Final Exam - Solutions
Math 102 Spring 2010

1. Find the integral
$$\int \frac{2x-3}{(x+1)^2(x^2+4)}dx$$
Solution: Use partial fractions decomposition.
$$\frac{2x-3}{(x+1)^2(x^2+4)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+4}$$
$$2x-3 = A(x+1)(x^2+4) + B(x^2+4) + (Cx+D)(x+1)^2$$
$$2x-3 = Ax^3 + 4Ax + Ax^2 + 4A + Bx^2 + 4B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D$$
Identifying the coefficients according to the powers of x
$x^3 : 0 = A + C$
$x^2 : 0 = A + B + 2C + D$
$x^1 : 2 = 4A + C + 2D$
$x^0 : -3 = 4A + 4B + D$
$A = C = 0, B = -1, D = 1$
$$\int \frac{2x-3}{(x+1)^2(x^2+4)}dx = \int -\frac{1}{(x+1)^2} + \frac{1}{x^2+4}dx = \frac{1}{x+1} + \frac{1}{2}\arctan\frac{x}{2} + C$$

2. Find the integral
$$I = \int \sin x \cdot e^x dx$$
Solution: Use integration by parts twice to recover I.
$u = \sin x$, $dv = e^x dx$
$du = \cos x$, $v = e^x$
$$I = \sin x \cdot e^x - \int \cos x \cdot e^x dx$$
$u = \cos x$, $dv = e^x dx$
$du = -\sin x$, $v = e^x$
$$I = \sin x \cdot e^x - [\cos x \cdot e^x + \int \sin x e^x dx] = \sin x \cdot e^x - \cos x \cdot e^x - I$$
$$2I = \sin x \cdot e^x - \cos x \cdot e^x$$
$$I = \frac{1}{2}(\sin x \cdot e^x - \cos x \cdot e^x) + C$$

3. Evaluate the definite integral

$$I = \int_0^4 \frac{\ln(x)}{\sqrt{x}} dx$$

Solution: This is an improper integral, as $\frac{\ln(x)}{\sqrt{x}}$ is not defined at $x = 0$.

$$I = \int_0^4 \frac{\ln(x)}{\sqrt{x}} dx = \lim_{b \to 0^+} \int_b^4 \frac{\ln(x)}{\sqrt{x}} dx$$

Use integration by parts.
$u = \ln x$, $dv = \frac{1}{\sqrt{x}} dx$
$du = \frac{1}{x} dx$, $v = 2\sqrt{x}$

$$I = \lim_{b \to 0^+} [2\sqrt{x} \ln x]_b^4 - \int_b^4 \frac{2}{\sqrt{x}} dx = \lim_{b \to 0^+} [2\sqrt{x} \ln x - 4\sqrt{x}]_b^4 =$$

$$= \lim_{b \to 0^+} (4 \ln 4 - 8 - 2\sqrt{b} \ln b + 4\sqrt{b}) = \lim_{b \to 0^+} (4 \ln 4 - 8 - 2\sqrt{b} \ln b)$$

Use L'Hospital's rule for the $\frac{\infty}{\infty}$ case

$$\lim_{b \to 0^+} 2\sqrt{b} \ln b = 2 \lim_{b \to 0^+} \frac{\ln b}{\frac{1}{\sqrt{b}}} = 2 \lim_{b \to 0^+} \frac{\frac{1}{b}}{-\frac{1}{2} \frac{1}{b\sqrt{b}}} = -4 \lim_{b \to 0^+} \sqrt{b} = 0$$

$$I = 4 \ln 4 - 8$$

4. Determine whether or not each of the following series converges. Note: before applying any test check (and show your work) that the hypotheses of the test are satisfied.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

Apply the integral test. Let $f(x) = \frac{1}{x \ln x}$.

$f(x)$ is positive, continuous and decreasing on $[2, \infty)$. Making the u-substitution $u = \ln x$, $du = \frac{1}{x} dx$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_2^{\infty} \frac{1}{u} du = \lim_{b \to \infty} [\ln(\ln x)]_2^b = +\infty$$

By the integral test, the series is divergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n^2}$

Apply the Alternating Series Test.

$b_n = \frac{\ln(n)}{n^2}$. Using L'Hospital's rule for the $\frac{\infty}{\infty}$ case,

$$i)\ \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln(n)}{n^2} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$$

$ii)$ To show $b_{n+1} \leq b_n$, show that the function $f(x) = \frac{\ln(x)}{x^2}$ is decreasing for sufficiently large x.

$f'(x) = \frac{x - 2x\ln x}{x^4} = \frac{1 - x\ln x}{x^3} < 0$, if $x \geq 2$.

By AST the series is convergent.

5. (a) Find the Maclaurin series of $\sin(2x)$.

Solution: Use the Maclaurin series of $\sin x$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, x \in \mathbb{R}$$

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

(b) Use the previous part to find $\sin(1)$ within an error, $|error| < 10^{-3}$.

Solution: Let $x = \frac{1}{2}$

$$\sin 1 = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} (1/2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$$

Use the estimation theorem for alternating series.

If $\sin 1 \sim 1 - \frac{1}{3!} + \frac{1}{5!} + ... + \frac{(-1)^n}{(2n+1)!}$, then the error $< \frac{1}{(2n+3)!}$

One wants the smallest n such that $\frac{1}{(2n+3)!} < \frac{1}{1000}$, equivalently, $(2n+3)! > 1000$

The smallest such n is $n = 2$, as $7! = 5040 > 1000$.

Thus, $\sin 1 \sim 1 - \frac{1}{3!} + \frac{1}{5!} = 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120}$

6. Compute the Taylor polynomial of degree 3 for $f(x) = x\ln(x)$ centered at $a = 1$.

Solution:

$$T_3(x) = f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!}$$

$f(1) = 0$
$f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x,\ f'(1) = 1$

$f''(x) = \frac{1}{x}$, $f''(1) = 1$
$f'''(x) = -\frac{1}{x^2}$, $f'''(1) = -1$

$$T_3(x) = \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} - \frac{(x-1)^3}{3!} = (x-1) + \frac{(x-1)^2}{2} - \frac{(x-1)^3}{6}$$

7. (a) Find the sum of the series $\sum_{n=1}^{\infty} nx^{n-1}$.

   Solution: Notice that $nx^{n-1} = \frac{d}{dx}(x^n)$

   Take term by term derivative in

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

   Then,

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1}$$

   (b) Use the previous part to evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$

   Let $x = \frac{1}{2}$. Then,

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{(1-\frac{1}{2})^2} = 4$$

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = 2$$

8. Consider the power series

$$\sum_{n=0}^{\infty} c_n(x-2)^n.$$

   It is known that the power series converges when $x = 5$ and diverges when $x = -3$. For each of the following questions mark the correct answer (No credit will be given if it is ambigous which answer is chosen).

   Solution:

   The hypothesis implies that the radius of convergence, R, is al least $5 - 2 = 3$ and at most $2 - (-3) = 5$. We conclude that the interval of convergence contains the interval $(-1, 5]$.

(a) The power series at $x = 4$

☐is convergent ☐is divergent ☐may or may not be convergent.

Convergent, as $4 \in (-1, 5]$

(b) The power series at $x = -1$

☐is convergent ☐is divergent ☐may or may not be convergent.

May or may not be convergent. Nothing is implied about $x = -1$.

(c) Could the radius of convergence R be

i. $R = 2$? ☐Yes ☐No, Answer: NO, $R \geq 3$

ii. $R = 7$? ☐Yes ☐No, Answer: NO, $R \leq 5$

iii. $R = 0$? ☐Yes ☐No, Answer: NO, $R \geq 3$

iv. $R = \infty$? ☐Yes ☐No, Answer: NO, $R \leq 5$

v. $R = 4$? ☐Yes ☐No Answer: YES

vi. $R = 3$? ☐Yes ☐No Answer: YES

9. Find the length of the curve $y = \ln(\cos(x))$ when $0 \leq x \leq \frac{\pi}{3}$.

Solution:

$\frac{dy}{dx} = \frac{-\sin x}{\cos x}$

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + (\frac{-\sin x}{\cos x})^2} dx = \int_0^{\frac{\pi}{3}} \sqrt{1 + \frac{\sin^2 x}{\cos^2 x}} dx = \int_0^{\frac{\pi}{3}} \sqrt{\frac{1}{\cos^2 x}} dx =$$

$$= \int_0^{\frac{\pi}{3}} \sec x dx = \ln |\sec x + \tan x|]_0^{\frac{\pi}{3}} = \ln(2 + \sqrt{3})$$

10. Find those values of t where the curve

$$x(t) = t^3 - 3t$$

$$y(t) = e^{2t} - e^t$$

has

(a) a horizontal tangent.

The slope along the curve is given by $m = \frac{dy/dt}{dx/dt} = \frac{2e^{2t} - e^t}{3t^2 - 3}$

The tangent is horizontal when $m = 0$, that is $2e^{2t} - e^t = e^t(2e^t - 1) = 0$.

This yields $2e^t = 1$, or $t = \ln \frac{1}{2}$

(b) a vertical tangent.

The tangent is horizontal when m is undefined, that is $3t^2 - 3 = 3(t^2 - 1) = 0$. This yields $t^2 = 1$, or $t = \pm 1$

11. Consider the curve which in polar coordinates is given by $r = 2 + \sin(3\theta)$.

(a) i. Find the range of θ for which r is increasing.

$(0, \frac{\pi}{6})$, $(\frac{\pi}{2}, \frac{5\pi}{6})$, $(\frac{7\pi}{6}, \frac{3\pi}{2})$, $(\frac{11\pi}{6}, 2\pi)$

ii. Find the range of θ for which r is decreasing.

$(\frac{\pi}{6}, \frac{\pi}{2}$, $(\frac{5\pi}{6}, \frac{7\pi}{6})$, $(\frac{3\pi}{2}, \frac{11\pi}{6})$

iii. Find the values of θ for which r is maximum.

$\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{3\pi}{2}$

iv. Find the values of θ for which r is minimum.

$\frac{\pi}{2}$, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$

(b) Use the information from the previous part to plot the curve.

To see the polar graph, please visit

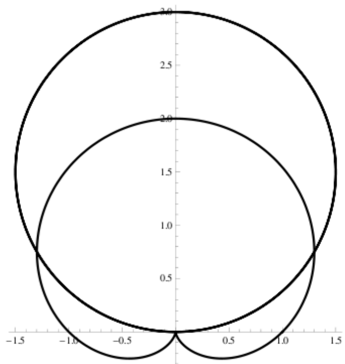
$http://www.ies.co.jp/math/java/calc/sg_kyok/sg_kyok.html$ and type in $2 + sin(3 * t)$ in the applet window.

12. Find the area of the region inside both the cardioid $r = 1 + \sin(\theta)$ and the circle $r = 3\sin(\theta)$.

Solution:

Find the values of θ where the two curves intersect.Then, use the symmetry about the y-axis.
$1 + \sin(\theta) = 3\sin(\theta)$, equivalently $\sin\theta = \frac{1}{2}$, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

$$A = 2\int_0^{\frac{\pi}{6}} \frac{1}{2}(3\sin\theta)^2 d\theta + 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1+\sin\theta)^2 d\theta = \int_0^{\frac{\pi}{6}} 9\sin^2\theta d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + 2\sin\theta + \sin^2\theta d\theta =$$



$$= 9\int_0^{\frac{\pi}{6}} \frac{1-\cos 2\theta}{2} d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 + 2\sin\theta + \frac{1-\cos 2\theta}{2} d\theta =$$

$$= 9[\frac{\theta}{2} - \frac{\sin 2\theta}{4}]_0^{\frac{\pi}{6}} + [\frac{3\theta}{2} - 2\cos\theta - \frac{\sin 2\theta}{4}]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{5\pi}{4}$$