Cartan-type estimates for potentials with the Cauchy kernel and with real kernels

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For a given $P > 0$, real numbers $\nu_1, \dots, \nu_N$ and points $z_1, \dots, z_N$ in the complex plane we define the set

$$\mathcal{Z}(P) = \left\{ z \in \mathbb{C} : \left| \sum_{j=1}^{N} \frac{\nu_j}{z - z_j} \right| > P \right\}$$

In the case $\nu_j \equiv 1$ the sum is the logarithmic derivative of the polynomial $\prod_{j=1}^{N}(z - z_j)$. It has also the following physical interpretation: if infinite wires carrying charge densities ν_j are placed perpendicular to the complex plane at each z_j then the resulting electrostatic field is given by the conjugate vector $2\overline{\mathcal{C}(z)}$.

Our goal is to estimate the size of $\mathcal{Z}(P)$ in terms of the radii of covering disks. The sharp estimate (up to an absolute constant) of the set

$$\mathcal{X}(P) = \left\{ z \in \mathbb{C} : \sum_{j=1}^{N} \frac{\nu_j}{|z - z_j|} > P \right\}$$

can be easily obtained using a method of Cartan (1928) (here the sum is the Newtonian potential with charges ν_j located at points $z_1, \dots, z_N$). Thus, the problem is to catch the effect of the mutual cancellation of the terms when we pass from the sum of moduli to the modulus of the sum.

In spite of the fact that the problem on estimation of $\mathcal{Z}(P)$ was posed by Macintyre and Fuchs in 1940 (a particular case was considered by Boole in 1857), it was solved only in 2005 by J. M. Anderson and the speaker using a tool which appeared only in the last 10 years in connection with the development of the theory of analytic capacity (Melnikov, Tolsa, Mattila, Nazarov, Treil, Volberg and others).

In the lecture I will speak about some of the notions and facts of this theory as well as about a very recent generalization of the problem described above. As an application we obtain results on the connection between the analytic capacity and Hausdorff measure, in particular, an analog of the Frostman theorem on classical capacities.