

Name:

Math 106: Practice Final Exam

You have 3 hours to complete the exam.

I. True/False questions.

Each question is worth 3 points. Circle the right answer.

1. A curve in \mathbb{R}^3 whose torsion is identically 0 lies on a plane.

T F

2. The differential of the form $y^2 dx + e^y dy$ is equal to $2dx \wedge dy$.

T F

3. Two unit-speed curves in \mathbb{R}^2 are congruent if and only if they have the same curvature.

T F

4. If $F : M \rightarrow N$ is a regular mapping of manifolds then $\dim M = \dim N$.

T F

5. Let M be the manifold obtained from \mathbb{R}^3 by removing the x -axis. Then M is simply connected.

T F

6. Let $M \subset \mathbb{R}^3$ be a surface and let $p \in M$ be a non-umbilic point of M . Then the two principal curves on M passing through p are always orthogonal to each other at p .

T F

7. Let M be a surface in \mathbb{R}^3 . If all geodesics on M are principal then M is flat.

T F

8. The matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

may be the matrix of the shape operator of a surface $M \subset \mathbb{R}^3$ at some point $p \in M$ (with respect to some basis of the tangent plane).

T F

9. Let M be a surface in \mathbb{R}^3 . Assume that for any closed curve α in M the parallel transport of any tangent vector \mathbf{v} along this curve is equal to \mathbf{v} . Then M is flat.

T F

10. The Euler characteristic of an ellipsoid is 0.

T F

II. (30 points)

a) (10 points) Show that the plane curvature of a curve $\alpha(t)$ parameterized as $\alpha(t) = (x(t), y(t))$ is given by

$$\tilde{\kappa} = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}.$$

You may only use the definition of the curvature for unit-speed curves.

b) (20 points) Consider the curve given by the equation $x^2 = y^3$ in \mathbb{R}^2 . Find a parametrization of this curve and its plane curvature with respect to this parametrization.

III. (30 points) Consider the function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by

$$F(x, y, z) = x^2y + yz + z^2$$

- a) (10 points) Find all points in \mathbb{R}^3 at which the above mapping is regular.
- b) (10 points) Show that the equation $F(x, y, z) = 1$ defines a smooth orientable surface in \mathbb{R}^3 .
Is this surface compact/connected?
- c) (10 points) Find all points of M at which the projection from M to the (x, z) -plane is not regular.

IV. (25 points) Consider the cylinder $x^2 + y^2 = r^2$ in \mathbb{R}^3 . Find the geodesics on this cylinder. You may use any method.

V. (25 points) Let M be a compact oriented geometric surface. Assume that there exists a nowhere vanishing tangent vector field on M . Prove that M is diffeomorphic to a torus (hint: use the Gauss-Bonnet theorem).