Characterizing projective spaces on deformations of Hilbert schemes of K3 surfaces

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Abstract

We seek to characterize homology classes of Lagrangian projective spaces embedded in irreducible holomorphic-symplectic manifolds, up to the action of the monodromy group. This paper addresses the case of manifolds deformation-equivalent to the Hilbert scheme of length-three subschemes of a K3 surface. The class of the projective space in the cohomology ring has prescribed intersection properties, which translate into Diophantine equations. Possible homology classes correspond to integral points on an explicit elliptic curve; our proof entails showing that the only such point is two-torsion.

1 Introduction

Let X be an irreducible holomorphic symplectic manifold, i.e., a compact Kähler simply-connected manifold admitting a unique nondegenerate holomorphic two-form. Let $(,)$ denote the Beauville–Bogomolov form on the cohomology group $\mathrm{H}^2(X,\mathbb{Z})$, normalized so that it is integral and primitive. When X is a K3 surface this coincides with the intersection form. In higher dimensions, the form induces an inclusion

$$\mathrm{H}^2(X,\mathbb{Z}) \subset \mathrm{H}_2(X,\mathbb{Z}), \tag{1.1}$$

which allows us to extend $(,)$ to a $\mathbb{Q}$-valued quadratic form.

Lagrangian projective spaces play a fundamental rôle in the birational geometry of these classes of manifolds. If X contains a holomorphically embedded projective space $\mathbb{P}^{\dim(X)/2}$ we can consider the *Mukai flop* of X, obtained by blowing up the projective space and blowing down the exceptional divisor

$$E \simeq \mathbb{P}(\Omega^1_{\mathbb{P}^{\dim(X)/2}})$$

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along the opposite ruling. Our goal is to characterize possible homology classes of such submanifolds, modulo the monodromy representation on the cohomology of X.

Assuming X contains a Lagrangian projective space $\mathbb{P}^{\dim(X)/2}$, let $\ell \in \mathrm{H}_2(X,\mathbb{Z})$ denote the class of a line in $\mathbb{P}^{\dim(X)/2}$, and $\lambda = N\ell \in \mathrm{H}^2(X,\mathbb{Z})$ a positive integer multiple. We can take N to be the index of $\mathrm{H}^2(X,\mathbb{Z}) \subset \mathrm{H}_2(X,\mathbb{Z})$. Hodge theory [19, 24] shows that the deformations of X containing a deformation of the Lagrangian space coincide with the deformations of X for which $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$ remains of type $(1,1)$. Infinitesimal Torelli implies this is a divisor in the deformation space, i.e.,

$$\lambda^{\perp} \subset \mathrm{H}^1(X,\Omega^1_X) \simeq \mathrm{H}^1(X,\mathscr{T}_X).$$

We seek to establish intersection theoretic properties of ℓ for various deformation-equivalence classes of holomorphic symplectic manifolds. Previous results in this direction include

(1) If X is a K3 surface then $(\ell,\ell) = -2$.
(2) If X is deformation equivalent to the Hilbert scheme of length-two subschemes of a K3 surface then $(\ell,\ell) = -5/2$. [11]
(3) If X is deformation equivalent to a generalized Kummer fourfold then $(\ell,\ell) = -3/2$. [12]

Here we prove

Theorem 1.1. *Let X be a six-dimensional Kähler manifold, deformation equivalent to the Hilbert scheme of length-three subschemes of a K3 surface. Let $\mathbb{P}^3 \subset X$ be a smooth subvariety and $\ell \subset \mathbb{P}^3$ a line. Then $(\ell,\ell) = -3$ and $\rho = 2\ell \in \mathrm{H}^2(X,\mathbb{Z})$. Furthermore, we have*

$$[\mathbb{P}^3] = \frac{1}{48}\left(\rho^3 + \rho^2 c_2(X)\right).$$

This uniquely characterizes the class of the Lagrangian plane, modulo the monodromy action, which acts transitively on the $\rho \in \mathrm{H}^2(X,\mathbb{Z})$ with $(\rho,\rho) = -12$ and $\left(\rho, \mathrm{H}^2(X,\mathbb{Z})\right) = 2\mathbb{Z}$ [9, §3].

In general, we conjectured in [13] that if X is of dimension $2n$ then

$$(\ell,\ell) = -(n+3)/2,$$

if X is deformation equivalent to a Hilbert scheme of a K3 surface. Our main motivation for making these conjectures is to achieve a classification of extremal rational curves on irreducible holomorphic symplectic varieties (i.e., generators of extremal rays of birational contractions) in terms of intersection properties under the Beauville-Bogomolov form.

The structure of this paper is as follows: Section 2 reviews the cohomology groups of Hilbert schemes of K3 surfaces; Section 3 focuses on the ring structure. We employ representation theory to get results on the Hodge classes in Section 4. The Hilbert scheme of length-three subschemes is studied in detail in Section 5. We extract the distinguished absolute Hodge class in the middle cohomology in

Section 6. Here 'absolute Hodge classes' are classes that remain Hodge under arbitrary deformations of complex structure; their span is stable under the monodromy representation, which is why we use this terminology (cf. [4, p. 28]). The computation of the class of the Lagrangian three planes is worked out in Section 7, modulo a number theoretic result. This is proved in Section 8.

2 Cohomology of Hilbert schemes

Let X be deformation equivalent to the punctual Hilbert scheme $S^{[n]}$, where S is a K3 surface. For $n > 1$ the Beauville-Bogomolov form can be written [1, §8]

$$\mathrm{H}^2(X,\mathbb{Z}) \simeq \mathrm{H}^2(S,\mathbb{Z})_{(,)} \oplus_\perp \mathbb{Z}\delta, \quad (\delta,\delta) = -2(n-1)$$

where 2δ is the class of the 'diagonal' divisor $\Delta^{[n]} \subset S^{[n]}$ parameterizing nonreduced subschemes. For each cohomology class $f \in \mathrm{H}^2(S,\mathbb{Z})$, let $f \in \mathrm{H}^2(X,\mathbb{Z})$ denote the class of the locus parameterizing subschemes with some support along f. This is compatible with the lattice embedding above. Duality gives a $\mathbb{Q}$-valued form on homology

$$\mathrm{H}_2(X,\mathbb{Z}) \simeq \mathrm{H}_2(S,\mathbb{Z})_{(,)} \oplus_\perp \mathbb{Z}\delta^\vee, \quad \left(\delta^\vee,\delta^\vee\right) = -\frac{1}{2(n-1)},$$

where $\delta^\vee$ is characterized as the homology class orthogonal to $\mathrm{H}^2(S,\mathbb{Z})$ and satisfying $\delta^\vee \cdot \delta = 1$.

Theorem 2.1. [8] *Let S be a K3 surface and $S^{[n]}$ its Hilbert scheme. Consider the Poincaré polynomial*

$$p(S^{[n]},z) = \sum_{j=0}^{4n} \beta_j(S^{[n]})z^j,$$

where $\beta_j(S^{[n]})$ denotes the jth Betti number of $S^{[n]}$. Then

$$\sum_{n=0}^{\infty} p(S^{[n]},z)t^n = \prod_{m=1}^{\infty}(1-z^{2m-2}t^m)^{-1}(1-z^{2m}t^m)^{-22}(1-z^{2m+2}t^m)^{-1}.$$

To save space, we write

$$q(S^{[n]},z) = \sum_{j=0}^{n} \beta_{2j}z^j,$$

which determines the Poincaré polynomial by Poincaré duality. We have

$$\begin{aligned} q(S,z) &= 1+22z \\ q(S^{[2]},z) &= 1+23z+276z^2 \\ q(S^{[3]},z) &= 1+23z+299z^2+2554z^3. \end{aligned}$$

A theorem of Verbitsky [23, Theorem 1.5] asserts that the homomorphism arising from the cup product

$$\mu_{k,n} : \mathrm{Sym}^k\mathrm{H}^2(S^{[n]},\mathbb{Q}) \to \mathrm{H}^{2k}(S^{[n]},\mathbb{Q})$$

is injective for $k \leq n$. Thus its image has dimension

$$\binom{22+k}{k}.$$

In light of the computations above, $\mu_{2,2}$ is an isomorphism, $\mu_{2,3}$ has cokernel of dimension 23, and $\mu_{3,3}$ has cokernel of dimension

$$2554 - \binom{25}{3} = 254 = \binom{23}{2} + 1.$$

The cup product also induces a homomorphism

$$\operatorname{coker}(\mu_{2,3}) \otimes \mathrm{H}^2(S^{[3]}, \mathbb{Q}) \to \operatorname{coker}(\mu_{3,3}).$$

This homomorphism has been observed by Markman [16, p. 80]. More generally, he analyzes what classes are needed to generate the cohomology ring $\mathrm{H}^*(S^{[n]}, \mathbb{Q})$, beyond those coming from $\mathrm{H}^2(S^{[2]}, \mathbb{Q})$. Markman uses Chern classes of universal sheaves over the product $S^{[n]} \times S$; a detailed discussion of the $n = 3$ case is given in [16, Ex. 14].

3 The ring structure on cohomology

Lehn-Sorger [15] and Nakajima [18] described $\mathrm{H}^*(S^{[n]}, \mathbb{Q})$ in terms of $\mathrm{H}^*(S, \mathbb{Q})$. We review the Lehn-Sorger formalism for the cup product on the cohomology ring.

Let S be a K3 surface and $A = \mathrm{H}^*(S, \mathbb{Q})(1)$, the cohomology ring shifted so that it has weights $-2, 0$, and 2; this is written as $\mathrm{H}^*(S, \mathbb{Q})[2]$ in their paper. Shifting the weights changes the sign of the intersection form, which is denoted by $\langle , \rangle$; this has signature $(20, 4)$. Let $T : A \to \mathbb{Q}$ denote the linear form

$$\gamma \mapsto -\int_S \gamma$$

and $\langle , \rangle$ the induced bilinear form

$$\langle \gamma_1, \gamma_2 \rangle = T(\gamma_1 \gamma_2) = -\int_S \gamma_1 \gamma_2.$$

For each $n \in \mathbb{N}$, we endow $A^{\otimes n}$ with an analogous structure. We shall use the fact that A has only graded pieces of *even* degrees to simplify the description in [15]. In this situation, graded commutative multiplication rules are in fact commutative, given by the rule

$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_n) = (a_1 b_1) \otimes \cdots \otimes (a_n b_n).$$

The linear form

$$T : A^{\otimes n} \to \mathbb{Q}$$

is defined by

$$T(a_1 \otimes \cdots \otimes a_n) = T(a_1) \cdots T(a_n).$$

Let $\langle , \rangle$ denote the associated bilinear form

$$\langle a, b \rangle = T(a \cdot b).$$

The symmetric group $\mathfrak{S}_n$ acts on $A^{\otimes n}$ by the rule

$$\pi(a_1 \otimes \cdots \otimes a_n) = a_{\pi^{-1}(1)} \otimes \cdots \otimes a_{\pi^{-1}(n)}.$$

Given a partition $n = n_1 + \ldots + n_k$ with $n_1, \ldots, n_k \in \mathbb{N}$, we have a generalized multiplication map

$$\begin{array}{rcl} A^{\otimes n} & \to & A^{\otimes k} \\ a_1 \otimes \cdots \otimes a_n & \mapsto & (a_1 \cdots a_{n_1}) \otimes \cdots \otimes (a_{n_1+\cdots+n_{k-1}+1} \cdots a_{n_1+\cdots+n_k}). \end{array}$$

Given a finite set $I \subset \{1, \ldots, n\}$, let $A^{\otimes I}$ denote the tensor power with factors indexed by elements of I. Given a surjection $\phi : I \to J$, there is an induced multiplication

$$\phi^* : A^{\otimes I} \to A^{\otimes J}$$

defined as above. Let

$$\phi_* : A^{\otimes J} \to A^{\otimes I}$$

denote the *adjoint* of ϕ^*, i.e.,

$$\langle \phi^* a, b \rangle = \langle a, \phi_* b \rangle$$

for $a \in A^{\otimes I}$ and $b \in A^{\otimes J}$.

We have the composite

$$A \xrightarrow{\Delta_*} A \otimes A \to A,$$

where the first map is adjoint comultiplication and the second is multiplication. Let $e := e(A)$ denote the image of 1 under the composed map.

Remark 3.1. We evaluate the signs of $\Delta_* 1$ and $e(A)$. Let Δ_S denote the fundamental class of the diagonal in $\mathrm{H}^*(S \times S, \mathbb{Z}) = \mathrm{H}^*(S, \mathbb{Z}) \otimes \mathrm{H}^*(S, \mathbb{Z})$. Using the adjoint property, we have

$$\begin{array}{rcl} \langle \Delta_* 1, \alpha \otimes \beta \rangle & = & \langle 1, \alpha\beta \rangle \\ & = & T(\alpha\beta) \\ & = & -\int_S \alpha\beta \end{array}$$

whereas

$$\begin{array}{rcl} \langle \Delta_S, \alpha \otimes \beta \rangle & = & \left\langle \sum_j e_j \otimes e_j^\vee, \alpha \otimes \beta \right\rangle \\ & = & \sum_j T(e_j \alpha) T(e_j^\vee \beta) \\ & = & \int_S \alpha\beta, \end{array}$$

where $\{e_j\}$ is a homogeneous basis for $\mathrm{H}^*(S, \mathbb{Q})$ with Poincaré-dual basis $e_j^\vee$. Therefore, we find

$$\Delta_* 1 = -[\Delta_S]. \tag{3.1}$$

Furthermore, we have

$$\int_S e(A) = -T(e(A)) = -\langle e(A), 1 \rangle = -\langle \Delta_* 1, \Delta_* 1 \rangle = -\chi(S) = -24,$$

so $e(A)$ is a *negative* multiple of the point class. Nevertheless, we still have (cf. [15, §2.2])

$$e(A) = \chi(S)\mathrm{vol}, \quad \text{where} \quad T(\mathrm{vol}) = 1,$$

but vol differs from the standard volume form by sign.

Let $\langle\pi\rangle\backslash[n]$ denote the set of orbits of $[n]=\{1,2,\ldots,n\}$ under the action of $\pi\in\mathfrak{S}_n$. Set

$$A\{\mathfrak{S}_n\}=\oplus_{\pi\in\mathfrak{S}_n}A^{\otimes\langle\pi\rangle\backslash[n]}\cdot\pi$$

which admits an action of $\mathfrak{S}_n$. First, note that $\sigma\in\mathfrak{S}_n$ induces a bijection

$$\begin{array}{rcl}\sigma:\langle\pi\rangle\backslash[n] & \rightarrow & \left\langle\sigma\pi\sigma^{-1}\right\rangle\backslash[n] \\ x & \mapsto & \sigma x.\end{array}$$

Thus we obtain an isomorphism

$$\begin{array}{rcl}\tilde{\sigma}:A\{S_n\} & \rightarrow & A\{S_n\} \\ a\pi & \mapsto & \sigma^*(a)\sigma\pi\sigma^{-1}.\end{array}$$

Example 3.2. [15, §2.9, 2.17] We have $A\{\mathfrak{S}_2\}=A^{\otimes 2}\mathrm{id}\oplus A(12)$ and

$$A\{\mathfrak{S}_3\}=A^{\otimes 3}\mathrm{id}\oplus A^{\otimes 2}(12)\oplus A^{\otimes 2}(13)\oplus A^{\otimes 2}(23)\oplus A(123)\oplus A(132).$$

Let $A^{[n]}\subset A\{\mathfrak{S}_n\}$ denote the invariants under this action. Then we have [15, §2]

$$A^{[n]}=\sum_{\|\alpha\|=n}\bigotimes_i \mathrm{Sym}^{\alpha_i}A,$$

where α corresponds to a partition

$$\underbrace{1+\cdots+1}_{\alpha_1\text{ times}}+\underbrace{2+\cdots+2}_{\alpha_2\text{ times}}+\cdots$$

and

$$n=\|\alpha\|=\alpha_1+2\alpha_2+\cdots+n\alpha_n.$$

Note that this is compatible with Hodge structures; in particular, $A^{[n]}$ is a representation of the Hodge group of S and the special orthogonal group G_S associated with the intersection form on $\mathrm{H}^2(S,\mathbb{R})$. We interpret this as acting on A, trivially on the summands $\mathrm{H}^0(S,\mathbb{R})$ and $\mathrm{H}^4(S,\mathbb{R})$.

Theorem 3.3. [15, Theorem 1.1] *Let S be a K3 surface. Then there is a canonical isomorphism of graded rings*

$$(\mathrm{H}^*(S,\mathbb{Q})[2])^{[n]}\xrightarrow{\sim}\mathrm{H}^*(S^{[n]},\mathbb{Q})[2n].$$

In the cohomology of the Hilbert scheme, the subring generated by $\mathrm{H}^2(S^{[n]})$ plays a special role. We have an isomorphism

$$\mathrm{H}^2(S^{[n]},\mathbb{Z})=\mathrm{H}^2(S,\mathbb{Z})\oplus\mathbb{Z}\delta,$$

where 2δ parameterizes the non-reduced subschemes of S. We express this in terms of our presentation. Given $D\in\mathrm{H}^2(S,\mathbb{Z})$, the class

$$\sum_{i=1}^n 1_{\{1\}}\otimes\cdots\otimes 1_{\{i-1\}}\otimes D_{\{i\}}\otimes 1_{\{i+1\}}\otimes\cdots\otimes 1_{\{n\}}(\mathrm{id})$$

is the corresponding class in $\mathrm{H}^2(S^{[n]},\mathbb{Q})[2n]$. Using the explicit form of the isomorphism in [15, §2.10 (2.7)] and Nakajima's correspondence (the isomorphism Ψ in [15, Thm. 3.5]), we find that

$$\delta = \sum_{1\le i<j\le n} 1_{\{1\}}\otimes\ldots\otimes 1_{\{i-1\}}\otimes 1_{\{i,j\}}\otimes 1_{\{i+1\}}\otimes\cdots\otimes 1_{\{j-1\}}\otimes 1_{\{j+1\}}\otimes\cdots\otimes 1_{\{n\}}(ij).$$

Here is the essence of the computation: the interpretation of the nonreduced subschemes via the correspondence

$$Z_2 = \{(\xi,x,\xi') : |\xi'|-|\xi| = 2x\} \subset S^{[n-2]}\times S\times S^{[n]}$$

allows us to express δ in terms of Nakajima's creation and annihilation operators, and thus in

$$\mathrm{H}^*(S^{[n]},\mathbb{Q})[2n].$$

We describe the general rule for evaluating the fundamental class in $A^{[n]}$. Let

$$[\mathrm{pt}]\in \mathrm{H}^4(S,\mathbb{Z})[2]\subset A$$

be the point class, which is of degree -2. Let

$$[\mathrm{pt}]_{\{1\}}\otimes\cdots\otimes[\mathrm{pt}]_{\{n\}}(\mathrm{id})\in A^{[n]}$$

denote the unique class of degree $-2n$ up to scalar. Then the class of a point in $S^{[n]}$ is equal to [15, §2.10]

$$[\mathrm{pt}_{S^{[n]}}] = n![\mathrm{pt}]_{\{1\}}\otimes\cdots\otimes[\mathrm{pt}]_{\{n\}}(\mathrm{id}). \tag{3.2}$$

4 Decomposition of the cohomology representation

We summarize general results on representations of complex (or split) orthogonal groups. Consider the orthogonal group of odd dimension $2r+1$, i.e., one associated with a quadratic form of rank $2r+1$. Let $V(\lambda)$ denote the representation with highest weight $\lambda=(\lambda_1,\ldots,\lambda_r)$, where λ is a vector consisting entirely of integers (or half integers) in the fundamental chamber

$$\{\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_{r-1}\ge\lambda_r\ge 0\}.$$

For orthogonal groups of even dimension $2r$, we also use $V(\lambda)$ to denote the representation of highest weight λ, taken from the fundamental chamber

$$\{\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_{r-1}\ge|\lambda_r|\ge 0\}.$$

We only consider representations where the λ_j are integers.

Recall that

- $V(1,0,\ldots)$ is the standard representation V.
- We have
$$V(\underbrace{1,\cdots,1}_{k\text{ times}},0,\cdots) = \bigwedge^k V,$$
provided $k<r$ (in the even case) or $k\le r$ (in the odd case); see, for instance, [7, Thms. 19.2 and 19.14].

- $V(k,0,\ldots) = \mathrm{Sym}^k(V)/\mathrm{Sym}^{k-2}(V)$, embedded via the dual to the quadratic form on V.
- For the odd orthogonal group, we have
$$\dim V(\lambda) = \prod_{i<j} \frac{\ell_i - \ell_j}{j-i} \prod_{i\leq j} \frac{\ell_i + \ell_j}{2n+1-i-j}$$
where $\ell_i = \lambda_i + n - i + \frac{1}{2}$ [7, p. 408].
- For the even orthogonal group, we have
$$\dim V(\lambda) = \prod_{i<j} \frac{\ell_i^2 - \ell_j^2}{(j-i)(2n-i-j)}$$
where $\ell_i = \lambda_i + n - i$ [7, p. 410].
- Let $V_X(\lambda)$ denote an irreducible representation of an orthogonal group G_X of dimension $2r+1$, $G_S \subset G_X$ the orthogonal subgroup of dimension $2r$ fixing a non-isotropic vector, and $V_S(\overline{\lambda})$ the representation of G_S with highest weight $\overline{\lambda}$. Then we have the branching rule [7, p. 426]
$$\mathrm{Res}_{G_S}^{G_X} V_X(\lambda) = \oplus_{\overline{\lambda}} V_S(\overline{\lambda}),$$
where the sum ranges over all $\overline{\lambda}$ with
$$\lambda_1 \geq \overline{\lambda}_1 \geq \lambda_2 \geq \overline{\lambda}_2 \geq \cdots \lambda_r \geq |\overline{\lambda}_r|,$$
with the λ_i and $\overline{\lambda}_i$ simultaneously all integers or half-integers.

Let X be a generic deformation of $S^{[n]}$, where S is a K3 surface. Our goal is to decompose $\mathrm{H}^*(X,\mathbb{Q})$ into irreducible representations for the action of the identity component G_X of the special orthogonal group associated with the Beauville-Bogomolov form on $\mathrm{H}^2(X,\mathbb{Q})$. Let G_S denote the identity component of the special orthogonal group associated with the intersection form on $\mathrm{H}^2(S,\mathbb{Q})$. The decomposition
$$\mathrm{H}^2(S^{[2]},\mathbb{Z}) = \mathrm{H}^2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}\delta$$
induces an inclusion $G_S \subset G_X$.

Proposition 4.1. *Let X be deformation equivalent to $S^{[n]}$ for some n, where S is a K3 surface. Then G_X admits a representation on the cohomology ring of X.*

Proof. Let Mon $\subset \mathrm{Aut}(\mathrm{H}^*(X,\mathbb{Z}))$ denote the monodromy group, i.e., the group generated by the monodromy representations of all connected families containing X. Let $\mathrm{Mon}^2 \subset \mathrm{Aut}(\mathrm{H}^2(X,\mathbb{Z}))$ denote its image under projection to the second cohomology group, so we have an exact sequence
$$1 \to K \to \mathrm{Mon} \to \mathrm{Mon}^2 \to 1.$$
Markman has shown [17, §4.3] that K is finite.

Note that G_X is a connected component of the Zariski closure of Mon^2 (see, for example [17, §1.8]). Since Mon and Mon^2 differ only by finite subgroups, it follows that the universal cover $\widetilde{G_X} \to G_X$ acts on the cohomology ring of X. Since the cohomology of X is nonzero only in even degrees, this representation passes to G_X. $\square$

In principle, we can decompose $\mathrm{H}^*(X,\mathbb{R})$ explicitly into isotypic components as follows:

(1) Fix an embedding $G_S \subset G_X$, e.g., using the isomorphism

$$\mathrm{H}^2(X,\mathbb{Z}) \simeq \mathrm{H}^2(S,\mathbb{Z}) \oplus_\perp \mathbb{Z}\delta,$$

and compatible maximal tori (both of which have rank 11).

(2) Identify the highest-weight irreducible G_S-representation $V_S(\lambda) \subset \mathrm{H}^*(S^{[n]},\mathbb{R})$, which is a summand of the restriction of an irreducible $V_X(\lambda) \subset \mathrm{H}^*(X,\mathbb{R})$. Decompose $V_X(\lambda)$ into irreducible G_S-representations.

(3) Repeat step two for $\mathrm{H}^*(X,\mathbb{R})/V_X(\lambda)$ and subsequent quotients.

First consider $X = S^{[2]}$. We have decompositions

$$\mathrm{H}^*(S^{[2]}) = A \oplus \mathrm{Sym}^2(A)$$

inducing

$$\begin{array}{rcl} \mathrm{H}^2(S^{[2]}) & = & \mathrm{H}^0(S) \oplus (\mathrm{H}^0(S) \otimes \mathrm{H}^2(S)) = \mathbf{1}_S \oplus V_S(1,0,\dots) \\ \mathrm{H}^4(S^{[2]}) & = & \mathrm{H}^2(S) \oplus (\mathrm{H}^0(S) \otimes \mathrm{H}^4(S)) \oplus \mathrm{Sym}^2(\mathrm{H}^2(S)) \\ & = & V_S(1,0,\dots) \oplus \mathbf{1}_S^{\oplus 2} \oplus V_S(2,0,\dots), \end{array}$$

where $\mathbf{1}_S$ is the trivial irreducible representation. Let $V_X(2,0,\dots,0)$ denote the highest-weight representation associated to $\mathrm{Sym}^2(\mathrm{H}^2(X))$ so that

$$\mathrm{Sym}^2(\mathrm{H}^2(X)) = V_X(2,0,\dots) \oplus \mathbf{1}_X.$$

The branching rule gives

$$V_X(1,0,\dots) = V_S(1,0,\dots) \oplus \mathbf{1}_S$$

and

$$V_X(2,0,\dots) = V_S(2,0,\dots) \oplus V_S(1,0,\dots) \oplus \mathbf{1}_S.$$

Therefore we obtain

$$\begin{array}{rcl} \mathrm{H}^2(X) & = & V_X(1,0,\dots) \\ \mathrm{H}^4(X) & = & V_X(2,0,\dots) \oplus \mathbf{1}_X. \end{array}$$

Now consider $X = S^{[3]}$. We have

$$\mathrm{H}^*(S^{[3]}) = A \oplus (A \otimes A) \oplus \mathrm{Sym}^3(A)$$

inducing the following decompositions (as described in [15, Example 2.9]):

$$\begin{array}{rcl}
\mathrm{H}^2(S^{[3]}) & = & (\mathrm{H}^0(S)^{\otimes 2}) \oplus (\mathrm{H}^2(S) \otimes \mathrm{H}^0(S)^{\otimes 2}) \\
& = & \mathbf{1}_S \oplus V_S(1,0\ldots) \\
\mathrm{H}^4(S^{[3]}) & = & \mathrm{H}^0(S) \oplus (\mathrm{H}^0(S) \otimes \mathrm{H}^2(S))^{\oplus 2} \\
& & \oplus(\mathrm{Sym}^2(\mathrm{H}^2(S)) \otimes \mathrm{H}^0(S)) \oplus (\mathrm{H}^4(S) \otimes \mathrm{H}^0(S)^{\otimes 2}) \\
& = & \mathbf{1}_S^{\oplus 3} \oplus V_S(1,0,\ldots)^{\oplus 2} \oplus V_S(2,0,\ldots) \\
\mathrm{H}^6(S^{[3]}) & = & \mathrm{H}^2(S) \oplus (\mathrm{H}^2(S) \otimes \mathrm{H}^2(S)) \oplus (\mathrm{H}^0(S) \otimes \mathrm{H}^4(S))^{\oplus 2} \\
& & \oplus\mathrm{Sym}^3(\mathrm{H}^2(S)) \oplus (\mathrm{H}^4(S) \otimes \mathrm{H}^2(S) \otimes \mathrm{H}^0(S)) \\
& = & \mathbf{1}_S^{\oplus 3} \oplus V_S(1,0,\ldots)^{\oplus 3} \oplus V_S(1,1,0,\ldots) \\
& & \oplus V_S(2,0,\ldots) \oplus V_S(3,0,\ldots).
\end{array}$$

Note that two of the trivial summands in $\mathrm{H}^4(S^{[3]})$ lie in the decomposable classes, i.e., the image of $\mathrm{Sym}^2(\mathrm{H}^2(S^{[2]}))$.

Let $V_X(1,1,0,\ldots) = \bigwedge^2 V_X(1,0,\ldots)$ and $V_X(3,0,\ldots)$ denote the highest weight representation in $\mathrm{Sym}^3(V_X(1,0,\ldots))$ so that

$$\mathrm{Sym}^3(V_X(1,0,\ldots)) = V_X(3,0,\ldots) \oplus V_X(1,0,\ldots).$$

Therefore we obtain

$$\begin{array}{rcl}
\mathrm{H}^2(X) & = & V_X(1,0,\ldots) \\
\mathrm{H}^4(X) & = & V_X(2,0,\ldots) \oplus V_X(1,0,\ldots) \oplus \mathbf{1}_X \\
\mathrm{H}^6(X) & = & V_X(3,0,\ldots) \oplus V_X(1,1,0\ldots) \oplus V_X(1,0,\ldots) \oplus \mathbf{1}_X.
\end{array}$$

The trivial factor in $\mathrm{H}^4(X)$ corresponds to the Chern class $c_2(X)$; indeed, $c_2(X)$ is a Hodge class and is nonzero because $c_2(X)^3 > 0$ [5, Remark 5.5]. Note that $c_2(X)$ is proportional to the trivial summand in $\mathrm{Sym}^2(\mathrm{H}^2(X))$ associated with the Beauville-Bogomolov form, regarded as an object in the symmetric algebra over $\mathrm{H}^2(X)$.

Our main task is to analyze the trivial summand in $\mathrm{H}^6(X)$.

5 Cohomology computations for length-three subschemes

The general rule for multiplication in $A\{\mathfrak{S}_n\}$ is fairly complicated, so we will only give a formula in the case ($n = 3$) we need. The fact that A only has terms of even degree simplifies the expressions of [15, §2.17]:

$$\begin{array}{rcl}
(\alpha_{\{1,2\}} \otimes \beta_{\{3\}})(12) \cdot (\gamma_{\{1,3\}} \otimes \delta_{\{2\}})(13) & = & \alpha\beta\gamma\delta(132) \\
(\alpha_{\{1,2\}} \otimes \beta_{\{3\}})(12) \cdot (\gamma_{\{1,2\}} \otimes \delta_{\{3\}})(12) & = & \Delta_*(\alpha\gamma) \otimes (\beta\delta)(\mathrm{id}) \\
\alpha_{\{1,2,3\}}(123) \cdot \beta_{\{1,2,3\}}(123) & = & (\alpha\beta e)(132) \\
\alpha_{\{1,2,3\}}(123) \cdot \beta_{\{1,2,3\}}(132) & = & (\Delta_*(\alpha\beta))_{\{1,2,3\}}(\mathrm{id}),
\end{array}$$

where Δ_* in the last line is the adjoint of the threefold multiplication $A \otimes A \otimes A \to A$.

The remaining products can be deduced as formal consequences using the associativity of the multiplication, e.g.,

$$\begin{aligned}
&(\alpha_{\{1,2\}} \otimes \beta_{\{3\}})(12) \cdot \gamma_{\{1,2,3\}}(132) \\
&\quad = (\alpha_{\{1,2\}} \otimes \beta_{\{3\}})(12) \cdot (\gamma_{\{1,2\}} \otimes 1_{\{3\}})(12) \cdot (13) \\
&\quad = (\Delta_*(\alpha\gamma)_{\{1,2\}} \otimes \beta_{\{3\}})(\mathrm{id}) \cdot (1_{\{1,3\}} \otimes 1_{\{2\}})(13) \\
&\quad = \alpha\beta\gamma(\Delta_*(1))_{\{1,3\},\{2\}}(13),
\end{aligned}$$

where α, β, and γ act on the diagonal via either the first or second variable. Thus in particular

$$(12) \cdot (132) = (\Delta_*(1))_{\{1,3\},\{2\}}(13).$$

We compute intersections among the absolute Hodge classes for $S^{[3]}$, i.e., classes that are Hodge for general K3 surfaces S. From now on, to condense notation we omit factors of the form $1_{\{i\}}, 1_{\{i,j\}}$, etc. from our expressions.

Based on the representation-theoretic analysis in Section 4, we expect one independent classes in codimension one, three in codimension two, and three in codimension three. We have the unique divisor

$$\delta = (12) + (13) + (23).$$

In codimension two, we have

$$\begin{aligned}
P &= [\mathrm{pt}]_{\{1\}} + [\mathrm{pt}]_{\{2\}} + [\mathrm{pt}]_{\{3\}} \\
Q &= \textstyle\sum_{j=1}^{22} e_{j\{1\}} \otimes e^\vee_{j\,\{2\}} + e_{j\{1\}} \otimes e^\vee_{j\,\{3\}} + e_{j\{2\}} \otimes e^\vee_{j\,\{3\}} \\
R &= (132) + (123).
\end{aligned}$$

We identify the linear combinations of P, Q, and R that are decomposable, i.e., lie in the symmetric algebra generated by $\mathrm{H}^2(S^{[3]})$. First, we have

$$\begin{aligned}
\delta^2 &= (\Delta_* 1)_{\{1,2\}}(12) + (\Delta_* 1)_{\{1,3\}}(13) + (\Delta_* 1)_{\{2,3\}}(23) \\
&\quad + 3((132) + (123)) \\
&= -2P - Q + 3R.
\end{aligned}$$

For the second decomposable class, we extract the intersection form on $\mathrm{H}^2(S)$ interpreted as an element in $\mathrm{Sym}^2(\mathrm{H}^2(S))$:

$$\begin{aligned}
&\textstyle\sum_{j=1}^{22} (e_{j\{1\}} + e_{j\{2\}} + e_{j\{3\}}) \cdot (e^\vee_{j\,\{1\}} + e^\vee_{j\,\{2\}} + e^\vee_{j\,\{3\}}) \\
&\quad = \textstyle\sum_{j=1}^{22} (e_j \cdot e^\vee_{j\,\{1\}} + e_j \cdot e^\vee_{j\,\{2\}} + e_j \cdot e^\vee_{j\,\{3\}}) \\
&\qquad + 2\textstyle\sum_{j=1}^{22} e_{j\{1\}} \otimes e^\vee_{j\,\{2\}} + e_{j\{1\}} \otimes e^\vee_{j\,\{3\}} + e_{j\{2\}} \otimes e^\vee_{j\,\{3\}} \\
&\quad = 22P + 2Q.
\end{aligned}$$

We now isolate the class in $\mathrm{Sym}^2(\mathrm{H}^2(S^{[3]})) \subset \mathrm{H}^4(S^{[3]})$ corresponding to the Beauville-Bogomolov quadratic form $(,)$. Recall this generates the unique trivial factor in the fourth cohomology, regarded as a representation of G_X. As we saw in Section 2, the Beauville-Bogomolov form agrees with the intersection form on $\mathrm{H}^2(S) \subset \mathrm{H}^2(S^{[3]})$ and $(\delta,\delta) = -4$. Thus we obtain

$$22P + 2Q - \frac{1}{4}\delta^2. \tag{5.1}$$

In codimension three, we have

$$\begin{aligned} U &= [\mathrm{pt}]_{\{1,2\}}(12) + [\mathrm{pt}]_{\{1,3\}}(13) + [\mathrm{pt}]_{\{2,3\}}(23) \\ V &= [\mathrm{pt}]_{\{3\}}(12) + [\mathrm{pt}]_{\{2\}}(13) + [\mathrm{pt}]_{\{1\}}(23) \\ W &= \textstyle\sum_{j=1}^{22} e_{j\{1,2\}} \otimes e^\vee_{j\ \{3\}}(12) + e_{j\{1,3\}} \otimes e^\vee_{j\ \{2\}}(13) + e_{j\{2,3\}} \otimes e^\vee_{j\ \{1\}}(23). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \delta \cdot P &= ((12)+(13)+(23)) \cdot ([\mathrm{pt}]_{\{1\}} + [\mathrm{pt}]_{\{2\}} + [\mathrm{pt}]_{\{3\}}) \\ &= 2U + V \\ \delta \cdot Q &= ((12)+(13)+(23)) \cdot (\textstyle\sum_{j=1}^{22} e_{j\{1\}} \otimes e^\vee_{j\ \{2\}} + e_{j\{1\}} \otimes e^\vee_{j\ \{3\}} \\ &\qquad + e_{j\{2\}} \otimes e^\vee_{j\ \{3\}}) \\ &= 22([\mathrm{pt}]_{\{1,2\}}(12) + [\mathrm{pt}]_{\{1,3\}}(13) + [\mathrm{pt}]_{\{2,3\}}(23)) \\ &\quad + 2(\textstyle\sum_{j=1}^{22} e_{j\{1,2\}} \otimes e^\vee_{j\ \{3\}} + e_{j\{1,3\}} \otimes e^\vee_{j\ \{2\}} + e_{j\{2,3\}} \otimes e^\vee_{j\ \{1\}}) \\ &= 22U + 2W \\ \delta \cdot R &= ((12)+(13)+(23))((132)+(123)) \\ &= 2(\Delta_* 1_{\{1,2\},\{3\}}(12) + \Delta_* 1_{\{1,3\},\{2\}} + \Delta_* 1_{\{2,3\},\{1\}}) \\ &= -2(U+V+W). \end{aligned}$$

We deduce then that

$$\delta^3 = \delta \cdot (-2P - Q + 3R) = -32U - 8V - 8W$$

and

$$\delta \cdot (22P + 2Q) = 88U + 22V + 4W.$$

Finally, we compute the intersection pairing on the subspace of the middle cohomology spanned by U, V, and W. Dimensional considerations give vanishing

$$U^2 = V^2 = U \cdot W = V \cdot W = 0.$$

For the remaining numbers, we get

$$\begin{aligned} U \cdot V &= ([\mathrm{pt}]_{\{1,2\}}(12) + [\mathrm{pt}]_{\{1,3\}}(13) + [\mathrm{pt}]_{\{2,3\}}(23)) \\ &\qquad \cdot([\mathrm{pt}]_{\{3\}}(12) + [\mathrm{pt}]_{\{2\}}(13) + [\mathrm{pt}]_{\{1\}}(23)) \\ &= -3[\mathrm{pt}]_{\{1\}} \otimes [\mathrm{pt}]_{\{2\}} \otimes [\mathrm{pt}]_{\{3\}}\mathrm{id} \end{aligned}$$

and

$$\begin{aligned} W^2 &= (\textstyle\sum_{j=1}^{22} e_{j\{1,2\}} \otimes e^\vee_{j\ \{3\}}(12) + e_{j\{1,3\}} \otimes e^\vee_{j\ \{2\}}(13) + e_{j\{2,3\}} \otimes e^\vee_{j\ \{1\}}(23))^2 \\ &= -3 \cdot 22 \cdot [\mathrm{pt}]_{\{1\}} \otimes [\mathrm{pt}]_{\{2\}} \otimes [\mathrm{pt}]_{\{3\}}\mathrm{id}. \end{aligned}$$

Remark 5.1. As a consistency check, we evaluate

$$\begin{aligned} \delta^6 &= (-32U - 8V - 8W)^2 = 2^6(8UV + W^2) \\ &= 2^6(-24 - 66)[\mathrm{pt}]_{\{1\}} \otimes [\mathrm{pt}]_{\{2\}} \otimes [\mathrm{pt}]_{\{3\}}\mathrm{id}. \end{aligned}$$

Using the formula for the point class (Equation 3.2), we obtain

$$\delta^6 = -\frac{2^7 \cdot 3^2 \cdot 5}{2 \cdot 3} = -2^6 \cdot 3 \cdot 5.$$

This is compatible with the Fujiki-type identity

$$D^6 = 15\,(D,D)^3, \quad D \in \mathrm{H}^2(S^{[3]}, \mathbb{Q}),$$

as $(\delta, \delta) = -4$.

6 Evaluation of the distinguished absolute Hodge class

Let S be a general K3 surface and X a general deformation of $S^{[3]}$. The computations above show that the middle cohomology of X admits one Hodge class

$$\mathrm{H}^6(X, \mathbb{Q}) \cap \mathrm{H}^{3,3}(X) = \mathbb{Q}\eta$$

and the middle cohomology of $S^{[3]}$ admits three Hodge classes

$$\mathrm{H}^6(S^{[3]}, \mathbb{Q}) \cap \mathrm{H}^{3,3}(S^{[3]}) = \mathbb{Q}\eta \oplus \mathbb{Q}\delta^3 \oplus \mathbb{Q}c_2(X)\delta.$$

Our goal is to compute the self-intersection of η, at least up to the square of a rational number. Note that η is orthogonal to δ^3 and $\delta c_2(X)$ under the intersection form, by the analysis in Section 4. The analysis here gives the one structure constant left open in [16, Ex. 14].

Proposition 6.1. *Let X be deformation equivalent to $S^{[3]}$, for S a K3 surface. Let $\eta \in \mathrm{H}^6(X, \mathbb{Q})$ denote the unique (up to scalar) absolute Hodge class. Then $\eta^2 = 4$.*

Proof. The argument relies heavily on the analysis in Section 5. Consider the decomposable classes in $\mathrm{H}^6(S^{[3]})$, i.e., those coming from $\mathrm{Sym}^3(\mathrm{H}^2(S^{[3]}))$. We have computed

$$\delta^3 = -32U - 8V - 8W, \quad \delta \cdot (22P + 2Q) = 88U + 22V + 4W,$$

hence the subspace

$$\mathrm{span}\{32U + 8V + 8W, 88U + 22V + 4W\} = \mathrm{span}\{4U + V, W\}$$

is spanned by decomposable classes. It has orthogonal complement spanned by $4U - V$. Thus we have

$$\eta = 4U - V$$

and

$$\begin{aligned} \eta^2 &= -8UV \\ &= 24([\mathrm{pt}]_{\{1\}} \otimes [\mathrm{pt}]_{\{2\}} \otimes [\mathrm{pt}]_{\{3\}})\mathrm{id} \\ &= 4. \end{aligned}$$

□

7 Proof of the main theorem

We compute the cohomology class of a Lagrangian subspace $\mathbb{P}^3 \subset X$, where X is deformation equivalent to the Hilbert scheme of length-three subschemes. As we shall see, the formula for $[\mathbb{P}^3]$ involves only decomposable classes, and not the absolute Hodge class η:

Lemma 7.1. *Let $\mathbb{P}^n \subset X$ be embedded in a general irreducible holomorphic symplectic variety of dimension $2n$. Then we have*

$$c_{2j}(\mathscr{T}_X|\mathbb{P}^n) = (-1)^j h^{2j}\binom{n+1}{j},$$

where h is the hyperplane class.

This is proved using the exact sequence

$$0 \to \mathscr{T}_{\mathbb{P}^n} \to \mathscr{T}_X|\mathbb{P}^n \to \mathscr{N}_{\mathbb{P}^n/X} \to 0$$

and

$$\mathscr{N}_{\mathbb{P}^n/X} \simeq \mathscr{T}_{\mathbb{P}^n}^{\vee},$$

reflecting the fact that $\mathbb{P}^n$ is a Lagrangian subvariety of X.

Regarding

$$\mathrm{H}^2(X,\mathbb{Z}) \subset \mathrm{H}_2(X,\mathbb{Z})$$

as a subgroup of index four, we can express $\ell = \lambda/4$ for some divisor class $\lambda \in \mathrm{H}^2(X,\mathbb{Z})$. (This might not be primitive.)

Given a deformation of X such that λ remains algebraic, the subvariety $\mathbb{P}^3$ deforms as well [10]. Without loss of generality, we can deform X to a variety containing a $\mathbb{P}^3$, but otherwise having a general Hodge structure. In particular, we have an injection

$$\mathrm{Sym}(\mathrm{H}^2(X,\mathbb{Q})) \hookrightarrow \mathrm{H}^*(X,\mathbb{Q}).$$

We expect to be able to write

$$\left[\mathbb{P}^3\right] = a\lambda c_2(X) + b\lambda^3 + d\eta$$

for some $a,b,d \in \mathbb{Q}$.

Furthermore, the Fujiki relations [6] imply that for each $f \in \mathrm{H}^2(X,\mathbb{Z})$,

$$f^6 = e_0\,(f,f)^3, \quad c_2(X)f^4 = e_2\,(f,f)^2, \quad c_4(X)f^2 = e_4\,(f,f)$$

for suitable rational constants e_0, e_2, e_4. Precisely, we have [5]

$$c_2^2(X)f^2 = \frac{5}{2}c_4(X)f^2.$$

The Riemann-Roch formula gives

$$\chi(\mathscr{O}_X(f)) = \frac{f^6}{6!} + \frac{c_2(X)f^4}{12\cdot 4!} + \frac{f^2(3c_2^2 - c_4)}{720\cdot 2!} + 4.$$

On the other hand, we know that

$$\chi(\mathscr{O}_X(f)) = \frac{1}{3!2^3}((f,f)+8)((f,f)+6)((f,f)+4).$$

Perhaps the quickest way to check this formula is to observe that if $X = S^{[3]}$ and f is a very ample divisor on S with no higher cohomology then the induced sheaf $\mathscr{O}_X(f)$ has no higher cohomology and

$$\dim \Gamma(\mathscr{O}_X(f)) = \dim \operatorname{Sym}^3(\Gamma(\mathscr{O}_S(f))) = \binom{\chi(\mathscr{O}_S(f))+2}{3}.$$

Equating coefficients, we find

$$\begin{array}{rcl} f^6 & = & 15\,(f,f)^3 \\ f^4c_2 & = & 108\,(f,f)^2 \\ f^2c_4 & = & 480\,(f,f) \\ f^2c_2^2 & = & 1200\,(f,f)\,. \end{array}$$

Remark 7.2. These structural constants allow us to normalize expression (5.1) to get the second Chern class

$$c_2(T_X) = \frac{4}{3}(22P+2Q-\frac{1}{4}\delta^2).$$

We generate Diophantine equations for $a, b, (\lambda,\lambda)$ and eventually, d. First, observe that

$$(\lambda,\ell) = \lambda\cdot\ell = \deg \lambda|\mathbb{P}^3$$

so that $\lambda|\mathbb{P}^3$ is $(\lambda,\lambda)/4$ times the hyperplane class. Thus we have

$$\left[\mathbb{P}^3\right]\lambda^3 = ((\lambda,\lambda)/4)^3$$

and

$$\left[\mathbb{P}^3\right]\lambda^3 = a\lambda^4c_2(X)+b\lambda^6.$$

Equating these expressions and evaluating the terms, we find

$$(\lambda,\lambda)\,(15b-1/64)+108a = 0.$$

We have divided out by (λ,λ); the solution $(\lambda,\lambda)=0$ is not possible for geometric reasons, and we shall exclude it algebraically below.

Second, the Lemma on restrictions of Chern classes implies

$$\left[\mathbb{P}^3\right]\lambda c_2(X) = -(\lambda,\lambda)$$

whereas the formula for the class of $\mathbb{P}^3$ yields

$$\left[\mathbb{P}^3\right]\lambda c_2(X) = a\lambda^2c_2(X)^2+b\lambda^4c_2(X).$$

Thus we obtain

$$108b\,(\lambda,\lambda)+(1200a+1) = 0.$$

Remark 7.3. The cup product of $\mathrm{H}^*(X)$ is compatible with the G_X-action, so the subring generated by Chern classes and elements of $\mathrm{H}^2(X)$ is orthogonal to η. Thus even if the decomposition of $[\mathbb{P}^3]$ were to involve η, the computations up to this point would not reflect this.

Finally, the fact that

$$\left[\mathbb{P}^3\right]^2 = c_3(\mathscr{N}_{\mathbb{P}^3/X}) = c_3(\mathscr{T}^\vee_{\mathbb{P}^3}) = -4$$

yields the *cubic* equation

$$15b^2\,(\lambda,\lambda)^3 + 216ab\,(\lambda,\lambda)^2 + 1200\,(\lambda,\lambda)\,a^2 + d^2\eta\cdot\eta = -4.$$

The proof of Proposition 6.1 implies that $\eta\cdot\eta = 4$. In particular, $(\lambda,\lambda) = 0$ is excluded.

Eliminating a and b from these equations and setting $L = (\lambda,\lambda)$, we obtain

$$-2^{16}\cdot 3\cdot 11d^2 = 5^2L^3 + 2^5\cdot 3^2L^2 + 2^8\cdot 5L + 2^{16}\cdot 3\cdot 11. \tag{7.1}$$

We know, *a priori*, that $L\in\mathbb{Z}$ and $d\in\mathbb{Q}$.

Proposition 7.4. *The only solution to (7.1) with $L\in\mathbb{Z}$ and $d\in\mathbb{Q}$ is $d=0$ and $L=-48$.*

We assume this for the moment; its proof can be found in Section 8.

Back-substitution yields

$$a = 1/96, \quad b = 1/384, \quad (\ell,\ell) = -3.$$

We claim that $\lambda/2\in\mathrm{H}^2(X,\mathbb{Z})$, i.e., λ is not primitive. Using the isomorphism

$$\mathrm{H}_2(X,\mathbb{Z}) = \mathrm{H}_2(S,\mathbb{Z})\oplus_\perp\mathbb{Z}\delta^\vee, \quad \left(\delta^\vee,\delta^\vee\right) = -1/4$$

we can express

$$\ell = D + m\delta^\vee, \quad D\in\mathrm{H}_2(S,\mathbb{Z}), m\in\mathbb{Z}.$$

If λ were primitive then m would have to be odd and

$$-3 = (\ell,\ell) = (D,D) - m^2/4.$$

Since $(D,D)\in 2\mathbb{Z}$, we have a contradiction.

8 Diophantine analysis

Theorem 8.1. *The only solution to*

$$-2^{16}\cdot 3\cdot 11d^2 = 5^2L^3 + 2^5\cdot 3^2L^2 + 2^8\cdot 5L + 2^{16}\cdot 3\cdot 11$$

with $L\in\mathbb{Z}$ and $d\in\mathbb{Q}$ is $L=-48$, $d=0$.

Proof. Put $x = -2^{-4}\cdot 3\cdot 5^2\cdot 11(L+48)$ and $y = 2^2\cdot 3^2\cdot 5^2\cdot 11^2 d$. The equation then takes the form

$$E : y^2 = x^3 + ax^2 + bx \tag{8.1}$$

where

$$a = 3^3\cdot 11\cdot 23, \qquad b = 2^2\cdot 3^2\cdot 5^2\cdot 11^3\cdot 13.$$

It suffices to prove the stronger statement that there are no solutions to (8.1) with $x, y \in \mathbb{Z}[\frac{1}{2}]$, apart from $x = y = 0$.

The proof is given in two steps. Proposition 8.2 below determines explicitly the structure of the Mordell–Weil group $E(\mathbb{Q})$. Proposition 8.3 then identifies the integral points. □

Algorithms for both of these steps are implemented in computer algebra systems such as Sage [22] and Magma [2], and the theorem may be verified this way. To avoid depending on the correctness of these systems, we give alternative proofs that use as little machine assistance as possible.

We first set notation and briefly recall some facts about point multiplication on elliptic curves given in the form (8.1). Let O denote the zero element of $E(\mathbb{Q})$ (the point at infinity). For nonzero $R \in E(\mathbb{Q})$ we write

$$R = (x(R), y(R)) = \left(\frac{\alpha(R)}{e(R)^2}, \frac{\beta(R)}{e(R)^3}\right),$$

where $\alpha, \beta, e \in \mathbb{Z}$, $e \geq 1$ and $(\alpha, e) = (\beta, e) = 1$. If p is a prime, then $p \mid e(R)$ if and only if R reduces to the identity in $E(\mathbb{F}_p)$. If $m \geq 1$ and $mR \neq O$, then $e(R) \mid e(mR)$ [14, Ch. III, Thm. 1.2]. The point $Q = (0,0)$ has order two, and addition with Q is given by the formula

$$R + Q = \left(\frac{b}{x(R)}, \frac{-b\cdot y(R)}{x(R)^2}\right) \qquad (R \neq O, Q). \tag{8.2}$$

Proposition 8.2. *We have $E(\mathbb{Q}) \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$, where the torsion part is generated by Q, and the free part by*

$$P = \left(\frac{2^2\cdot 3\cdot 5\cdot 277^2}{53^2}, \frac{2^3\cdot 3^2\cdot 5\cdot 29\cdot 277\cdot 11311}{53^3}\right).$$

Proof. The discriminant of the Weierstrass equation (8.1) is given by

$$\Delta = 16b^2(a^2 - 4b) = -2^8\cdot 3^6\cdot 5^4\cdot 11^8\cdot 13^2\cdot 113\cdot 127,$$

so the model is minimal, and the primes of bad reduction are 2, 3, 5, 11, 13, 113 and 127.

We first check that the torsion subgroup is as described. For ℓ prime, by [20, Prop. VII.3.1] we see that $E(\mathbb{Q})[\ell]$ injects into $E(\mathbb{F}_{17}) = \mathbb{Z}/18\mathbb{Z}$ for $\ell \neq 17$ and into $E(\mathbb{F}_{19}) = \mathbb{Z}/14\mathbb{Z}$ for $\ell \neq 19$. These facts force $E(\mathbb{Q})[2] = \mathbb{Z}/2\mathbb{Z}$ and $E(\mathbb{Q})[\ell] = 0$ for $\ell \neq 2$. Hence $E_{\mathrm{tors}}(\mathbb{Q}) = \langle Q \rangle$.

Now we consider the free part. The point P was found using Cremona's mwrank library [3] via the Sage computer algebra system [22]. We may check that $P \in E(\mathbb{Q})$ using a computer (or by hand with some patience); this shows that $\operatorname{rank} E \geq 1$.

To show that $\operatorname{rank} E \leq 1$ we use a standard 2-descent strategy (see for example [21, Ch. III]). Consider the auxiliary curve

$$E' : y^2 = x^3 - 2ax^2 + (a^2 - 4b)x.$$

There are isogenies $\phi : E \to E'$ and $\hat{\phi} : E' \to E$ of degree 2, and injections

$$E(\mathbb{Q})/\hat{\phi}(E'(\mathbb{Q})) \overset{\psi}{\hookrightarrow} S \subset \mathbb{Q}^*/(\mathbb{Q}^*)^2,$$
$$E'(\mathbb{Q})/\phi(E(\mathbb{Q})) \overset{\psi'}{\hookrightarrow} S' \subset \mathbb{Q}^*/(\mathbb{Q}^*)^2,$$

where S consists of the cosets $\delta(\mathbb{Q}^*)^2$ for $\delta \mid 2{\cdot}3{\cdot}5{\cdot}11{\cdot}13$, and S' of the cosets for $\delta \mid 3{\cdot}11{\cdot}113{\cdot}127$ (these are the primes dividing b and $a^2 - 4b$ respectively).

We must determine which elements of S and S' arise from points in $E(\mathbb{Q})$ and $E'(\mathbb{Q})$. This is achieved by testing for the existence of $\mathbb{Q}$-rational points on the two families of quartic curves

$$C_\delta : \delta w^2 = \delta^2 z^4 + \delta a z^2 + b, \qquad \delta \in S,$$
$$C'_\delta : \delta w^2 = \delta^2 z^4 - 2\delta a z^2 + (a^2 - 4b), \qquad \delta \in S'.$$

First consider C_δ, which may be rewritten as

$$\delta w^2 = \delta^2 z^4 + 3^3{\cdot}11{\cdot}23\delta z^2 + 2^2{\cdot}3^2{\cdot}5^2{\cdot}11^3{\cdot}13,$$

or

$$4\delta w^2 = (2\delta z^2 + 3^3{\cdot}11{\cdot}23)^2 + 3^2{\cdot}11^2{\cdot}113{\cdot}127,$$

for $\delta \mid 2{\cdot}3{\cdot}5{\cdot}11{\cdot}13$.

For $\delta = 11{\cdot}13$ there is a trivial rational point $z = 0$, $w = 2{\cdot}3{\cdot}5{\cdot}11$, corresponding to the class of Q in $E(\mathbb{Q})/\hat{\phi}(E'(\mathbb{Q}))$. For $\delta = 3{\cdot}5$ there is a nontrivial rational point corresponding to P, namely $z = 2{\cdot}277/53$, $w = 2^2{\cdot}3{\cdot}29{\cdot}11311/53^2$. Rational points are automatic for $\delta = 1$ and $3{\cdot}5{\cdot}11{\cdot}13$ since the image of ψ is a subgroup of S. We will show that $C_\delta(\mathbb{Q}) = \emptyset$ for all other δ, by identifying local obstructions. We only summarize the obstructions encountered, omitting detailed arguments (which are straightforward, but in some cases lengthy).

Consider the conic $4\delta w^2 = u^2 + 3^2{\cdot}11^2{\cdot}113{\cdot}127$. If $\delta < 0$ then the conic has no points over $\mathbb{R}$. If $(\delta/113) = -1$ then the conic has no points over $\mathbb{Q}_{113}$. Since $(2/113) = (11/113) = (13/113) = 1$ and $(3/113) = (5/113) = -1$, this shows that $3 \mid \delta$ if and only if $5 \mid \delta$. No further information can be obtained from local analysis of the conic, so we proceed to C_δ itself. If $2 \mid \delta$ then C_δ has no points over $\mathbb{Q}_2$. If $(\delta/11) = -1$, or if $\delta = 11\varepsilon$ with $(\varepsilon/11) = 1$, then C_δ has no points over $\mathbb{Q}_{11}$. These conditions rule out all but the four values for δ noted in the previous paragraph.

Next we give a similar analysis for C'_δ, which becomes

$$\delta w^2 = \delta^2 z^4 - 2{\cdot}3^3{\cdot}11{\cdot}23\delta z^2 - 3^2{\cdot}11^2{\cdot}113{\cdot}127,$$

or

$$\delta w^2 = (\delta z^2 - 3^3{\cdot}11{\cdot}23)^2 - 2^4{\cdot}3^2{\cdot}5^2{\cdot}11^3{\cdot}13,$$

for $\delta \,|\, 3{\cdot}11{\cdot}113{\cdot}127$. The values $\delta = 1$ and $\delta = -113{\cdot}127$ correspond to the classes in $E'(\mathbb{Q})/\phi(E(\mathbb{Q}))$ of O and the unique two-torsion point of $E'(\mathbb{Q})$; both have trivial image in $\hat{\phi}(E'(\mathbb{Q}))/2E(\mathbb{Q})$. We will show that $C'_\delta(\mathbb{Q}) = \emptyset$ for all other possible δ.

If $3 \,|\, \delta$, then the conic $\delta w^2 = u^2 - 2^4{\cdot}3^2{\cdot}5^2{\cdot}11^3{\cdot}13$ has no points over $\mathbb{Q}_3$. If $11 \,|\, \delta$, then $(\delta/13) = -1$ (because $(-1/13) = (113/13) = (127/13) = 1$) and the conic has no points over $\mathbb{Q}_{13}$. If $(\delta/11) = -1$ the conic has no points over $\mathbb{Q}_{11}$. If $(\delta/5) = -1$, then C'_δ has no points over $\mathbb{Q}_5$. These conditions rule out all but $\delta = 1$ and $\delta = -113{\cdot}127$.

This completes the 2-descent. In particular, we have found that

$$|E(\mathbb{Q})/2E(\mathbb{Q})| = |E(\mathbb{Q})/\hat{\phi}(E'(\mathbb{Q}))| \cdot |\hat{\phi}(E'(\mathbb{Q}))/2E(\mathbb{Q})| = 4 \cdot 1 = 4,$$

and that $E(\mathbb{Q})/2E(\mathbb{Q})$ is generated by the classes of P and Q. Moreover, for $R \neq O, Q$ the image of $x(R)$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ is one of $\{1, 3{\cdot}5, 11{\cdot}13, 3{\cdot}5{\cdot}11{\cdot}13\}$.

At this stage we know that $\langle P, Q\rangle$ is of finite index in $E(\mathbb{Q})$; we must still check that it exhausts $E(\mathbb{Q})$. Suppose not; then for some odd prime ℓ and some $R \in E(\mathbb{Q})$ we have $\ell R = \pm P$ or $\ell R = \pm P + Q$. In the latter case we replace R by $R + Q$, and we may also switch the sign of R if necessary, so now may assume that $\ell R = P$ and $\ell(R + Q) = P + Q$.

In this case $e(R) \,|\, e(P) = 53$ and $e(R+Q) \,|\, e(P+Q) = 277$. Moreover, by (8.2) we have $\alpha(R)\alpha(R+Q) = b{\cdot}e(R)^2 e(R+Q)^2$. Since

$$(\alpha(R), e(R)) = (\alpha(R+Q), e(R+Q)) = 1$$

this implies that $\alpha(R) = b_1 e(R+Q)^2$ and $\alpha(R+Q) = (b/b_1)e(R)^2$ for some $b_1 \,|\, b$. But also $P = R$ and $P + Q = R + Q$ in $E(\mathbb{Q})/2E(\mathbb{Q})$, so comparing with the result of the 2-descent shows the only possibilities are $b_1 = 3{\cdot}5$, $2^2{\cdot}3{\cdot}5$, $3{\cdot}5{\cdot}11^2$, or $2^2{\cdot}3{\cdot}5{\cdot}11^2$. This leaves only 16 choices for $x(R)$, and it is easy to check that only one of them defines a point on the curve (namely for $R = P$). $\square$

Proposition 8.3. *The only solution to* (8.1) *with* $x, y \in \mathbb{Z}[\frac{1}{2}]$ *is* $x = y = 0$.

Proof. First consider the points nP for $n \neq 0$. Since $53 \,|\, e(\pm P)$, also $53 \,|\, e(nP)$, so $x(nP) \notin \mathbb{Z}[\frac{1}{2}]$.

Now consider $nP + Q$. Write $n = 2^i r$ for some $i \geq 0$ and odd r. We may assume r is positive. Then $nP + Q = 2^i rP + Q = r(2^i P + Q)$, so it suffices to show that $e(2^i P + Q)$ is divisible by some prime $q \neq 2$. By (8.2), it suffices to show that $\alpha(2^i P)$ is divisible by some prime $q \neq 2, 3, 5$. For $i = 0$ we may take $q = 277$.

To establish the result for $i \geq 1$, we prove the following series of claims. Let N_p denote the set of points $R \in E(\mathbb{Q})$ such that R reduces to a nonsingular point in

$E(\mathbb{F}_p)$; it is a subgroup of $E(\mathbb{Q})$. For $p = 2,3,5,11,13$ the only singular point on $E(\mathbb{F}_p)$ is $(0,0)$, so for these primes we have $N_p = \{P \in E(\mathbb{Q}) : p \mid \alpha(P)\}$.

Claim 1: $P \in N_p$ for all p except $p = 2,3,5$.

Claim 2: $2P \in N_p$ for all p except $p = 5$.

Claim 3: $4P \in N_p$ for all p, and hence $2^i P \in N_p$ for all p and all $i \geq 2$.

Claim 4: $\alpha(2P) = 5^2 U$, where $U = 4 \pmod{11}$ and $2,3,5 \nmid U$.

Claim 5: $\alpha(4P) = 5 \pmod{11}$.

Claim 6: For $i \geq 2$, $\alpha(2^i P) = \begin{cases} 5 \pmod{11} & i \text{ even}, \\ 9 \pmod{11} & i \text{ odd}. \end{cases}$

Claims 1–5 can easily be proved with a computer, by directly computing the coordinates of $2P$ and $4P$. This approach is infeasible by hand, as the coordinates are rather large (for example $\alpha(4P)$ has almost 100 decimal digits). In Remark 8.4 we sketch an indirect method that is amenable to manual verification.

For Claim 6, we will need the following doubling formula, valid for $R \in E(\mathbb{Q})$, $2R \neq O$:

$$x(2R) = \left(\frac{x(R)^2 - b}{2y(R)}\right)^2 = \frac{(\alpha(R)^2 - b{\cdot}e(R)^4)^2}{(2e(R)\beta(R))^2}. \tag{8.3}$$

Moreover, if $R \in N_p$ then p cannot divide both the numerator and denominator of the fraction on the right side of (8.3). In other words, there is no cancellation locally at p. One proof of this is given in [25, Prop. IV.2]; as pointed out in that paper, it can also be proved from properties of real-valued non-archimedean local heights.

Using this non-cancellation result, if $R \in N_p$ for all p, we obtain $\alpha(2R) = (\alpha(R)^2 - b{\cdot}e(R)^4)^2$. Claim 6 follows by induction, the base case being Claim 5.

The main result for $i = 1$ follows from Claim 4, since U must be divisible by a suitable prime. (In fact the only primes dividing U are 159319 and 709141!) For $i \geq 2$ it follows from Claim 6. □

Remark 8.4. Claims 1–5 can all be checked by using low-precision p-adic approximations, together with the non-cancellation result for (8.3), without needing to compute the full coordinates of $2P$ or $4P$. For example, one calculates that

$$\begin{aligned}(\alpha(P)^2 - b{\cdot}e(P)^4)^2 &= 2^4 \bmod 2^5,\ 3^4 \bmod 3^5,\ 5^4 \bmod 5^5, \\ (2e(P)\beta(P))^2 &= 2^8 \bmod 2^9,\ 3^4 \bmod 3^5,\ 4{\cdot}5^2 \bmod 5^3.\end{aligned}$$

Thus the cancellation between the numerator and denominator of (8.3) is exactly $2^4 3^4 5^2$. It follows that

$$\begin{aligned}\alpha(2P) &= 2^{-4}3^{-4}5^{-2}(\alpha(P)^2 - b{\cdot}e(P)^4)^2, \\ e(2P)^2 &= 2^{-4}3^{-4}5^{-2}(2e(P)\beta(P))^2,\end{aligned}$$

and Claims 2 and 4 follow easily. A similar but more involved calculation establishes Claim 5.

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