SOME RATIONAL CUBIC FOURFOLDS

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1. Introduction

The purpose of this paper is to give new examples of rational cubic fourfolds. Let $\mathcal{C}$ denote the moduli space of cubic fourfolds, a twenty-dimensional quasi-projective variety. The cubic fourfolds containing a plane form a divisor $\mathcal{C}_8 \subset \mathcal{C}$. We prove the following theorem:

Main Theorem (Theorem 4.2) *There is a countably infinite collection of divisors in $\mathcal{C}_8$ which parametrize rational cubic fourfolds. Each of these is a codimension two subvariety in the moduli space of cubic fourfolds $\mathcal{C}$.*

Experimental evidence strongly suggests that the general cubic fourfold containing a plane is not rational, but no smooth cubic fourfold has yet been proven to be irrational.

The key to our construction is the following observation: a cubic fourfold containing a plane is birational to a smooth quadric surface over $k(\mathbb{P}^2)$. Indeed, projecting from the plane gives a rational map to $\mathbb{P}^2$ whose fibers are quadric surfaces. The cubic fourfold is rational if the quadric surface over $k(\mathbb{P}^2)$ is rational. Using Hodge theory, we prove that the rational quadric surfaces correspond to a countably infinite union of divisors in $\mathcal{C}_8$. This is quite natural from an arithmetic point of view. Over $\mathbb{Q}$, the rational quadric surfaces form a countably infinite union of divisors in the Hilbert scheme of quadric surfaces in $\mathbb{P}^3$.

We conclude this introduction by listing the cubic fourfolds known to be rational. There is an irreducible divisor $\mathcal{C}_{14} \subset \mathcal{C}$ parametrizing rational cubic fourfolds. This divisor has many geometric characterizations. For example, it is the closure of the cubic fourfolds containing rational normal scrolls of degree four [Fa] [Tr1] [Ha1] and of the locus of Pfaffian cubic fourfolds [BD]. All the examples of rational cubic fourfolds known to the author are contained in $\mathcal{C}_{14}$ or one of the subvarieties of $\mathcal{C}_8$. Furthermore, the birational map from $\mathbb{P}^4$ involves blowing up a surface birational to a $K3$ surface (see §5 for details).

We work over the complex numbers $\mathbb{C}$ unless mentioned otherwise. Here *generic* means 'in the complement of a Zariski closed proper subset' and *general* means 'in the complement of a countable union of Zariski closed proper subsets'. A *lattice* is a finitely generated free $\mathbb{Z}$-module equipped with a nondegenerate integral quadratic form.

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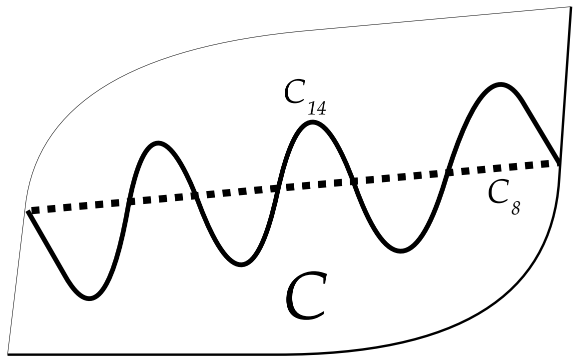

FIGURE 1. Known Rational Cubic Fourfolds

2. Geometry of quadric surface bundles

For our purposes, a *quadric surface bundle* is a flat projective morphism $q : \mathcal{Q} \to B$ of regular connected schemes, such that the generic fiber is a smooth quadric surface. The *relative Fano scheme* $\mathcal{F} \to B$ of a quadric surface bundle parametrizes the lines contained in the fibers of q. For a smooth quadric surface over a field, this consists of two disjoint smooth genus zero curves, corresponding to the rulings of the surface.

Proposition 2.1. Let $q : \mathcal{Q} \to \mathrm{Spec}(k)$ be a smooth quadric surface over a field k. Then the following are equivalent:

1. $\mathcal{Q}$ is rational over k
2. The Fano scheme $\mathcal{F}$ of $\mathcal{Q}$ has a divisor defined over k, with degree one on each component
3. $\mathcal{Q}$ has a zero-cycle of odd degree defined over k

Proof. Let $\mathcal{Z}$ denote the universal line over $\mathcal{F}$, so that we have a correspondence

$$\begin{array}{ccccc} & p & \mathcal{Z} & q & \\ & \swarrow & & \searrow & \\ \mathcal{Q} & & & & \mathcal{F} \end{array}$$

and an induced Abel-Jacobi map

$$\alpha = q_* p^* : \mathrm{Ch}^2(\mathcal{Q}) \to \mathrm{Pic}(\mathcal{F})$$

where $\mathrm{Ch}^2(\mathcal{Q})$ denotes the Chow group of zero-cycles on $\mathcal{Q}$.

The quadric $\mathcal{Q}$ is rational if and only if it has a point over k. This point is mapped by α to a pair of points defined over k, one on each component of $\mathcal{F}$. Conversely, given such a pair of points, the intersection of the corresponding lines gives a k-point of $\mathcal{Q}$. This proves the equivalence of the first two conditions.

Clearly either of the first two conditions implies the third; we prove the converse. Let z be a cycle of odd degree $2n+1$ on $\mathcal{Q}$ and defined over k. The cycle $\alpha(z)$ has degree $2n+1$ on each component of $\mathcal{F}$. The canonical class $K_{\mathcal{F}}$ is defined over k and has degree -2 on each component of $\mathcal{F}$. Consequently, $nK_{\mathcal{F}} + \alpha(z)$ has degree one on each component of $\mathcal{F}$. Given a nonzero section $s \in H^0(\mathcal{F}, nK_{\mathcal{F}} + \alpha(z))$, the locus $s = 0$ consists of a pair of points on $\mathcal{F}$, one on each component. $\square$

The proposition has the following consequence:

Corollary 2.2. Let $q : \mathcal{Q} \to B$ be a quadric surface bundle and assume B is rational over the base field. Let Q denote the class of the generic fiber of q and assume there is a cycle $T \in \mathrm{Ch}^2(\mathcal{Q})$, defined over the base field, such that $\langle T, Q\rangle$ is odd. Then $\mathcal{Q}$ is rational over the base field.

We apply the proposition to $k = k(B)$, the function field of B. Note that $\langle , \rangle$ denotes the intersection product on $\mathcal{Q}$.

In our analysis of cubic fourfolds, we shall use a transcendental version of this result:

Proposition 2.3. Let $q : \mathcal{Q} \to B$ be a quadric surface bundle over a rational projective variety. Assume there is a class $T \in H^4(\mathcal{Q}, \mathbb{Z}) \cap H^{2,2}(\mathcal{Q})$ such that $\langle T, Q\rangle$ is odd. Then X is rational over $\mathbb{C}$.

Proof. By the previous proposition, it suffices to construct a divisor on the relative Fano scheme $\mathcal{F} \to B$ intersecting the components of the generic fiber in $\langle Q, T\rangle$ points. We may discard any components of $\mathcal{F}$ that fail to dominate B. Choose a resolution of singularities $\sigma : \tilde{\mathcal{F}} \to \mathcal{F}$ and set $\tilde{\mathcal{Z}} = \tilde{\mathcal{F}} \times_{\mathcal{F}} \mathcal{Z}$. We again obtain a correspondence of smooth varieties

$$\begin{array}{ccccc} & \tilde{p} & \tilde{\mathcal{Z}} & \tilde{q} & \\ & \swarrow & & \searrow & \\ \mathcal{Q} & & & & \tilde{\mathcal{F}}. \end{array}$$

and an induced map on cohomology

$$\tilde{\alpha} = \tilde{q}_* \tilde{p}^* : H^4(\mathbb{Q}, \mathbb{Z}) \cap H^{2,2}(\mathcal{Q}) \to H^2(\tilde{\mathcal{F}}, \mathbb{Z}) \cap H^{1,1}(\tilde{\mathcal{F}}).$$

By the Lefschetz Theorem on $(1,1)$ classes, $\tilde{\alpha}(T)$ is a divisor on $\tilde{\mathcal{F}}$. The image of this divisor in $\mathcal{F}$ has the desired properties. □

3. Cubic fourfolds containing a plane

First we fix some notation. The Hilbert scheme of cubic hypersurfaces in $\mathbb{P}^5$ is a projective space $\mathbb{P}^{55}$. The smooth hypersurfaces form an open subset $\mathcal{U} \subset \mathbb{P}^{55}$ and are called *cubic fourfolds*. Two cubic fourfolds are isomorphic if and only if they are equivalent under the action of SL_6. Consequently, the isomorphism classes of cubic fourfolds correspond to elements of the orbit space

$$\mathcal{C} := \mathcal{U} // \mathrm{SL}_6 .$$

Applying Geometric Invariant Theory [GIT] §4.2, one may prove that $\mathcal{C}$ has the structure of a twenty-dimensional quasi-projective variety; $\mathcal{C}$ is called the *moduli space of cubic fourfolds*.

Now consider a cubic fourfold X containing a plane P. A dimension count shows the isomorphism classes of such cubic fourfolds form a divisor $\mathcal{C}_8 \subset \mathcal{C}$. We shall restrict our attention to these special cubic fourfolds; for more details see [V], [Ha1], or [Ha2]. Let h denote the hyperplane class of X, and let Q denote the class of a quadric surface residual to P in a three-dimensional linear space, so that $h^2 = P + Q$. Let $\tilde{X}$ denote the blow-up of X along P. Projecting from the plane P, we obtain a morphism

$$q : \tilde{X} \to \mathbb{P}^2.$$

The fibers of this morphism correspond to quadric surfaces in the class Q. In particular, a cubic fourfold containing a plane is birational to a quadric surface bundle over $\mathbb{P}^2$. Applying the results of the previous section, we obtain the following theorem:

Theorem 3.1. Let X be a cubic fourfold containing a plane P, and let Q be the class of a quadric surface residual to P. Assume there is a class $T \in H^4(X,\mathbb{Z}) \cap H^{2,2}(X)$ such that $\langle Q, T\rangle$ is odd. Then X is rational over $\mathbb{C}$.

4. Analysis of the periods

Our next goal is to determine when the hypotheses of the theorem are satisfied. We retain the terminology of the previous section. The methods we use are explained in more detail in [Ha1] §2 and [Ha2]; [V] also contains a detailed discussion of the periods of cubic fourfolds containing a plane. Let $K_8 \subset H^4(X,\mathbb{Z})$ denote the sublattice spanned by h^2 and Q. The intersection form on $H^4(X,\mathbb{Z})$ restricts to

$$\begin{array}{c|cc} & h^2 & Q \\ \hline h^2 & 3 & 2 \\ Q & 2 & 4 \end{array}.$$

Let $K_8^{\perp}$ denote the orthogonal complement to K_8 in $H^4(X,\mathbb{Z})$.

We now recall some results about the periods of cubic fourfolds containing a plane. More general statements are proved in [Ha1] §1,2 and [Ha2]. Let L be a lattice isomorphic to the middle cohomology of a cubic fourfold; L has signature $(21,2)$. Fix distinguished elements h^2 and P in L, corresponding to the hyperplane class squared and a plane contained in some cubic fourfold. Let X be a cubic fourfold containing a plane P, and let $\phi : H^4(X,\mathbb{Z}) \to L$ be a complete marking of its cohomology preserving the classes h^2 and P. This induces a map

$$\phi : H^4(X,\mathbb{C}) \to L \otimes \mathbb{C}.$$

Now the Hodge structure on the middle cohomology of X is entirely determined by the one-dimensional subspace $\phi(H^{3,1}(X)) \subset K_8^{\perp} \otimes \mathbb{C}$, which is isotropic with respect to the intersection form. Consequently, each completely marked cubic fourfold containing a plane yields a point on the quadric hypersurface of $\mathbb{P}(K_8^{\perp} \otimes \mathbb{C})$ where the intersection form is zero. The *local period domain* for cubic fourfolds containing a plane is a topologically open subset of this hypersurface, consisting of one of the connected components of the open set where the Hermitian form $-\langle u, \bar{v}\rangle$ is positive. This manifold has the structure of a nineteen-dimensional bounded symmetric domain of type IV [Sa] (§6 of the appendix); it is denoted $\mathcal{D}_8'$.

Let Γ_8^+ denote the automorphisms of L which preserve the intersection form, act trivially on K_8, and respect the orientation on the negative definite part of $K_8^{\perp}$. The group Γ_8^+ acts from the left on $\mathcal{D}_8' \subset \mathbb{P}(K_8^{\perp} \otimes \mathbb{C})$. The quotient $\Gamma_8^+ \backslash \mathcal{D}_8'$ is called the *global period domain* for cubic fourfolds containing a plane. This is the quotient of a bounded symmetric domain by an arithmetic group and so is a normal quasi-projective variety[BB]. Let $\mathcal{C}_8^{\mathrm{mar}}$ be the variety parametrizing the pairs (X, P), where X is a cubic fourfold and P is plane contained in X. The period map

$$\tau_8 : \mathcal{C}_8^{\mathrm{mar}} \to \Gamma_8^+ \backslash \mathcal{D}_8'$$

is an algebraic open immersion of quasi-projective varieties. This follows from the Torelli theorem for cubic fourfolds [V] and the Borel extension theorem [Bo].

We now state our main technical result:

Proposition 4.1. Consider the Hodge structures in the global period domain $\Gamma_8^+\backslash\mathcal{D}_8'$ for which there exists some $T \in L \cap H^{2,2}$ such that $\langle T, Q\rangle$ is odd. These Hodge structures form a countable union of divisors, indexed by the discriminant of the saturation of $K_8 + \mathbb{Z}T$. This discriminant may be any positive integer $n \equiv 5 (\text{mod } 8)$.

Here the *discriminant* of a lattice means the determinant of its intersection form. Later we shall introduce a more refined notion.

We are interested in the restriction of these divisors to the Zariski open subset $\mathcal{C}_8^{\text{mar}}$ of the period domain. These are also irreducible divisors and only finitely many of them are contained in the complement to $\mathcal{C}_8^{\text{mar}}$. By Theorem 3.1, these divisors parametrize rational cubic fourfolds. Thus we obtain our main theorem:

Theorem 4.2 (Main Theorem)**.** There is a countably infinite collection of divisors in $\mathcal{C}_8$ which parametrize rational cubic fourfolds. Each of these is a codimension two subvariety in the moduli space of cubic fourfolds $\mathcal{C}$.

Remark 4.3. 1)There are rational cubic fourfolds X containing a plane such that the quadric surface bundle $\tilde{X} \to \mathbb{P}^2$ does not have a rational section. For example, certain components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$ are of this type.
2)It seems likely that the only discriminants corresponding to divisors in the boundary $(\Gamma_8^+\backslash\mathcal{D}_8')\backslash\mathcal{C}_8^{\text{mar}}$ are $n = 5$ and 13.

The remainder of this section is the proof of Proposition 4.1. We first establish the following lemma:

Lemma 4.4. Let x be a Hodge structure in $\mathcal{D}_8'$. Then the following conditions are equivalent.

1. There exists a cohomology class $T \in L \cap H^{2,2}(x)$ with $\langle T, Q\rangle$ odd.
2. There exists a saturated rank one sublattice $S \subset K_8^\perp \cap H^{2,2}(x)$ such that the orthogonal complement to S in $K_8^\perp$ has odd discriminant.

Furthermore, the discriminant n of the saturation of $K_8 + \mathbb{Z}T$ is congruent to 5 modulo 8. This is equal to the discriminant of the orthogonal complement to S in $K_8^\perp$.

Proof. The key ingredient of this lemma is the following fact: let K be a saturated nondegenerate sublattice of a unimodular lattice and let $K^\perp$ be its orthogonal complement. Then K and $K^\perp$ have the same discriminant (up to sign). This is proved in [Ni] §1.6.

Assume the first condition holds. We may find an element T_1 in the saturation of $K_8 + \mathbb{Z}T$ so that $K_8 + \mathbb{Z}T_1$ is saturated and $\langle Q, T_1\rangle = 1$. The intersection form on $K_8 + \mathbb{Z}T_1$ takes the form

	h^2	Q	T_1
h^2	3	2	a
Q	2	4	1
T_1	a	1	b

with discriminant $n = -3 + 4(a - a^2) + 8b$, which is congruent to 5 modulo 8. Let S be the intersection of $K_8 + \mathbb{Z}T_1$ with $K_8^\perp$. By construction S is saturated and contained in $H^{2,2}(x)$. The orthogonal complement to S in $K_8^\perp$ coincides with the orthogonal complement to $K_8 + \mathbb{Z}T_1$ in the full cohomology lattice. In particular, this lattice has discriminant n, which is odd.

Now assume the second condition holds. Again, the saturation of K_8+S has the same discriminant as the orthogonal complement to S in $K_8^{\perp}$. If the discriminant of the saturation of K_8+S is odd, then it contains a class T with $\langle T, Q\rangle$ odd. Otherwise, the intersection form could be written

$$\begin{array}{c|ccc} & h^2 & Q & T_1 \\ \hline h^2 & 3 & 2 & a \\ Q & 2 & 4 & 0 \\ T_1 & a & 0 & b \end{array}$$

for some T_1 in the saturation of K_8+S. But the discriminant would then be even, a contradiction. $\square$

We now analyze the geometry of these Hodge structures in the global period domain. Suppose we are given a positive definite rank one saturated sublattice $S \subset K_8^{\perp}$, such that the orthogonal complement has odd discriminant. These are precisely the sublattices arising from the lemma, since the Hodge-Riemann bilinear relations imply that the intersection form on $L \cap H^{2,2}(x)$ is positive definite. We determine the $x \in \mathcal{D}'_8$ for which $S \subset H^{2,2}(x)$. Because x corresponds to $H^{3,1}$, it is necessary and sufficient that x be orthogonal to S with respect to the intersection form. The locus of such x forms a hyperplane section of $\mathcal{D}'_8$ with respect to its imbedding in $\mathbb{P}(K_8^{\perp}\otimes\mathbb{C})$. It is not difficult to see that this is a nonempty irreducible divisor in the period domain. Like $\mathcal{D}'_8$, it is a bounded symmetric domain of type IV (see the appendix of [Sa], §6 for more details). Its image in $\Gamma_8^+\backslash\mathcal{D}'_8$ parametrizes those periods with algebraic classes of type S. This subvariety is an algebraic divisor in the global period domain, because both the global period domain and the normalization of the subvariety are arithmetic quotients of bounded symmetric domains [BB] [Bo].

As we choose various S satisfying the conditions stipulated above, we obtain divisors in the global period domain. Now S_1 and S_2 determine the same divisor if and only if $S_2=\gamma(S_1)$ for some $\gamma \in \Gamma_8^+$; we then say that S_1 and S_2 are *equivalent* modulo Γ_8^+. Hence we obtain a bijective correspondence between the following two types of data

1. irreducible divisors in $\Gamma_8^+\backslash\mathcal{D}'_8$ parametrizing Hodge structures x satisfying the conditions of Lemma 4.4
2. Γ_8^+ equivalence classes of rank one positive definite saturated sublattices $S \subset K_8^{\perp}$, such that the orthogonal complement has odd discriminant

We shall prove that these equivalence classes are classified by the discriminants of their orthogonal complements. Our argument relies heavily on ideas of Nikulin [Ni]. If K is a lattice then the bilinear form induces an inclusion $K \hookrightarrow K^*=\mathrm{Hom}(K,\mathbb{Z})$. The *discriminant group* of K is defined as the quotient

$$d(K) := K^*/K.$$

The order of $d(K)$ is equal to the absolute value of the discriminant of K, so $d(K)$ is trivial iff K is unimodular. For example

$$d(K_8)=\frac{1}{8}(2h^2+Q)\mathbb{Z}/(2h^2+Q)\mathbb{Z} \cong \mathbb{Z}/8\mathbb{Z}.$$

The bilinear form on K extends to a $\mathbb{Q}$-valued bilinear form on K^*, which induces a $\mathbb{Q}/\mathbb{Z}$-valued bilinear form on $d(K)$. Furthermore, if K is even then the quadratic form induces a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form on $d(K)$, denoted q_K. We use Γ_K to denote the automorphisms of K acting trivially on $d(K)$.

Under certain conditions, imbeddings of lattices with compatible signatures are classified by homomorphisms of the corresponding discriminant groups. The following special case of results from [Ni] §1.15 illustrates this principle:

> Assume K is an even lattice of signature $(19,2)$ and S is a rank one positive definite lattice. A saturated imbedding of S into K corresponds to the following data:
> 1. A subgroup $H_S \subset d(S)$
> 2. A subgroup $H_K \subset d(K)$
> 3. An isomorphism $h : q_S|H_S \to q_K|H_K$
> 4. An even lattice $S^{\perp}$ of signature $(18,2)$ and an isomorphism $\eta : d(S^{\perp}) \to -\delta$ of discriminant quadratic forms. Here $\delta := ((q_S \oplus -q_K)|_{H^{\perp}})/H$ where H is the graph of h in $d(S) \oplus d(K)$.
>
> Two such imbeddings (denoted S_1 and S_2) are conjugate under the action of Γ_K if the following conditions are satisfied
> 1. $H_{S_1} = H_{S_2}$
> 2. $h_1(H_{S_1}) = h_2(H_{S_2})$
> 3. There exist isomorphisms $\psi : S_1^{\perp} \to S_2^{\perp}$ and $\xi : d(K) \to d(K)$ such that $\overline{\xi} \circ \eta_1 = \eta_2 \circ \overline{\psi}$. Here $\overline{\psi} : d(S_1^{\perp}) \to d(S_2^{\perp})$ and $\overline{\xi} : \delta \to \delta$ are the maps induced by ψ and ξ respectively.

In the same spirit, we can often construct isomorphisms between lattices by constructing isomorphisms between their discriminant groups. The following statement is a special case of results from [Ni] §1.14:

> Let M be an even indefinite lattice. Let ℓ be the minimal number of generators of the discriminant group $d(M)$ and assume that the rank of M is greater than $\ell + 2$. Let M' be another even lattice with the same signature and assume there is an isomorphism $\overline{\psi} : d(M) \to d(M')$ respecting the discriminant quadratic forms q_M and $q_{M'}$. Then $\overline{\psi}$ is induced by an isomorphism $\psi : M \to M'$.

We use these results to prove the following lemma, which completes the proof of Propostion 4.1:

Lemma 4.5. Let S_1 and S_2 be rank one positive definite saturated sublattices of $K_8^{\perp}$, and assume that their orthogonal complements have the same odd discriminant n. Then S_1 and S_2 are equivalent modulo Γ_8^+. Moreover, such lattices exist for each positive integer $n \equiv 5 (\mathrm{mod}\ 8)$.

Proof. Set $K = K_8^{\perp}$, which is even by the results of [Ha1] §1.3 or [Ha2]. For an imbedding $i : S \hookrightarrow K_8^{\perp}$ of the type we are considering, $H \cong d(K_8^{\perp})$ is a cyclic group of order eight and $d(S)$ is cyclic of order $8n$. Let Γ_8 denote the group $\Gamma_{K_8^{\perp}}$, which contains Γ_8^+ as an index two subgroup. We first claim that for any saturated imbeddings $i_1, i_2 : S \to K_8^{\perp}$ there exists a $\gamma \in \Gamma_8$ such that $i_1(S) = \gamma(i_2(S))$. This is a consequence of the first result of Nikulin quoted above. The existence of the desired isomorphism ψ follows from the second result.

This result also implies that

$$K_8^{\perp} \cong \begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \oplus U \oplus E_8 \oplus E_8$$

where U is the unique even unimodular quadratic form of signature $(1,1)$ and E_8 is the positive definite quadratic form associated to the corresponding Dynkin diagram. For each positive integer a there exists a saturated $\mathbb{Z}v \subset E_8$ with $\langle v,v\rangle = 2a$ [Se] §VII.6.6. Consider elements of

$$\begin{pmatrix} -2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \oplus E_8$$

of the form $(3,1,1)+8v$ (respectively $(1,3,3)+8v$); let S' denote the sublattice generated by such an element. The discriminant of S' is $8(-3+16a)$ (respectively $8(5+16a)$), and by the first result of Nikulin the orthogonal complement to S' in $K_8^{\perp}$ has discriminant $-3+16a$ (respectively $5+16a$). Thus for each positive integer $n \equiv 5(\text{mod } 8)$ there is a rank one saturated sublattice of $K_8^{\perp}$ such that the orthogonal complement has discriminant n. Moreover, this sublattice is equivalent modulo Γ_8 to one of the sublattices S'.

Now consider $\gamma \in \Gamma_8$ acting by multiplication by -1 on U and trivially on the other components. We find that $\gamma|S' = 1$ and $\gamma \notin \Gamma_8^+$, so we conclude that S is also Γ_8^+-equivalent to S'. In particular, the arguments of the first paragraph are still valid if we replace Γ_8 by Γ_8^+. This completes the proof of the lemma and the proposition. □

5. Geometry of the rational maps

Our first goal is to prove the following result about the geometry of our rational parametrizations:

Proposition 5.1. Let X be a cubic fourfold containing a plane P such that the quadric surface bundle $q : \tilde{X} \to \mathbb{P}^2$ has a rational section, and let $\psi : \tilde{X} \dashrightarrow \mathbb{P}^4$ denote the birational map obtained by projecting from this section. Then ψ blows down a family of lines parametrized by a surface birational to a degree two $K3$ surface.

Proof. Recall that the discriminant curve $C \subset \mathbb{P}^2$ is defined as the locus over which q fails to be smooth. In our case, C is a sextic plane curve such that the double cover of $\mathbb{P}^2$ branched over C is birational to a $K3$ surface [V] §1.

Let $T \subset \tilde{X}$ be a closed integral subscheme meeting the generic fiber of q in a single reduced point. The induced map $T \to \mathbb{P}^2$ is an isomorphism except over a codimension two subset of the base. Let R denote the lines in the fibers of q incident to T (or the irreducible component of this locus dominating the base). The induced map from R to $\mathbb{P}^2$ is generically finite of degree two and is ramified over the discriminant curve. In particular, R is birational to a degree two $K3$ surface.

If $\mathcal{Q}$ is a smooth quadric surface and $t \in \mathcal{Q}$, then projecting from t blows down the two lines incident to t. In particular, the birational map from $\tilde{X}$ to $\mathbb{P}^4$ constructed from T blows down a family of lines birational to R. □

To conclude this section, we suggest constructions of explicit linear series for some of the rational maps constructed above. Consider a $K3$ surface S such that

Pic(S) is generated by two ample divisors h_1 and h_2 with intersection matrix

$$\begin{array}{c|cc} & h_1 & h_2 \\ \hline h_1 & 2 & 2k+1 \\ h_2 & 2k+1 & 2 \end{array}$$

where $k > 1$. Set $n = -4 + (2k+1)^2 = 4k^2 + 4k - 3$, which is positive and satisfies $n \equiv 5 (\mathrm{mod}\ 8)$. The sections of $\mathcal{O}_S(h_i)$ give branched double covers $s_i : S \to \mathbb{P}^2$. The pair (s_1, s_2) induces a map $s : S \to \mathbb{P}^2 \times \mathbb{P}^2$ and the image S' has $2k^2 - 7k + 6$ double points [Fu] §9.3. We assume that the double points of S' are analytically equivalent to the transverse intersection of two smooth surfaces.

Let H_1 and H_2 be the divisors on $\mathbb{P}^2 \times \mathbb{P}^2$ such that $h_i = s^* H_i$. Let Y be the blow up of $\mathbb{P}^2 \times \mathbb{P}^2$ along the surface S' with exceptional divisor E. Our assumption on the singularities of S' implies that Y is smooth. We set

$$L = 2H_1 + kH_2 - E$$

and consider the linear series $H^0(Y, \mathcal{O}_Y(L))$. We compute that $L^4 = 3$ and a dimension count suggests that $\dim |L| = 5$. We assume this linear series defines a morphism

$$\phi : Y \to \mathbb{P}^5$$

so that the image $X = \phi(Y)$ is a smooth cubic fourfold.

Under these assumptions, we can compute numerical invariants for some of the surfaces contained in X. Set $F_1 = 2H_1 + (k-1)H_2 - E$ and $F_2 = H_1 + 2(k-1) - E$ so that

$$L^3 F_1 = L^3 F_2 = 0.$$

A dimension count suggests that F_1 and F_2 are effective, and we assume they are exceptional divisors for ϕ obtained by blowing up surfaces P and T in X. Under these assumptions, we find that h^2, P, T span a rank three sublattice of $H^4(X, \mathbb{Z})$ with discriminant $n = 4k^2 + 4k - 3$, and also that P has degree one and so is a plane. Since $n \equiv 5 (\mathrm{mod}\ 8)$, the intersection $\langle T, h^2 - P \rangle$ is necessarily odd.

To lend some credibility to this numerology, we should point out the geometry of these examples is understood in the cases $k = 2, 3$. The case $k = 2$ corresponds to the cubic fourfolds containing two disjoint planes. The assumptions above are easily verified in this case. The case $k = 3$ corresponds to the cubic fourfolds containing a plane P and Veronese surface T such that $\langle P, T \rangle = 3$. This example has been worked out by Tregub [Tr2].

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