Solutions, Exam 1

Exercise 1. (a) Since $\|\mathbf{u}\| = \sqrt{1+4+4} = 3$ and $\|\mathbf{v}\| = \sqrt{1+0+1} = \sqrt{2}$, the normalized vectors are

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \quad \text{and} \quad \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) .$$

(b) Since

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1+2}{3\sqrt{2}} = \frac{1}{\sqrt{2}}, \quad \theta = \frac{\pi}{4} .$$

(c) A vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u}\times\mathbf{v} = (2,1,-2)$. Since the length of this vector is $\sqrt{4+1+4} = 3$, a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is the normalized vector $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

(d) We can use the vector from (c) to see that the general equation of a plane parallel to both $\mathbf{u}$ and $\mathbf{v}$ is $c = (2,1,-2)\cdot(x,y,z) = 2x+y-2z$ Inserting $A = (0,0,1)$ gives $c = 2(0)+1(1)-2(1) = -1$. So $2x+y-2z = -1$.

Exercise 2. (a) Here $\|\mathbf{a}\| = \sqrt{4+12} = 4$ and $\|\mathbf{b}\| = \sqrt{3+1} = 2$. Also

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{4\sqrt{3}}{4\cdot 2} = \frac{\sqrt{3}}{2} .$$

So $\sin\theta = \frac{1}{2}$, and Area $= \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta = 4\cdot 2\cdot\frac{1}{2} = 4$.

(b) The volume is the absolute value of scalar triple product

$$|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = |(4,0,0)\cdot(4, 2\sqrt{3}, \sqrt{5})| = 16 .$$

(c) Let $\mathbf{x} = (x,y,z)$. Since $0 = (x,y,z)\cdot(1,0,3) = x+3z$, $x = -3z$. Then we find

$$(0,2,1) = 2\mathbf{j}+\mathbf{k} = (-3z,y,z)\times(1,0,0) = (0,z,-y) .$$

Thus, $z = 2$, $y = -1$, $x = -3(2) = -6$ and $\mathbf{x} = (-6,-1,2)$.

Exercise 3. Since $Q - P = (0-4, -2-4, 2-0) = (-4,-6,2)$, a parameterization for the line is

$$\mathbf{c}(t) = (4,4,0) + t(-4,-6,2) = (4-4t, 4-6t), 2t) .$$

(b) Using the normal direction $(-4,-6,2)$ from (a), we get $0 = (-4,-6,2)\cdot(x,y,z)$ or $-2x-3y+z = 0$.

(c) The desired plane is parallel to the plane of (b) and passes through the midpoint between P and Q, which is $\frac{1}{2}(P+Q) = (2,1,1)$. So the equation is $-2x-3y+z = c$ where $c = -2(2)-3(1)+1 = -6$. So $-2x-3y+z = -6$ is the desired equation.

Exercise 4. See attachment.

Exercise 5. (a) This set is open. [A point (a, b) with $a > 0$ and $b > 0$ is the center of an open ball of radius $r = \min\{a, b\}$ which also lies in the set].

The boundary of the set is the union of the origin, the positive X-axis and the positive Y-axis. That is $\{(0, y) \ : \ y \geq 0\} \cup \{(x, 0) \ : \ x \geq 0\}$.

(b) This set is also open. [A point (a, b) with $a > 0$ and $b > 0$ is the center of an open ball of radius $r = |a|$ which also lies in the set].

It's boundary is the entire Y axis.

(c) This set is not open. It contains all its boundary points which forms the unit circle $x^2 + y^2 = 1$. Any ball centered at one of these boundary points intersects both points in the set and points not in the set.

Exercise 6. (a) $\frac{\partial f}{\partial x} \ = \frac{1}{2}(x^2y + 1)^{-1/2}(2yx) \ = \ \frac{xy}{\sqrt{x^2y+1}}$.

(b) $\frac{\partial f}{\partial y} \ = \frac{1}{2}(x^2y + 1)^{-1/2}(x^2) \ = \ \frac{x^2}{2\sqrt{x^2y+1}}$.

(b) These partial derivatives exist for all (x, y) such that $x^2y + 1 + 1 > 0$, that is, $x^2y > -1$.

(c) $\frac{\partial f}{\partial x}(2, 2) \ = \ \frac{2 \cdot 2}{\sqrt{2^2 2+1}} \ = \ \frac{4}{3}$, $\frac{\partial f}{\partial y}(2, 2) \ = \ \frac{2^2}{2\sqrt{2^2 2+1}} \ = \ \frac{2}{3}$.

Exercise 7. (a) Here we note that $t \ = \ x^2 + y^4 \ \downarrow 0$ as $x, y \to 0$. Since $\lim_{t \to 0} \cos t = \cos 0 \ = 1$, we find

$$\lim_{x,y \to 0} \frac{x^2 + y^4}{\cos(x^2 + y^4)} \ = \ \lim_{t \downarrow 0} \frac{t}{\cos t} \ = \ \frac{0}{1} \ = \ 0 \ .$$

(b) Similarly, using L'hôpital's rule, we find

$$\lim_{x,y \to 0} \frac{x^2 + y^4}{\sin(x^2 + y^4)} \ = \ \lim_{t \downarrow 0} \frac{t}{\sin t} \ = \ \lim_{t \downarrow 0} \frac{1}{\cos t} \ = \ \frac{1}{1} \ = \ 1 \ .$$

(c) The difference between $x^2 + y^4$ and $x^4 + y^2$ makes us suspicious that the limit may not exist. To show the limit does not exist, it suffices to find two different ways of approaching $(0, 0)$ that give different limiting behavior of f.

If we take $x = 0$ and let $y \downarrow 0$, we find that

$$\lim_{y \downarrow 0} \frac{y^4}{\sin(y^2)} \ = \ \lim_{y \downarrow 0} y^2 \frac{y^2}{\sin y^2} \ = \ 0 \cdot 1 \ = \ 0 \ .$$

On the other hand, if we take $y = 0$ and let $x \downarrow 0$, we find that

$$\lim_{x \downarrow 0} \frac{x^2}{\sin(x^4)} \ = \ \lim_{x \downarrow 0} x^{-2} \frac{x^4}{\sin x^4} \ = \ (\infty) \cdot 1 \ = \ \infty \ .$$

So the limit in part (c) does not exist.