

Math 212 Spring 2008 Exam 2

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Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You have **two hours**. Do all 8 problems. You must show your work to receive full credit on a problem. An answer with no supporting work or explanation will receive little to no credit.

Please do all your work on standard letter size sheets. At the end write your name on each sheet, attach this page to the front and staple!

The exam is due at the **beginning of class on April 10.**

Please print you name clearly here.

Print name: _____

Upon finishing please sign the pledge below:

On my honor I have neither given nor received any aid on this exam and have observed the time limit given. I started working on this exam at ____:____ and finished at ____:____ on the ____th day of April.

Grader's use only:

1. _____ /20

2. _____ /15

3. _____ /25

4. _____ /20

5. _____ /20

6. _____ /15

7. _____ /15

8. _____ /20

1. [20 points]

Find the extreme values of

$$f(x, y) = x^2 + 3y^2 - 4x + 3$$

over the region bounded by the curves $y^2 = x$ and $x = 4$.

We first find the critical points inside the region. $\frac{\partial f}{\partial x} = 2x - 4$ and $\frac{\partial f}{\partial y} = 6y$ so the only critical point is $(2, 0)$ which is inside the region.

The boundary is made up of two pieces. The first piece is parametrized by (t^2, t) where $-2 \leq t \leq 2$ (drawing a picture would help). On this piece f restricts to

$$g_1(t) = f(t^2, t) = t^4 + 3t^2 - 4t^2 + 3$$

Now $g_1'(t) = 4t^3 - 2t$ so the critical values are at $t = 0$ and $t = \pm \frac{1}{\sqrt{2}}$. This corresponds to points $(0, 0)$, $(\frac{1}{2}, \pm \frac{1}{\sqrt{2}})$.

The second piece is parametrized by $(4, t)$ where $-2 \leq t \leq 2$. On this piece f restricts to

$$g_2(t) = f(4, t) = 16 + 3t^2 - 16 + 3 = 3t^2 + 3$$

Now $g_2'(t) = 6t$ so the only critical point is $t = 0$ which is the point $(4, 0)$.

Finally, we also need to check the special points $(4, 2)$ and $(4, -2)$ which are the end-points of these two pieces.

Comparing all these points we have $f(2, 0) = -1$, $f(0, 0) = 3$, $f(\frac{1}{2}, \frac{1}{\sqrt{2}}) = 1/4 + 3/2 - 2 + 3 = 11/4$ and similarly $f(\frac{1}{2}, -\sqrt{1}\sqrt{2}) = 11/4$ while $f(4, 2) = 16 + 12 - 16 + 3 = 15$ and similarly $f(4, -2) = 15$.

Thus the absolute minimum is -1 and occurs at the point $(2, 0)$ while the absolute maximum is 15 and occurs at the two points $(4, \pm 2)$.

2. a)[5 points] Sketch the vector field $\mathbf{F}(x, y) = (-y, x)$ at the eight points $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 1)$, $(-1, 0)$, $(-1, -1)$, $(0, -1)$ and $(1, -1)$. The vector fields points in the following directions at the eight points above: $(0, 1)$, $(-1, 1)$, $(-1, 0)$, $(-1, -1)$, $(0, -1)$, $(1, -1)$, $(1, 0)$ and $(1, 1)$.
- b)[5 points] Sketch the flow line through the point $(1, 0)$. The flow line is the unit circle (with center $(0, 0)$) oriented in the counter-clockwise direction.
- c)[5 points] Find an explicit parametrization of the flow line through the point $(1, 0)$ and check that it is indeed a flow line. An explicit parametrization is $c(t) = (\cos t, \sin t)$ where $-\infty \leq t \leq \infty$. To check this notice that $c'(t) = (-\sin t, \cos t)$ while $\mathbf{F}(c(t)) = (-\sin t, \cos t)$. Thus $c'(t) = \mathbf{F}(c(t))$ which proves that $c(t)$ is the flow line through $(1, 0)$.

3. a)[10 points] If $\mathbf{F}(x, y, z)$ is a vector field prove that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. This is a direct application of the definitions. Suppose $\mathbf{F} = (F_1, F_2, F_3)$. Then we have

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}) &= \nabla \cdot \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, -\frac{\partial F_3}{\partial x_1} + \frac{\partial F_1}{\partial x_3}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) \\ &= \left(\frac{\partial^2 F_3}{\partial x_2 \partial x_1} - \frac{\partial^2 F_2}{\partial x_3 \partial x_1} \right) - \left(\frac{\partial^2 F_3}{\partial x_1 \partial x_2} - \frac{\partial^2 F_1}{\partial x_3 \partial x_2} \right) + \left(\frac{\partial^2 F_2}{\partial x_1 \partial x_3} - \frac{\partial^2 F_1}{\partial x_2 \partial x_3} \right) \\ &= 0 \end{aligned}$$

where to obtain the last equality we just noticed that all the terms cancelled out in pairs.

- b)[5+5 points] Compute the curl and divergence of $\mathbf{F}(x, y, z) = (-xy, xz, -yz)$.

The curl is

$$\begin{aligned} \nabla \times (-xy, xz, -yz) &= \left(\left(\frac{\partial(-yz)}{\partial y} - \frac{\partial(xz)}{\partial z} \right), -\left(\frac{\partial(-yz)}{\partial x} - \frac{\partial(-xy)}{\partial z} \right), \left(\frac{\partial(xz)}{\partial x} - \frac{\partial(-xy)}{\partial y} \right) \right) \\ &= (-z - x, 0 + 0, z + x) \end{aligned}$$

While the divergence is

$$\begin{aligned} \nabla \cdot (-xy, xz, -yz) &= \frac{\partial(-xy)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(-yz)}{\partial z} \\ &= -y + 0 - y \\ &= -2y \end{aligned}$$

- c)[5 points] Can $\mathbf{F}(x, y, z) = (-xy, xz, -yz)$ be the curl of a vector field? (i.e. does there exist a vector field \mathbf{V} such that $\nabla \times \mathbf{V} = \mathbf{F}$.) If yes then find such a \mathbf{V} – otherwise justify why not.

If such a vector field existed then $\mathbf{F} = \nabla \times \mathbf{V}$ so applying divergence we get $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{V})$. Now the right hand side is zero by part (a) while the left hand side is $-2y$ by part (b). So no such \mathbf{V} can exist.

4. a)[10 points] Consider the triangle T with vertices $(1, 1)$, $(3, 2)$ and $(2, 4)$ and the triangle T' with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Find a change of variables $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ which maps the triangle T' onto the triangle T .

If we shift the triangle T by $(-1, -1)$ then we get the triangle S with vertices $(0, 0)$, $(2, 1)$ and $(1, 3)$. Now it is easy to write down a map $f_1 : T' \rightarrow S$ since we can use the linear map

$$f_1 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + y \\ x + 3y \end{pmatrix}$$

which maps $(1, 0)$ to $(2, 1)$ and $(0, 1)$ to $(1, 3)$. To get T' we just have to shift S by $(1, 1)$ – i.e. to compose with the map $f_2 : (x, y) \mapsto (x + 1, y + 1)$. Thus one change of variables we can use is the composition

$$f = f_2 \circ f_1 : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + y + 1 \\ x + 3y + 1 \end{pmatrix}$$

which is nothing more than a linear map f_1 composed with a translation f_2 .

- b)[10 points] Let E be the ellipse given by $x^2 + 4y^2 = 4$. Find a change of coordinates $f : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ which maps the square $[0, 1] \times [0, 1]$ onto the (interior of the) ellipse E .

If E were a circle then we could use our favourite parametrizations $(\cos t, \sin t)$. Since an ellipse is a squished circle we should look for a parametrization of the form

$$(a \cos t, b \sin t)$$

where a, b are real numbers. To find a, b just plug in $t = 0$ to get $(a, 0)$ which ought to be one of the x intercepts of $x^2 + 4y^2 = 4$. Since the x -intercepts are $x = \pm 2$ we can take $a = 2$. Similarly, plugging in $t = \pi/2$ we get $(0, b)$ which should be the y -intercept $(0, 1)$ so $b = 1$. Thus our parametrization should be

$$(2 \cos t, \sin t)$$

where $0 \leq t \leq 2\pi$ just like in the case of a circle. To check that this parametrization works we substitute into $x^2 + 4y^2$ to get

$$(2 \cos t)^2 + 4(\sin t)^2 = 4(\cos^2 t + \sin^2 t) = 4$$

as required.

Thus the parametrization we want is

$$(2r \cos(2\pi t), r \sin(2\pi t))$$

where $0 \leq r \leq 1$ and $0 \leq t \leq 1$ (we added the factor of 2π so that t varies between 0 and 1 instead of 0 and 2π).

5. [20 points] Compute the integral $\int_0^2 \int_0^{x^2} (y + 3x^2) dy dx$ by **first** changing the order of integration and **then** integrating.

First one needs to determine what the region D of integration is. Checking the endpoints of the integral we find that D is bounded above and below by $y = x^2$ and the x -axis while on the left and the right by the lines $x = 0$ and $x = 2$ (a figure at this point would be very helpful).

Integrating in the x -direction first we need to integrate from $y = x^2$ (or equivalently $x = \sqrt{y}$) to $x = 2$ so we get

$$\int_{\sqrt{y}}^2 (y + 3x^2) dx$$

In the y direction we must integrate from $y = 0$ to $y = 4$. We got $y = 4$ here since the point of intersection of $x = 2$ and $y = x^2$ is $(2, 4)$ (the y -coordinate being 4). Thus the integral we get is

$$\begin{aligned} \int_0^4 \int_{\sqrt{y}}^2 (y + 3x^2) dx dy &= \int_0^4 [xy + x^3]_{\sqrt{y}}^2 dy \\ &= \int_0^4 (2y + 8 - y\sqrt{y} - y\sqrt{y}) dy \\ &= [y^2 + 8y - \frac{4}{5}y^{5/2}]_0^4 \\ &= 16 + 32 - \frac{4}{5}(32) = \frac{16 \cdot 7}{5} \\ &= \frac{112}{5} \end{aligned}$$

6. [15 points]

Let D be the region in \mathbf{R}^2 determined by conditions $4x^2 + 9y^2 \leq 36$ and $x, y \geq 0$. Compute the integral

$$\int \int_D y dx dy$$

Like in problem 4(b) we can parametrize the ellipse $4x^2 + 9y^2 = 36$ as $(3 \cos t, 2 \sin t)$ except that now $0 \leq t \leq \pi/2$ since $x, y \geq 0$ (so we're in the first quadrant). Thus we can use coordinates on D given by $T(r, t) = (3r \cos t, 2r \sin t)$ where $0 \leq t \leq \pi/2$ and $0 \leq r \leq 1$.

The Jacobian of this change of coordinates is

$$\frac{\partial(x, y)}{\partial(r, t)} = \det \begin{pmatrix} -3r \sin t & 2r \cos t \\ 3 \cos t & 2 \sin t \end{pmatrix} = -6r \sin^2 t - 6r \cos^2 t = -6r$$

Thus, by the change of coordinate formula the integral is equal to

$$\begin{aligned} \int \int_D y dx dy &= \int_0^1 \int_0^{\pi/2} (2r \sin t) | -6r | dt dr \\ &= \int_0^1 \int_0^{\pi/2} 12r^2 \sin t dt dr \\ &= \int_0^1 12r^2 [-\cos t]_0^{\pi/2} dr \\ &= \int_0^1 12r^2 dr \\ &= [4r^3]_0^1 \\ &= 4 \end{aligned}$$

7. Let $f(x, y) = e^{x^2+y} - 1$.

a)[6 points] Find the second order Taylor series expansion of $f(x, y)$ around the point $(0, 0)$.

We have $f(0, 0) = 0$ while $\frac{\partial f}{\partial x} = 2xe^{x^2+y}$ so $\frac{\partial f}{\partial x}(0, 0) = 0$ and $\frac{\partial f}{\partial y} = e^{x^2+y}$ so $\frac{\partial f}{\partial y}(0, 0) = 1$.

Next

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} \end{pmatrix} = \begin{pmatrix} e^{x^2+y}(2 + 4x^2) & 2xe^{x^2+y} \\ 2xe^{x^2+y} & e^{x^2+y} \end{pmatrix}$$

so

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x}(0, 0) & \frac{\partial^2 f}{\partial x \partial y}(0, 0) \\ \frac{\partial^2 f}{\partial y \partial x}(0, 0) & \frac{\partial^2 f}{\partial y \partial y}(0, 0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus the 2nd order Taylor series expansion around $(0, 0)$ is

$$(x, y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2}(x, y) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = y + \frac{1}{2}(2x^2 + y^2)$$

b)[6 points] Find the second order Taylor series expansion of $f(x, y)$ around the point $(0, 1)$.

We have $f(0, 1) = e - 1$. Using the computations in part (a) we have $\frac{\partial f}{\partial x}(0, 1) = 0$ and $\frac{\partial f}{\partial y}(0, 1) = e$ while

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x \partial x}(0, 1) & \frac{\partial^2 f}{\partial x \partial y}(0, 1) \\ \frac{\partial^2 f}{\partial y \partial x}(0, 1) & \frac{\partial^2 f}{\partial y \partial y}(0, 1) \end{pmatrix} = \begin{pmatrix} 2e & 0 \\ 0 & e \end{pmatrix}$$

Thus the 2nd order Taylor series expansion around $(0, 1)$ is

$$(e-1) + (x, y-1) \begin{pmatrix} 0 \\ e \end{pmatrix} + \frac{1}{2}(x, y-1) \begin{pmatrix} 2e & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} x \\ y-1 \end{pmatrix} = (e-1) + e(y-1) + \frac{e}{2}(2x^2 + (y-1)^2)$$

c)[3 points] Give an example of a smooth function $g(t)$ whose Taylor series around 0 does **not** converge to $g(t)$?

One such function is given by $g(t) = 0$ if $t \leq 0$ and $g(t) = e^{-\frac{1}{t^2}}$ if $t > 0$. We saw in one of the homework problems that this function is smooth and that its Taylor series around $t = 0$ is 0 (which obviously does not equal $g(t)$).

8. [20 points]

Let W be the region in \mathbf{R}^3 determined by conditions $x^2 + y^2 + z^2 \leq 1$ and $z^2 \geq x^2 + y^2$. Compute the volume of W . (hint: it may be helpful for you as well as the grader if you drew a picture of W). You should be able to compute the integral and get a number at the end.

W is made up of two pieces corresponding to $z \geq 0$ and $z \leq 0$. These two pieces are the same and so have the same volume. We will calculate the volume of the piece when $z \geq 0$. To do this we first integrate in the z direction to get

$$\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 1 dz = \sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}$$

Next we integrate in the $x-y$ plane. The intersection of $x^2 + y^2 + z^2 = 1$ and $z^2 = x^2 + y^2$ (where $z \geq 0$) is the circle with z -coordinate $\frac{1}{\sqrt{2}}$ and radius $\frac{1}{\sqrt{2}}$. So in the $x-y$ plane we need to integrate over the disk D of radius $\frac{1}{\sqrt{2}}$. We'll use polar coordinates so we get

$$\begin{aligned} \iint_D \sqrt{1-x^2-y^2} - \sqrt{x^2+y^2} dx dy &= \int_0^{1/\sqrt{2}} \int_0^{2\pi} (\sqrt{1-r^2} - r)(r d\theta dr) \\ &= 2\pi \int_0^{1/\sqrt{2}} r\sqrt{1-r^2} - r^2 dr \\ &= 2\pi \left[-\frac{1}{3}(1-r^2)^{3/2} - r^3/3 \right]_0^{1/\sqrt{2}} \\ &= 2\pi \left[-\frac{1}{3} \frac{1}{2\sqrt{2}} - \frac{1}{6\sqrt{2}} + \frac{1}{3} \right] \\ &= \frac{2\pi}{3} \cdot \left(1 - \frac{1}{\sqrt{2}} \right) \end{aligned}$$

Thus the total volume is twice this which is $\frac{2\pi}{3}(2 - \sqrt{2})$.