Solutions to Third Midterm Exam, Math 212 Spring 2012

1. What is the length of the path $\gamma(t) \;=\; (2\cos t, 2\sin t, \frac{2}{3}t^{3/2})$ for $0 \le t \le 5$?

Since $\gamma'(t) \;=\; (-2\sin t, 2\cos t, t^{1/2})$

$$\|\gamma'(t)\| \;=\; \sqrt{4\sin^2 t + 4\cos^2 t + t} \;=\; \sqrt{4+t}\;,$$

and the length is

$$\int_0^5 \|\gamma'(t)\|\,dt \;=\; \int_0^5 \sqrt{4+t}\,dt \;=\; \int_{4+0}^{4+5} \sqrt{u}\,du$$
$$=\; (\frac{2}{3})u^{3/2}|_4^9 \;=\; (\frac{2}{3})(9^{3/2} - 4^{3/2}) \;=\; (\frac{2}{3})(27-8) \;=\; \frac{38}{3}\;.$$

2. Suppose f, g and h are three smooth functions on $\mathbf{R}^3$ and that $\mathbf{F} \;=\; (f,g,h)$. Prove that $\mathrm{div}(\mathrm{curl}\mathbf{F}) \;=\; \mathbf{0}$.

Since $\mathrm{curl}\,\mathbf{F} \;=\; (h_y - g_z, f_z - h_x, g_x - f_y)$,

$$\begin{aligned}\mathrm{div}(\mathrm{curl}\,\mathbf{F}) \;&=\; (h_{yx} - g_{zx}) + (f_{zy} + h_{xy}) + (g_{xz} + f_{yz})\\ &=\; (h_{yx} - h_{xy}) + (f_{zy} - f_{yz}) + (g_{xz} - g_{zx}) \;=\; 0+0+0\end{aligned}$$

because of the equality of mixed partial derivatives.

3. Let $\mathbf{G}$ be the vector field $\mathbf{G}(x,y,z) = (y+z, y, z+z^2)$.

(a) Find the curl of $\mathbf{G}$.

$$\begin{aligned}\mathrm{curl}\,\mathbf{G} \;&=\; \big((z+z^2)_y - y_z, (y+z)_z - (z+z^2)_x, y_x - (y+z)_y\big)\\ &=\; (0-0, 1-0, 0-1) \;=\; (0,1,-1)\;.\end{aligned}$$

(b) Is $\mathbf{G}$ a gradient field (i.e., the gradient of some function)? Explain why/why not.
No, because $\mathrm{curl}\,\mathbf{G} \neq 0$ while, for any smooth function f, $\mathrm{curl}\,(\mathrm{grad} f) \;=\; \mathbf{0}$. In fact,

$$\mathrm{curl}(\,f_x, f_y, f_z) \;=\; (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}) \;=\; (0,0,0)\;.$$

4. Let A be the region in the plane given by $y \ge 0$ and $1 \le x^2 + y^2 \le 4$. Evaluate the integral $\iint_A (x^2+y^2)dx\,dy$.

Since x^2+y^2 and the description of A both involve some rotational symmetry, it is best to use polar coordinates. Here A is given by $1 \le r \le 2$ and $0 \le \theta \le \pi$. Also $x^2+y^2 = r^2$, and $dx\,dy \;=\; r\,dr\,d\theta$. So

$$\begin{aligned}\iint_A (x^2+y^2)dx\,dy \;&=\; \int_0^\pi \int_1^2 r^2\,r\,dr\,d\theta\\ &=\; \int_0^\pi (\frac{r^4}{4})|_1^2\,d\theta \;=\; 2\pi(4 - \frac{1}{4}) \;=\; \frac{15\pi}{2}\;.\end{aligned}$$

5. Find the volume of the region W obtained as the intersection of the sets

$$x^2 + y^2 + z^2 \leq 1 \text{ and } x^2 + y^2 \leq z^2.$$

Here spherical coordinates are good because $0 \leq \rho = \sqrt{x^2 + y^2 + z^2} \leq 1$ and the other inequality $x^2 + y^2 \leq z^2$ becomes $r^2 \leq z^2$ so that, since

$$-1 \leq \tan\phi = \frac{r}{z} \leq 1 \text{ hence, } 0 \leq \phi \leq \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \leq \phi \leq \pi .$$

The top and bottom volumes are the same. Thus, since $dx\,dy\,dz = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$,

$$\begin{aligned} \text{Volume}\, W &= \iiint_W dx\,dy\,dz = 2\int_0^{\pi/4}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\ &= 2\int_0^{\pi/4}\int_0^{2\pi} (\frac{\rho^3}{3})\Big|_0^1 \sin\phi\, d\theta\, d\phi = 2\int_0^{\pi/4} 2\pi(\frac{1}{3}) \sin\phi\, d\phi \\ &= \frac{4\pi}{3}(-\cos\phi)\Big|_0^{\pi/4} = \frac{4\pi}{3}(-\frac{\sqrt{2}}{2} + 1) = \frac{2\pi}{3}(2 - \sqrt{2}) . \end{aligned}$$

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $T(u, v) = (u, uv)$.

(a) Let $D \subset \mathbb{R}^2$ be the region given by $1 \leq x \leq 2$ and $|y| \leq x$. Find a region $D^* \subset \mathbb{R}^2$ such that $D = T(D^*)$ and T is one-to-one on D^*.

Sketching D we find that it is the quadrilateral with vertices $(1, 1), (2, 2), (2, -2), (1, -1)$. Since T doesn't change the first coordinate, T preserves each vertical line $\{(x_0, y) : y \in \mathbb{R}\}$ for $1 \leq x_0 \leq 2$. Also it expands the Y coordinate of points on this line by a factor x_0. Since $T(D^*) = D$, we may find D^* by noting that

$$(x_0, y) \in D^* \iff (x_0, x_0 y) \in T(D^*) = D \iff |x_0 y| \leq x_0 \iff |y| \leq 1 .$$

Thus D^* is the rectangle defined by $1 \leq x \leq 2$ and $-1 \leq y \leq 1$, that is, $D^* = [1, 2] \times [-1, 1]$.
To check that T is one-to-one, assume (x, y), $(\tilde{x}, \tilde{y}) \in D^*$ and

$$(x, xy) = T(x, y) = T(\tilde{x}, \tilde{y}) = (\tilde{x}, \tilde{x}\tilde{y}) .$$

Then $x = \tilde{x}$ and $y = x^{-1}(xy) = (\tilde{x})^{-1}(\tilde{x}\tilde{y}) = \tilde{y}$. So $(x, y) = (\tilde{x}, \tilde{y})$, and T is one-to-one.

(b) Using the change of variables formula, rewrite the integral $\iint_D \cos\left(\frac{\pi y}{2x}\right) dxdy$ as an integral over the region D^*. Then find the value of the integral.
To follow pur previous notations for the change-of-variable theorem, we use (u, v) for points in D^*, we have $T(u, v) = (x(u, v), y(u, v))$ with the real-valued functions $x(u, v) = u$ and

$y(u,v) = uv$. Then $T_u = (1,v)$, $T_v = (0,u)$ and the integration factor is found by taking the determinant so that

$$\frac{\partial(x,y)}{\partial(u,v)} = \|T_u \times T_v\| = |1 \cdot u - 0 \cdot v| = u ,$$

that is, $dx\,dy = u\,du\,dv$.

(b) Using the change of variables formula, rewrite the integral $\iint_D \cos(\frac{\pi x}{2y})dxdy$ as an integral over the region D^*. Then find the value of the integral.

$$\begin{aligned}\iint_D \cos(\frac{\pi x}{2y})dx\,dy &= \iint_{D^*} \cos(\frac{\pi uv}{2u})u\,du\,dv \\ &= \int_{-1}^{1}\int_{1}^{2} \cos(\frac{\pi}{2}v)\cdot u\,du\,dv = \int_{-1}^{1} \cos(\frac{\pi}{2}v)\cdot(\frac{u^2}{2})\Big|_1^2\,dv \\ &= \frac{3}{2}(\frac{2}{\pi})\sin(\frac{\pi}{2}v)\Big|_{-1}^{1} = \frac{6}{\pi} .\end{aligned}$$

7. Let C be the curve obtained as the intersection of the surfaces $x^2+y^2=1$ and $x+y+z=1$.

(a) Find a parametrization for C.

$x^2+y^2=1$ suggests using $x(t)=\cos t, y(t)=\sin t$ for $0 \le t \le 2\pi$. Since $z = 1-x-y$, take $z(t) = 1-\cos t - \sin t$, that is

$$\vec{c}(t) = (\cos t, \sin t, 1-\cos t - \sin t) \quad \text{for } 0 \le t \le 2\pi .$$

(b) Find the value of the path integral $\int_C \sqrt{1-xy}\,ds$.
Here $\vec{c}\,'(t) = (-\sin t, \cos t, \sin t - \cos t)$ and

$$\begin{aligned}\|\vec{c}\,'(t)\| &= \sqrt{\sin^2 t + \cos^2 t + \sin^2 t - 2\sin t\cos t + \cos^2 t} \\ &= \sqrt{2 - 2\sin t \cos t} = \sqrt{2}\sqrt{1-\sin t\cos t} .\end{aligned}$$

$$\begin{aligned}\int_C \sqrt{1-xy}\,ds &= \int_0^{2\pi} \sqrt{1-x(t)y(t)}\|\vec{c}\,'(t)\|\,dt \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t\sin t}\,\sqrt{1-\sin t\cos t}\,dt \\ &= \sqrt{2}\int_0^{2\pi} (1-\sin t\cos t)\,dt = 2\sqrt{2}\pi - \sqrt{2}\int_0^{2\pi} d(\frac{\sin^2 t}{2}) = 2\sqrt{2}\pi .\end{aligned}$$