Math 212 *Multivariable Calculus* **- Final Exam**

Instructions: You have **3 hours** to complete the exam (**12 problems**). This is a closed book, closed notes exam. Use of calculators is not permitted. Show all your work for full credit. **Please do not forget to write your name and your instructor's name on the blue book cover, too.**

Print your instructor's name : ______________________________

Print your name : ______________________________

Upon finishing please sign the pledge below:

On my honor I have neither given nor received any aid on this exam.

Signature : ______________________________

Problem	Max Points	Your Score	Problem	Max Points	Your Score
1	8		7	8	
2	8		8	8	
3	8		9	9	
4	8		10	9	
5	8		11	9	
6	8		12	9	
			Total	100	

[**1**] (8 points) Find the equation of the tangent line to the path $\mathbf{c}(t) = (\cos t, \sin t, t^2)$ at $t = \pi$.

Solution. The derivative of this path is given by

$$c'(t) = (-\sin(t), \cos(t), 2t)$$

The tangent line to $c(t)$ at the point $t = \pi$ is given by

$$\begin{aligned} L(t) &= c(\pi) + t \cdot c'(\pi) \\ &= \left(\cos(\pi), \sin(\pi), \pi^2\right) + t \cdot (-\sin(\pi), \cos(\pi), 2\pi) \\ &= \left(-1, 0, \pi^2\right) + t \cdot (0, -1, 2\pi) \\ &= \left(-1, -t, \pi^2 + 2\pi t\right) \end{aligned}$$

(Or you may use $L(t) = c(\pi) + (t - \pi) \cdot c'(\pi)$.)

[**2**] (8 points) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by:

$$f(x, y, z) = 3z + e^{x^2 - y^2}$$

Let C be the set of the heads of unit vectors v in $\mathbb{R}^3$ such that f increases at $1/3$ of its maximum rate of change in the direction v starting from $(0, 0, 1)$. Find the equation(s) which determine(s) the set C. (Hint : C is a circle in $\mathbb{R}^3$.)

Solution. The gradient of f is given by

$$\nabla f = \left(2x \exp\left(x^2 - y^2\right), -2y \exp\left(x^2 - y^2\right), 3\right)$$

$\nabla f(0, 0, 1)$ is a vector in the direction of greatest increase for the function f at the point $(0, 0, 1)$. So

$$\begin{aligned} \nabla f(0, 0, 1) &= (0, 0, 3) \\ &= 3 \cdot (0, 0, 1) \end{aligned}$$

Calculate the maximimum rate of change of f by evaluating the directional derivative of f.

$$\begin{aligned} Df_{(0,0,1)} &= \nabla f \cdot (0, 0, 1) \\ &= (0, 0, 3) \cdot (0, 0, 1) \\ &= 3 \end{aligned}$$

$1/3$ of this rate is just 1, so the goal of this problem is to find and equation for the unit vectors $v = (x, y, z)$ such that $Df_v = 1$ at the point $(0, 0, 1)$.

$$\begin{aligned} 1 &= Df_v \\ &= \nabla f \cdot v \\ &= (0, 0, 3) \cdot (x, y, z) \\ &= 3z \\ 1/3 &= z \end{aligned}$$

Now applying the constraint that v be a unit vector

$$\begin{aligned} v \cdot v &= 1 \\ (x, y, 1/3) \cdot (x, y, 1/3) &= 1 \\ x^2 + y^2 + 1/9 &= 1 \\ x^2 + y^2 &= 8/9 \end{aligned}$$

The final equations are : $x^2 + y^2 = 8/9$ and $z = 1/3$.

[**3**] (8 points) Find the absolute minimum and maximum for the function $f(x, y) = x + 2y^2 + 1$ on the unit disk $D = \{(x, y) |\, x^2 + y^2 \leq 1\}$.

Solution. On the interior of D, which is $\{(x, y) |\, x^2 + y^2 < 1\}$, if (x, y) is a critical point of f, $\nabla f(x, y) = (0, 0)$. But $\nabla f(x, y) = (1, 4y) \neq (0, 0)$. Hence f has its extrema on ∂D.

On $\partial D = \{(x, y) |\, x^2 + y^2 = 1\}$, we use Lagarange Multiplier. Let $g(x, y) = x^2 + y^2$, and solve $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $x^2 + y^2 = 1$ simultaneously. Then we have $1 = 2\lambda x, 4y = 2\lambda y$, and $x^2 + y^2 = 1$.

If $y = 0$, $x = \pm 1$, hence we have $(x, y) = (1, 0), (-1, 0)$. If $y \neq 0$, then $\lambda = 2$ and $x = \frac{1}{4}$, hence $y = \pm\frac{\sqrt{15}}{4}$. Thus $(x, y) = (\frac{1}{4}, \frac{\sqrt{15}}{4}), (\frac{1}{4}, -\frac{\sqrt{15}}{4})$.

Now $f(1, 0) = 2, f(-1, 0) = 0, f(\frac{1}{4}, \frac{\sqrt{15}}{4}) = f(\frac{1}{4}, -\frac{\sqrt{15}}{4}) = \frac{25}{8}$. Therefore f has the absolute maxima $(\frac{1}{4}, \frac{\sqrt{15}}{4})$ and $(\frac{1}{4}, -\frac{\sqrt{15}}{4})$, and the absolute minimum $(-1, 0)$.

[**4**] (8 points) Let C be a simple closed curve in $\mathbb{R}^3$, and $\mathbf{c} : [a, b] \to \mathbb{R}^3$ be a parametrization of C such that $||\mathbf{c}'(t)|| = 1$ for all $t \in [a, b]$. The *curvature* of C at $\mathbf{c}(t)$ is defined to be

$$\kappa(t) = ||\mathbf{c}''(t)||$$

and the *total curvature* of C is defined by

$$\int_{\mathbf{c}} \kappa \text{ ds}.$$

Compute the total curvature of a circle $C = \{(x, y, z) \in \mathbb{R}^3 |\, x^2 + y^2 = R^2, z = 0\}$.

Solution. Let $\mathbf{c}(s) = (R\cos(s/R), R\sin(s/R), 0)$, $s \in [0, 2\pi]$. Note that

$$\mathbf{c}'(s) = (-\sin(s/R), \cos(s/R), 0)$$

is of unit length for all s. We have $\mathbf{c}''(s) = (-\frac{\cos(s/R)}{R}, -\frac{\sin(s/R)}{R}, 0)$ and $\kappa(s) = ||\mathbf{c}''(s)|| = 1/R$ for all s. Hence the total curvature is

$$\int_{\mathbf{c}} \kappa \, ds = \int_0^{2\pi} \frac{1}{R} ds = \frac{1}{R} \cdot \text{length}(C) = \frac{1}{R} \cdot 2\pi R = 2\pi.$$

[**5**] (8 points) Let C be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Evaluate the line integral $\int_C \mathbf{F}\cdot\ d\mathbf{s}$ where $\mathbf{F}(x,y)=(x/a^2,y/b^2)$.

Solution. Method I: Consider the region

$$S=\{(x,y,)\,|\,\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1\}.$$

By Stokes' theorem, we have

$$\int_C \mathbf{F}\cdot\ d\mathbf{s}=\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}.$$

But $\nabla\times\mathbf{F}=0$, hence the integral is 0.

Method II: The ellipse can be parametrized by

$$\mathbf{c}(t)=(a\,\text{cos}t,b\,\text{sin}t),\ \ t\in[0,2\pi].$$

Since we have

$$\mathbf{F}(\mathbf{c}(t))\cdot\mathbf{c}'(t)=\frac{\text{cos}t}{a}(-a\,\text{sin}t)+\frac{\text{sin}t}{b}b\,\text{cos}t=0,$$

the line integral is zero.

[**6**] (8 points) Evaluate the line integral $\int_{\mathbf{c}} e^{x^2}\ dx-xy\ dy+y^2\ dz$, where $\mathbf{c}(t)=(1,t,t^2)$, $0\leq t\leq 1$.

Solution. $\int_{\mathbf{c}} e^{x^2}\ dx-xy\ dy+y^2\ dz=\int_0^1(e\cdot 0-1\cdot t\cdot 1+t^2\cdot 2t)\ dt=\int_0^1(-t+2t^3)\ dt=\left[-\frac{1}{2}t^2+\frac{2}{4}t^4\right]_0^1=0.$

[**7**] (8 points) Find the area of the portion of the unit sphere inside the cylinder $x^2+y^2=\frac{1}{2}$ and $z>0$.

Solution. The intersection of the unit sphere and the cylinder is a circle, and the angle between the z-axis and a line from the origin to the circle is $\frac{\pi}{4}$. Denote the surface by S. We parametrize S by $\Phi(\phi,\theta)=(\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi)$ where $0\leq\phi\leq\frac{\pi}{4},0\leq\theta\leq 2\pi$. Then $||\Phi_\phi\times\Phi_\theta||=\sin\phi$.

Hence $\iint_S\ dS=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\sin\phi\ d\phi\ d\theta=[-\text{cos}\phi]_0^{\pi/4}\cdot 2\pi=(2-\sqrt{2})\pi.$

[**8**] (8 points) W is the volume defined by $x^2+y^2+z^2\leq 1$ and $y\leq x$. Find the flux of (x^3-3x,y^3+xy,z^3-xz) out of W.

Solution. Calculate that

$$\begin{aligned}\nabla\cdot F &= 3x^2-3+3y^2+x+3z^2-x\\ &= 3\left(x^2+y^2+z^2-1\right)\end{aligned}$$

By Gauss Theorem the flux out of W is given by

$$\begin{aligned}\int\int_{\delta W} F\cdot dS &= \int\int\int_{W} \nabla\cdot F\cdot dW\\ &= 3\int\int\int x^2+y^2+z^2-1dxdydz\end{aligned}$$

In spherical coordinates, the defining equations of the volume W are $\rho^2 \leq 1$ and $\sin\theta \leq \cos\theta$. The latter inequality implies $-3\pi/4 \leq \theta \leq \pi/4$.

The Jacobian determinant for spherical coordinates is $\rho^2\sin\phi$, so the integral is just:

$$\begin{aligned}3\int\int\int x^2+y^2+z^2-1dxdydz &= 3\int_0^1\int_{-3\pi/4}^{\pi/4}\int_0^{\pi}\left(\rho^2-1\right)\rho^2\sin\phi d\phi d\theta d\rho\\ &= -4\pi/5\end{aligned}$$

[9] (9 points) Let S be the surface of the tetrahedron whose vertices are $(1,0,0),(0,1,0),(0,0,2)$ and the origin. Evaluate $\iint_S f \text{ d}S$ where $f(x,y,z)=xz$.

Solution. Let S', S'', S''', T denote the faces of the tetrahedron in the xz-plane, yz-plane, xy-plane, and the plane $2x+2y+z=2$ (the frontal face), respectively. Since $x=0$ on S'' and $z=0$ on S''', it follows that the integral vanishes on those faces. Hence we have

$$\iint_S xz \text{ d}S = \iint_{S'} xz \text{ d}S + \iint_T xz \text{ d}S.$$

(1)

$$\begin{aligned}\iint_{S'} xz \text{ d}S &= \int_0^1\int_0^{1-x} xz \text{ d}z \text{ d}x\\ &= \int_0^1 x(1-x)^2/2 \text{ d}x = 1/24.\end{aligned}$$

(2) T is the graph of $z=2-2x-2y$ over the triangle Δ with vertices $(1,0,0)$, $(0,1,0)$, and the origin. Let $g(x,y)=2-2x-2y$. Then we have

$$\begin{aligned}\iint_T xz \text{ d}S &= \iint_\Delta xg(x,y)\sqrt{1+g_x^2+g_y^2} \text{ d}A\\ &= \int_0^1\int_0^{1-x} x(2-2x-2y)3 \text{ d}y \text{ d}x\\ &= \int_0^1 3x(1-x)^2 \text{ d}x = 1/4.\end{aligned}$$

(3) $\iint_S xz \text{ d}S = 1/24 + 1/4 = 7/24.$

[10] (9 points) Evaluate $\iint_S \mathbf{F}\cdot \text{ d}\mathbf{S}$ where

$$\mathbf{F}(x,y,z) = \left(1,1,z\left(x^2+y^2\right)^2\right)$$

and S is the surface of the cylinder $x^2+y^2 \le 1$, $0 \le z \le 1$ (oriented with the outward unit normal).

Solution. To use Gauss Theorem, write $\delta W = S$ and so

$$\begin{aligned}\int\int_S F\cdot dS &= \int\int_{\delta W} F\cdot dS \\ &= \int\int\int_W div\,(F)\cdot dW\end{aligned}$$

$$\begin{aligned}div\,(F) &= 0+0+\left(x^2+y^2\right)^2 \\ &= \left(x^2+y^2\right)^2\end{aligned}$$

The Jacobian determinant of cylindrical coordinates is r, so

$$\begin{aligned}\int\int\int_W div\,(F)\cdot dW &= \int\int\int_W \left(x^2+y^2\right)^2\cdot dW \\ &= \int_0^1\int_0^{2\pi}\int_0^1 \left(r^2\right)^2 r dz d\theta dr \\ &= \pi/3\end{aligned}$$

[11] (9 points) Let S be the portion of the unit sphere centered at the origin that is cut out by the cone $z \ge \sqrt{x^2+y^2}$. Evaluate $\iint_S \mathbf{F}\cdot\ d\mathbf{S}$ where

$$\mathbf{F}(x,y,z) = (xy+\cos z, -yx+x^2+z^3, 2z^2+x).$$

Solution. The boundary C of S is the circle of radius $1/\sqrt{2}$ on $z=1/\sqrt{2}$ and with its center on the z-axis. Let D be the disc in the plane $z=1/\sqrt{2}$ bounded by C, and $T=S\cup D$. Then we have

$$\iint_S \mathbf{F}\cdot\ d\mathbf{S} = \iint_T \mathbf{F}\cdot\ d\mathbf{S} - \iint_D \mathbf{F}\cdot\ d\mathbf{S}.$$

(1) T is a surface without boundary and we can apply Gauss' theorem to the first integral $\iint_T \mathbf{F}\cdot\ d\mathbf{S}$. Let W be the (3-dimensional) region bounded by T.

$$\begin{aligned}\iint_T \mathbf{F}\cdot\ d\mathbf{S} &= \iiint_W \nabla\cdot\mathbf{F}\ dV \\ &= \iiint_W y-x+4z\ dV.\end{aligned}$$

But $\iiint_W y\ dV = \iiint_W x\ dV = 0$ because of the apparent symmetry.

$$\begin{aligned}\iiint_W 4z\ dV &= \int_0^{2\pi}\int_{1/\sqrt{2}}^1\int_0^{\sqrt{1-z^2}} 4z\,r\ dr\ dz\ d\theta \\ &= 8\pi\int_{1/\sqrt{2}}^1 z\frac{1-z^2}{2}\ dz = \frac{\pi}{4}.\end{aligned}$$

(2) Since $\mathbf{n}=-\mathbf{k}$ on D, we have

$$\begin{aligned}\iint_D \mathbf{F}\cdot\ d\mathbf{S} &= \iint_D \mathbf{F}\cdot(-\mathbf{k})\ dS \\ &= -\iint_D 2z^2+x\ dA.\end{aligned}$$

But $\iint_D x \, \mathrm{d}A = 0$ because of symmetry.

$$\begin{aligned} -\iint_D 2z^2 \, \mathrm{d}A &= -\iint_D 2 \cdot \tfrac{1}{2} \, \mathrm{d}A \\ &= -\iint_D \, \mathrm{d}A = -\pi/2. \end{aligned}$$

(3) $\iint_S \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \pi/4 - (-\pi/2) = 3\pi/4.$

[12] (9 points) Evaluate $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \, \mathrm{d}\mathbf{S}$, where $\mathbf{F}(x, y, z) = (xy, y, z)$ and S is the surface described by $x^2 + y^2 + z^2 = 2$ and $z \geq 1$. (S is oriented with the upward unit normal.)

Solution. ∂S is a circle with the equations $x^2 + y^2 = 1$ and $z = 1$. Use Stokes' Theorem.

$\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \, \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s}$. We parametrize ∂S by $\mathbf{c}(t) = (\cos t, \sin t, 1)$, $0 \leq t \leq 2\pi$.

$\displaystyle\int_{\partial S} \mathbf{F} \cdot \, \mathrm{d}\mathbf{s} = \int_0^{2\pi} (\cos t \sin t(-\sin t) + \sin t \cos t + 1 \cdot 0) \, \mathrm{d}t = \int_0^{2\pi} (-\sin^2 t \cos t + \frac{1}{2} \sin 2t) \, \mathrm{d}t = \left[-\frac{1}{3} \sin^3 t - \frac{1}{4} \cos 2t \right]_0^{2\pi} = 0.$