have $W = X - \{E_1, E_2, E_3, l_{123}\} \cong \mathbb{A}^2$, and X is an equivariant compactification of $\mathbb{G}_a^2$, acting on $\mathbb{A}^2$ by translation.

2. The ample cone, which is the dual of the effective cone, is generated by

$$\{l_{123} + E_1 + E_2 + E_3, l_{123} + E_1 + E_2, l_{123} + E_1 + E_3, l_{123} + E_2 + E_3\}.$$

3. The anticanonical divisor $-K_X$ is nef and big. Therefore, X is (log) Del Pezzo.

Next, we are looking for generators and relations of $\mathrm{Cox}(X)$. Generators are $\lambda_{123} \in \Gamma(\mathscr{O}_X(l_{123})) \subset \mathrm{Cox}(X)_{l_{123}}$ which is vanishing exactly along l_{123}, and $\eta_j \in \Gamma(\mathscr{O}_X(E_j)) \subset \mathrm{Cox}(X)_{E_j}$ for $j \in \{1, 2, 3\}$.

These sections do not generate the Cox ring – in cases where they generate it, the method of Colliot-Thélène works well, but not here. We must choose additional generators: $\Gamma(\mathscr{O}_X(l_{123} + E_1 + E_2))$ corresponds to linear forms in x, y, z vanishing at P_3, i.e., it is $K^2 \cong \langle x, z \rangle$. Besides $\lambda_{123}\eta_1\eta_2$, which can be identified as z, we can choose another section ξ_3 such that $\xi_3\eta_3 = -x$.

Similarly, we have $\xi_1 \in \Gamma(\mathscr{O}_X(l_{123} + E_2 + E_3))$ such that $y = \xi_1\eta_1$ and $\xi_2 \in \Gamma(\mathscr{O}_X(l_{123} + E_1 + E_3))$ such that $x - y = \xi_2\eta_2$.

This gives a homomorphism

$$\psi : K[\lambda_{123}, \eta_1, \eta_2, \eta_3, \xi_1, \xi_2, \xi_3]/\langle \eta_1\xi_1 + \eta_2\xi_2 + \eta_3\xi_3 \rangle \to \mathrm{Cox}(X),$$

and since the dimension of both of these is 6 ($\dim(X) = 2$ and $\mathrm{Rank}(\mathrm{Pic}(X)) = 4$), it is reasonable to hope that this is an isomorphism.

Remark 10. Then $\eta_1\xi_1 + \eta_2\xi_2 + \eta_3\xi_3$ is the equation of the universal torsor $T_X \to \mathscr{U} \to X$ in the sense that

$$\mathscr{U} \subset \mathscr{V} := \mathrm{Spec}\, K[\lambda_{123}, \eta_1, \eta_2, \eta_3, \xi_1, \xi_2, \xi_3]/\langle \eta_1\xi_1 + \eta_2\xi_2 + \eta_3\xi_3 \rangle.$$

Strategy of the proof. First, consider ψ in degrees ν corresponding to a nef line bundles on X. Such line bundles are semi-ample and in this case even globally generated. By induction on the effective monoid or by application of a vanishing theorem, we can prove that ψ is surjective in these nef degrees.

In degrees ν corresponding to not necessarily nef divisors ν, we reduce to the nef case the following way: Given $s \in \mathrm{Cox}(X)_\nu = \Gamma(X, \mathscr{L}_\nu)$, there exists a nef line bundle m, a section $\mu \in \mathrm{Cox}(X)_m$ and $a, b_1, b_2, b_3 \in \mathbb{Z}_{\geq 0}$ so that $s = \mu\lambda_{123}^a\eta_1^{b_1}\eta_2^{b_2}\eta_3^{b_3}$. This follows from the geometric fact that, given effective D on X, we can write $D = M + F$ for a base point free divisor M and a fixed divisor F supported in $\{l_{123}, E_1, E_2, E_3\}$. □

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