Boundaries of Teichmüller spaces and end-invariants for hyperbolic 3-manifolds

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Abstract. We study two boundaries for the Teichmüller space of a surface Teich(S) due to Bers and Thurston. Each point in Bers’ boundary is a hyperbolic 3-manifold with an associated geodesic lamination on S, its end-invariant, while each point in Thurston’s is a measured geodesic lamination, up to scale. We show that when dimc(Teich(S)) > 1 the end-invariant is not a continuous map to Thurston’s boundary modulo forgetting the measure with the quotient topology. We recover continuity by allowing as limits maximal measurable sub-laminations of Hausdorff limits and enlargements thereof.

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1. Introduction

In celebrated boundaries for Teichmüller space due to Bers and Thurston, geodesic laminations arise in natural ways:

• A point M in Bers’ boundary, a hyperbolic 3-manifold, has an associated geodesic lamination \( \mathcal{E}(M) \) that has been pinched. The lamination \( \mathcal{E}(M) \) is an invariant of the quasi-isometry class \([M]\) of \(M\).

• A point \([\mu]\) in Thurston’s boundary, a measured lamination \(\mu\) up to scale, records the asymptotic stretching of divergent hyperbolic metrics \(X_i \to [\mu]\). Its support \(|\mu|\) is a geodesic lamination.

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Thurston’s *ending lamination conjecture* predicts that the map $[M] \to \mathcal{E}(M)$ from quasi-isometry classes in Bers’ boundary to the quotient of Thurston’s boundary by forgetting the measure is an injection. In other words, if one knows the lamination $\mathcal{E}(M)$, one knows the manifold $M$ up to quasi-isometry. The map $\mathcal{E}$ gives a bijection between dense subsets: the dense family of *maximal cusps* $M$ (a maximal family of simple closed curves is pinched in $M$) is mapped by $\mathcal{E}$ to the dense set of *maximal partitions* of $S$ by simple closed curves (which are analogous to rational points of $S^1$). Thus, given Thurston’s conjecture, it is natural to ask whether $\mathcal{E}$ is a homeomorphism. Or, as a starting point, how do sequences $\mathcal{E}(M_n)$ behave under limits $M_n \to M$?

In this paper we show $\mathcal{E}$ has the following continuity properties:

I. $\mathcal{E}$ is strictly lower-semi-continuous in the quotient topologies,

II. $\mathcal{E}$ is continuous in a new *end-invariant topology*, based on the Hausdorff topology, which predicts new information about its limiting values, and

III. $\mathcal{E}$ cannot have a continuous inverse in the end-invariant topology, nor do Hausdorff limits completely encode the limiting end-invariant in general.

To state our results more precisely, we review terminology.

Let $S$ be an oriented surface, closed for simplicity, and let $Q(X, Y)$ denote the quasi-Fuchsian *Bers simultaneous uniformization* of the pair of surfaces $(X, Y) \in \text{Teich}(S) \times \text{Teich}(\overline{S})$ (where $\overline{S}$ is $S$ with the reverse orientation). Such uniformizations sit in the closed subset $AH(S)$ of the *representation variety*

$$\mathcal{V}(S) = \text{Hom}(\pi_1(S), \text{PSL}_2(\mathbb{C}))/\text{conjugation}$$

consisting of representations that are discrete and faithful.

The map $Q: \text{Teich}(S) \times \text{Teich}(\overline{S}) \to AH(S)$ is a homeomorphism onto its image, the quasi-Fuchsian space $QF(S) \subset AH(S)$. Fixing $Y$ in the second factor gives the *Bers slice* $BY \cong \text{Teich}(S)$ of $QF(S)$. Bers proved $BY$ has compact closure in $AH(S)$, giving rise to a *Bers compactification* $\overline{BY}$ and a *Bers boundary* $\partial BY$.

The measured laminations $\mathcal{ML}(S)$ on $S$ are a natural completion of the isotopy classes of essential simple closed curves on $S$ with positive real weights. Projectivizing, one obtains a sphere $P\mathcal{L}(S) = \mathcal{ML}(S) - \{0\}/\mathbb{R}_+$ of *projective measured laminations* with which Thurston compactifies $\text{Teich}(S)$. On any hyperbolic surface $X$, each measured lamination $\mu$ determines a *geodesic lamination*, a closed subset of $X$ foliated by geodesics, as its *support* $|\mu|$.

Representations $\rho \in AH(S)$ are in bijection with marked hyperbolic 3-manifolds $(f: S \to M)$ up to homotopy, where $M = \mathbb{H}^3/\rho(\pi_1(S))$ and $f_* = \rho$. Thurston associates an *end-invariant* $\mathcal{E}(M)$ to each $M \in \partial BY$, namely, the geodesic lamination consisting of all non-peripheral parabolics and laminations on which any measure has ‘length-zero’ in $M$ (see §2). Since any such geodesic lamination is *measurable* (it arises in the quotient of Thurston’s boundary by forgetting the measure), $\mathcal{E}$ gives a mapping

$$\mathcal{E}: \partial BY \to P\mathcal{L}(S)/|.|.$$  

The lamination $\mathcal{E}(M)$ is an invariant of the marked quasi-isometry class $[M]$ of $M$. Letting $\partial BY/\sim$ denote the quotient of $\partial BY$ by marking preserving quasi-isometry, $\mathcal{E}$ descends to a mapping $\mathcal{E}: \partial BY/\sim \to P\mathcal{L}(S)/|.|$ which we also denote by $\mathcal{E}$.

Our first theorem is the following.

**Theorem 1.1.** The mapping $\mathcal{E}$ is strictly lower-semi-continuous in the quotient topologies on domain and range.
Here, lower-semi-continuity means:

for \([M_n] \to [M]\) any limit \(\mathcal{E}_\infty\) of \(\{\mathcal{E}([M_n])\}\) satisfies \(\mathcal{E}_\infty \subset \mathcal{E}([M])\). Strict lower-semi-continuity means there exists \(M_n \to M\) for which the final containment is proper (see theorem 4.1).

Note that maximal families of pairwise disjoint, essential simple closed curves are dense in \(\mathcal{PL}(S)/\|\cdot\|\). These are the images under \(\mathcal{E}\) of maximal cusps: 3-manifolds \(M \in \partial B_Y\) for which the curves in such a maximal family are parabolic. The invariant \(\mathcal{E}(M)\) determines the maximal cusp \(M\) up to isometry. The question of the continuity properties of \(\mathcal{E}\) is then motivated by

**Theorem 1.2** (McMullen). \textit{Maximal cusps are dense in \(\partial B_Y\).}

Theorem 1.1 contrasts the behavior of maximal families as measures and as parabolics in the passage to limits.

Before recovering continuity, we give a characterization of the laminations that can arise in the image of \(\mathcal{E}\). A measurable lamination \(\nu \in \mathcal{PL}(S)/\|\cdot\|\) fills a compact surface \(S\) if for any essential simple closed curve \(\alpha\) on \(S\) that is not parallel to \(\partial S\), \(\alpha\) intersects \(\nu\). Decompose \(\nu\) into the union \(\nu = P \sqcup E\) of its simple closed curve components \(P\) and its infinite \textit{minimal} components \(E\) for which every leaf is infinite and dense in its component. We say \(\nu\) \textit{relatively fills} \(S\) if any component \(\nu'\) of \(E\) fills the subsurface of \(S - P\) that it meets. Let \(\mathcal{EL}(S)\) be the quotient of the quotient \(\mathcal{PL}(S)/\|\cdot\|\) obtained assigning to \(\nu \in \mathcal{PL}(S)/\|\cdot\|\) the lamination \(\tilde{\nu} \in \mathcal{PL}(S)/\|\cdot\|\) given by adding to \(\nu\) the minimal set of simple closed curves required to obtain a lamination that relatively fills \(S\).

Compactness theorems for Thurston’s \textit{pleated surfaces} show that \(\mathcal{E}\) takes values in \(\mathcal{EL}(S)\) (§3). Given \(\nu \in \mathcal{EL}(S)\), we may use theorem 1.1 to find an \(M \in \partial B_Y\) for which \(\mathcal{E}(M) = \nu\): pinching \(P\) and families of simple closed curves approximating \(E\) to cusps, we extract a limit \(M\) with \(\mathcal{E}(M) = \nu\). This gives a new proof\(^1\) of:

**Theorem 1.3.** \textit{The mapping} \(\mathcal{E}\) \textit{is a surjection onto} \(\mathcal{EL}(S)\).

We introduce a new topology on \(\mathcal{EL}(S)\): the \textit{end-invariant topology} is the topology of convergence for which

\(\nu_n \to \nu\) \textit{if for any subsequence} \(\nu_{n_j}\) \textit{converging to} \(\lambda_H\) \textit{in the Hausdorff topology,} \(\nu\) \textit{contains the maximal measurable sub-lamination} \(\eta\) \textit{of} \(\lambda_H\).

(The end-invariant topology, like the quotient topologies, is non-Hausdorff). Then we obtain the following strengthening of theorem 1.1 (theorem 5.3):

**Theorem 1.4.** \textit{The mapping} \(\mathcal{E}\) \textit{is continuous from the quotient topology on} \(\partial B_Y/\{\mathcal{E}\}\) \textit{to} \(\mathcal{EL}(S)\) \textit{with the end-invariant topology.}

In general, given a convergent sequence \(M_n \to M\) in \(\partial B_Y\), the end-invariants \(\mathcal{E}(M_n)\) need not converge in the Hausdorff topology. Theorem 1.4 forces the measurable sub-laminations of any pair Hausdorff limits of \(\mathcal{E}(M_n)\) into alignment.

The main techniques in this paper are developed in [Br1] where we prove a bi-continuity theorem for the \textit{lengths} of measured laminations realized by pleated surfaces in hyperbolic 3-manifolds. The end invariant \(\mathcal{E}(M)\) is the zero-set of this length function when \(M\) is fixed.

These questions relate to the following

\(^1\)K. Ohshika gave a proof of surjectivity of \(\mathcal{E}\) in [Ohs1] but his proof assumed a special case of the main result of [Br1]. This special case was claimed by Thurston but had not appeared.
Conjecture 1.5 (Thurston). The map \( \mathcal{E} : \partial B_Y / \mathcal{q}_i \to \mathcal{E}\mathcal{L}(S) \) is a bijection.

One may speculate as to whether \( \mathcal{E} \) gives a homeomorphism in any reasonable topology on \( \mathcal{E}\mathcal{L}(S) \). Theorems 1.2 and 1.4 show \( \mathcal{E} \) cannot have a continuous inverse in the end-invariant topology (\S 7).

Convergence in a Bers compactification. The possibility of pinching in the conformal boundary of \( M \) means the end-invariant topology must allow for the constant sequence to enlarge in the limit. We record this extra information by considering maximal families of disjoint simple closed curves on \( \partial M - Y \) whose lengths in \( M \) and on \( Y \) are in small ratio. Indeed, given \( M_n \to M \) in the Bers compactification \( \mathcal{B}_Y \) there is a family \( \Pi(M_n) \) of such curves so that \( \mathcal{E}(M_n) \sqcup \Pi(M_n) \) is a geodesic lamination and

\[
\lim_{n \to \infty} \max_{\gamma \in \Pi(M_n)} \frac{\text{length}_{M_n}(\gamma)}{\text{length}_Y(\gamma)} = 0.
\]

Then we prove the following (see corollary 6.3):

**Theorem 1.6.** The laminations \( \mathcal{E}(M_n) \sqcup \Pi(M_n) \) converge to \( \mathcal{E}(M) \) in the end-invariant topology.

In the case when each \( \mathcal{E}(M_n) \) is maximal (a maximal partition, say) it is reasonable to ask whether given the maximal measurable sub-lamination \( \eta \) of the Hausdorff limit \( \lambda_H \) of \( \mathcal{E}(M_n) \), the lamination \( \hat{\eta} \) is the full end-invariant \( \mathcal{E}(M) \). Though the answer is yes in many cases, we conclude this paper with a negative answer to this question in general (see theorem 7.1):

**Theorem 1.7.** Implicit Cusps Let \( \gamma \) be an essential simple closed curve in \( S \). Then for any other essential simple closed curve \( \alpha \) in \( S - \gamma \), there are maximal partitions \( C_n \to \lambda_H \) in the Hausdorff topology and associated maximal cusps \( C_n \to M \) in \( \partial B_Y \) for which:

1. \( \gamma \) is the maximal measurable sub-lamination of \( \lambda_H \), and
2. \( \alpha \) lies in \( \mathcal{E}(M) \).

The curve \( \alpha \) is an “implicit cusp” forced by 3-dimensional hyperbolic geometry that, somewhat surprisingly, goes undetected by the Hausdorff topology. The example producing theorem 1.7 reveals a new geometric phenomenon that complicates the relationship between hyperbolic surfaces and the 3-manifolds they parameterize.

**History and references.** The density of maximal cusps in Bers’ boundary is proven by McMullen in [Mc2]. Whether or not appropriate quotients of Bers’ and Thurston’s boundaries are homeomorphic is asked by McMullen in [Mc3]. For informative discussions of the end-invariant see [Mc4] and [Min2].

In general, we allow \( S \) to be compact with nonempty boundary. Indeed, when \( \dim_C(\text{Teich}(S)) = 1 \), Y. Minsky has shown (see [Min3]) that that \( \mathcal{E} \) is a homeomorphism from \( \partial B_Y \) to \( \mathcal{P}\mathcal{L}(S) \) (passing to quotients is redundant as the support \( |\mu| \) of any measured lamination \( \mu \in \mathcal{M}\mathcal{L}(S) \) admits a unique transverse measure up to scale, and Minsky proves that \( \mathcal{E}(M) \) determines \( M \) up to isometry). Note that in this setting \( \mathcal{E}(M) \) is always connected, while when \( \dim_C(\text{Teich}(S)) > 1 \), the invariant \( \mathcal{E}(M) \) can be disconnected.

Thurston introduces pleated surfaces and lengths of laminations in [Th1], [Th2], and [Th4]. Various versions of Thurston’s length function are discussed.
in [Th4], [Bon3] and [Ohs2]; we prove a general bi-continuity theorem (see theorem 2.3) in [Br1] where the key lemmas on nearly-straight train tracks employed in the proof of theorem 1.4 ([Br1, Lem. 5.2, Cor. 5.3]) also appear.

We have chosen to work in the Bers slice to avoid certain technicalities that arise in more general deformation spaces of hyperbolic 3-manifolds. We remark that work of J. Anderson and R. Canary [AC] reveals a different type of possible discontinuity in the analogous end-invariant mapping for general deformation spaces (see [Min3, §12]). We plan to merge these two perspectives in a sequel.

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2. Preliminaries

Let $S$ be an oriented compact topological surface of negative Euler characteristic. We allow $S$ to have non-empty boundary; let int($S$) = $S - \partial S$ denote its interior.

**Teich($S$).** The *Teichmüller space* $\text{Teich}(S)$ is the space of finite-area hyperbolic surfaces $X$ equipped with homeomorphisms $f : \text{int} (S) \to X$ such that

$$(f : \text{int} (S) \to X) \sim (g : \text{int} (S) \to Y)$$

if there is an isometry $\phi : X \to Y$ so that $\phi \circ f \simeq g$.

The topology on $\text{Teich}(S)$ is induced by the natural distance $d(X,Y)$ obtained by taking the infimum $K$ over all $k$ for which there is a $k$-bi-Lipschitz diffeomorphism $\phi$ homotopic to $g \circ f^{-1}$ and setting $d(X,Y) = \log(K)$. The Teichmüller space is homeomorphic to an open ball and carries a natural complex structure of dimension $\dim\mathbb{C}(\text{Teich}(S)) = 3g - 3 + n$, where $S$ has genus $g$ with $n$ boundary components.

**AH($S$).** Let $\mathcal{D}(S)$ denote the space of discrete faithful representations $\rho : \pi_1 (S) \to \text{Isom}^+(\mathbb{H}^3)$ so that $\rho(\gamma)$ is parabolic for each peripheral element $\gamma \in \pi_1(S)$ (i.e. $\gamma$ is boundary-parallel), with the compact-open topology, or the topology of algebraic convergence. Let

$$\text{AH}(S) = \mathcal{D}(S) / \text{Isom}^+(\mathbb{H}^3)$$

be its quotient by conjugation.

By a theorem of Thurston and Bonahon [Th1, Ch. 9] [Bon1] $M = \mathbb{H}^3 / \rho(\pi_1(S))$ is a complete hyperbolic manifold homeomorphic to int($S$) $\times \mathbb{R}$. The complete hyperbolic manifold $M$ is prolonged to its *Kleinian manifold* $\overline{M}$ by adding its conformal boundary $\partial M$: namely, the quotient of the domain $\Omega(M) \subset \mathbb{C}$ where $\rho(\pi_1(S))$ acts properly discontinuously.

The set of hyperbolic 3-manifolds $M$ marked by homotopy equivalences $(f : S \to M)$ up to marking-preserving isometry is in bijection with conjugacy classes of representations $\rho \in \text{AH}(S)$ via the association $f \mapsto f_*$. Thus we will often speak of $\text{AH}(S)$ as a space of marked hyperbolic manifolds and write $M \in \text{AH}(S)$, assuming an implicit marking homotopy equivalence $(f : S \to M)$.

One may formulate algebraic convergence in this context: $\{(f_n : S \to M_n)\}$ converges to $(f : S \to M)$ if for any compact set $K \subset M$ there are smooth, marking-preserving homotopy equivalences $q_n : M \to M_n$ that converge to a local isometry.
on \( K \) in the \( C^\infty \) topology (see [Mc5, §3.1]; we refer the reader to [Mc5], [Th1], or [Br2] for details about hyperbolic 3-manifolds and Kleinian groups).

\( QF(S) \). By a theorem of Bers [Bers1] there is unique quasi-Fuchsian manifold \( Q(X,Y) \in AH(S) \) interpolating between any pair of hyperbolic surfaces \((X,Y) \in \text{Teich}(S) \times \text{Teich}(\overline{S})\) in its conformal boundary. Given \( Y \in \text{Teich}(S) \), the Bers slice 

\[
B_Y = \{Q(X,Y) : X \in \text{Teich}(S)\}
\]

is an embedded copy of \( \text{Teich}(S) \) in \( AH(S) \). The embedding depends on \( Y \), but for any \( Y \) the slice \( B_Y \) is precompact in \( AH(S) \). One obtains a Bers compactification \( \overline{B_Y} \) by forming the closure, and an associated Bers boundary for Teichmüller space as its boundary \( \partial B_Y \) (see also [KT], [Mc5], or [Bers2]).

\( ML(S) \). Let \( S \) be the set of isotopy classes of essential non-peripheral simple closed curves on \( S \). The geometric intersection number 

\[
i : S \times S \to \mathbb{Z}_{\geq 0}
\]

counts the minimal number \( i(\alpha, \beta) \) of intersections of curves in distinct isotopy classes \((\alpha, \beta) \in S \times S\) and takes the value zero on the diagonal.

Attaching a positive real weight to each isotopy class, let 

\[
i : \mathbb{R}_+ \times S \to \mathbb{R}^\mathbb{R}
\]

be defined by 

\[
(i_\gamma)_\alpha = ti(\alpha, \gamma).
\]

Then we define the measured laminations \( ML(S) = i(\mathbb{R}_+ \times S) \) by taking the closure of the image (note that weighted simple closed curves are naturally dense in \( ML(S) \)). The intersection number extends to a symmetric continuous function 

\[
i : ML(S) \times ML(S) \to \mathbb{R}_{\geq 0} \text{ so that } i(s \alpha, t \beta) = s \cdot t(i(\alpha, \beta)) \text{ for } \alpha, \beta \in S \text{ and } s, t \in \mathbb{R}_{\geq 0} \text{ [Bon1, Prop. 4.5].}
\]

The measured lamination space \( ML(S) \) is a cell of the same real dimension as \( \text{Teich}(S) \). The projective measured laminations \( P\mathcal{L}(S) = ML(S) - \{0\}/\mathbb{R}_+ \) form a sphere of one dimension lower. The sphere \( P\mathcal{L}(S) \) is Thurston’s boundary for Teichmüller space - the topology on Thurston’s compactification \( \text{Teich}(S) \sqcup P\mathcal{L}(S) \) is determined by the conditions that \( \text{Teich}(S) \) is open in \( \text{Teich}(S) \sqcup P\mathcal{L}(S) \) and \( X_n \to [\mu] \in P\mathcal{L}(S) \) if and only if 

\[
\frac{\text{length}_{X_n}(\alpha)}{\text{length}_{X_n}(\beta)} \to \frac{i(\mu, \alpha)}{i(\mu, \beta)}
\]

for any pair \( \alpha \) and \( \beta \) in \( S \) for which \( i(\mu, \beta) \neq 0 \). (For more on measured and projective laminations, and Thurston’s compactification see [FLP], [Th1], or [Bon2]).

**Subsurfaces.** A subsurface is a compact 2-submanifold of \( S \). An essential subsurface \( T \subset S \) is a subsurface so that each curve in \( \partial T \) is homotopically essential. Given an essential subsurface \( T \subset S \), let \( \mathcal{S}(T) \subset \mathcal{S} \) be isotopy classes of simple closed curves in \( \mathcal{S} \) isotopic into \( T \) that are non-peripheral in \( T \). Then \( ML(T) \) is naturally a closed subspace of \( ML(S) \).

\( GL(S) \). Given \( X \in \text{Teich}(S) \), a geodesic lamination \( \lambda \) on \( X \) is a closed subset of \( X \) that admits a decomposition into complete simple geodesics called leaves of \( \lambda \). The set of geodesic laminations \( GL(X) \) on \( X \) is a compact subspace of the space of closed subsets \( \text{Cl}(X) \) in the Hausdorff topology.
Via a natural circle at infinity for $S$, geodesic laminations are canonically associated to the surface $S$ and can be realized geodesically on any $X \in \text{Teich}(S)$ via its implicit marking (see [Bon2], [Fl], or [CEG, §4.1]). Thus we will speak of a point $\lambda \in \mathcal{GL}(S)$, which determines a geodesic lamination on any particular hyperbolic surface $X \in \text{Teich}(S)$. Given $\lambda \in \mathcal{GL}(S)$, let $S(\lambda) \subset S$ be the essential subsurface obtained by realizing $\lambda$ on $(f : S \to X) \in \text{Teich}(S)$ and pulling back by $f^{-1}$ the smallest subsurface with geodesic boundary containing $\lambda$.

A measured lamination $\mu \in \mathcal{ML}(S)$ determines a transverse measure on a geodesic lamination $|\mu|$. The geodesic lamination $|\mu|$ is called the support of $\mu$. A geodesic lamination $\nu$ is measurable if there is some $\mu \in \mathcal{ML}(S)$ for which $\nu = |\mu|$; $\nu$ admits a transverse measure of full support.

Given $\lambda, \nu \in \mathcal{GL}(S)$, the notation $\lambda \subset \nu$ will mean that $\lambda$ is a sub-lamination of $\nu$, while the notation $\lambda \cap \nu$ will refer to any common sublamination of $\lambda$ and $\nu$ together with the set of transverse intersections of leaves of $\lambda$ and $\nu$, well defined on any hyperbolic surface $X \in \text{Teich}(S)$.

**Pleated surfaces.** Let $(f : S \to M) \in AH(S)$ and let $\lambda \in \mathcal{GL}(S)$ be a geodesic lamination. We say $\lambda$ is realizable in $M$ if there is a hyperbolic surface $X \in \text{Teich}(S)$, and a path-isometry $^2g : X \to M$, compatible with markings on $X$ and $M$, so that $g|_\lambda$ is a local isometry. If $g$ is totally geodesic on the complement of some geodesic lamination $\lambda'$ containing $\lambda$, the triple $(g, X, M)$ is called a pleated surface in $M$, and we say the pleated surface realizes $\lambda$. A measured lamination $\mu \in \mathcal{ML}(S)$ is realizable in $M$ if its support $|\mu|$ is realizable. Any realizable lamination can be realized by a pleated surface.

Let $\mathcal{PS}(f)$ denote the set of all pairs $(g, X)$, where $(\phi : S \to X) \in \text{Teich}(S)$, and $g : X \to M$ is a pleated surface with $f \simeq g \circ \phi$. Let $\mathcal{PS}_{np}(f) \subset \mathcal{PS}(f)$ be the subset for which $f_* (\gamma)$ is parabolic only if $\gamma$ is a peripheral element of $\pi_1(S)$.

We topologize $\mathcal{PS}(f)$ by the Teichmüller distance on the underlying surfaces and the topology of uniform convergence on compact sets on the pleated mappings. In other words, $(g_n, X_n) \to (g, X)$ if there are marking-preserving bi-Lipschitz diffeomorphisms $q_n : X \to X_n$ with bi-Lipschitz constant tending to 1 so that the composition $g_n \circ q_n$ converges uniformly on compact subsets to $g$. Then we have the following compactness result due to Thurston (see [CEG, 5.2.18]):

**Theorem 2.1** (Thurston). **Pleated Surfaces Compact** Let $(f : S \to M) \in AH(S)$, and let $K \subset M$ be a compact subset. Then the set of all $(g, X) \in \mathcal{PS}_{np}(f)$ with the property that $g(X) \cap K \neq \emptyset$ is compact.

Also relevant is the following theorem which we restate in a form useful to us.

**Theorem 2.2** (Thurston). **Limits Realized** Let $\{(g_n, X_n)\} \subset \mathcal{PS}_{np}(f)$ converge to $(g, X)$ and let $(g_n, X_n)$ realize convergent measured laminations $\mu_n \to \mu$. Then $(g, X)$ realizes $\mu$.

(The theorem is a direct consequence of [CEG, 5.3.2]).

**Lengths of laminations.** Given $X \in \text{Teich}(S)$, any isotopy class $\gamma \in S$ has a well defined length by taking the arclength $\ell_X(\gamma^*)$ of its geodesic representative $\gamma^*$. By a theorem of Thurston and Bonahon (see [Th4] [Bon1, Prop. 4.5]) there is a unique continuous function

$$\text{length} : \text{Teich}(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

[2]The map $g$ sends geodesic arcs in $X$ to rectifiable arcs in $M$ of the same length.
that restricts to $\mathbb{R}^+ \times \mathcal{S}$ by

$$\text{length}_X(t\gamma) = t\ell_X(\gamma^*).$$

Let $\mathcal{R} \subset \mathcal{AH}(S) \times \mathcal{ML}(S)$ denote the set of pairs $(M, \mu)$ such that $\mu$ is realizable in $M$. We define the length function

$$\text{length}: \mathcal{R} \to \mathbb{R}$$

by setting $\text{length}_M(\mu) = \text{length}_X(\mu)$ where $g: X \to M$ is any pleated surface realizing $|\mu|$ (the length in $M$ does not depend on the realizing pleated surface; see [Th4] [Bon4]).

When $\mu$ is not realizable in $M$, proper sub-laminations may still be realizable. Define the projection map $R_M: \mathcal{ML}(S) \to \mathcal{ML}(S)$ to be the identity on laminations realizable in $M$ and to associate to any non-realizable lamination $\mu$ the maximal sub-lamination $R_M(\mu)$ of $\mu$ that is realizable in $M$.

Then we have the following from [Br1]:

**Theorem 2.3. Length Continuous** The function

$$\text{length}: \mathcal{AH}(S) \times \mathcal{ML}(S) \to \mathbb{R}$$

given by $(M, \mu) \mapsto \text{length}_M(R_M(\mu))$ is continuous.

In particular, we have the following corollary:

**Corollary 2.4.** Let pairs $\{(M_n, \mu_n)\}$ converge to $(M, \mu)$ in $\mathcal{AH}(S) \times \mathcal{ML}(S)$ so that $\text{length}_{M_n}(\mu_n) \to 0$. Then $R_M(\mu) = 0$.

In other words, if $\mu$ lies in $\mathcal{ML}(S)_+$, the non-zero elements of $\mathcal{ML}(S)$, and $\text{length}_M(\mu) = 0$, then each component of $\mu$ is non-realizable in $M$.

**The end invariant $\mathcal{E}(M)$.** We make the following definition.

**Definition 2.5.** Let $M \in \partial B$ be a point in a Bers’ boundary. Then its end invariant $\mathcal{E}(M)$ is the union of all connected geodesic laminations $\lambda$ such that for some $\mu \in \mathcal{ML}(S)_+$ we have,

$$\lambda = |\mu| \quad \text{and} \quad \text{length}_M(\mu) = 0.$$ 

By a theorem of Thurston and Bonahon (the geometric tameness of $M$ [Th1], [Bon1]), $\mathcal{E}(M)$ lies in $\mathcal{PL}(S)/|.|$; i.e. $\mathcal{E}(M)$ is itself a measurable geodesic lamination.

**Notation:** Throughout, the notation $n \gg 0$ will mean ‘all $n$ sufficiently large.’ Unless otherwise stated, constants will depend only on $S$.

### 3. Surjectivity onto measurable laminations that relatively fill

In this section, we reprise implications of compactness of pleated surfaces on the basic structure of $\mathcal{E}(M)$ (this theory is developed in [Th1, Ch. 9]) and go on to give a characterization of laminations that arise in the image of $\mathcal{E}$.

**Decomposing laminations.** A partition $P$ of $S$ is a collection $P \subset \mathcal{S}$ of distinct isotopy classes of pairwise-disjoint, essential, non-peripheral, simple closed curves on $S$. A maximal partition is a partition that cannot be enlarged. The partition $P$
determines a collection of essential subsurfaces in its complement as the complement of pairwise embedded open annular neighborhoods of each curve in $P$. Let $S - P$ denote their union, abusing notation.

Each measurable lamination $\nu$ (i.e. $\nu \in \mathcal{P} \mathcal{L}(S)/\|\|$) admits a decomposition
\[
\nu = P(\nu) \sqcup E(\nu)
\]
where $P(\nu) \subset S$ is a partition, and each component of $E(\nu)$ is infinite and minimal: each leaf of $E(\nu)$ is bi-infinite and dense in its component. A general geodesic lamination $\lambda$ decomposes into its maximal measurable sub-lamination $\nu \subset \lambda$ and a finite collection of bi-infinite leaves each end of which is either asymptotic to $\nu$ or to a puncture of $S$ (see [Otal, §A]).

The measurable lamination $\nu$ fills $S$ if for each $\alpha \in \mathcal{C}B$ and any measure $\mu \in \mathcal{ML}(S)$ with $|\mu| = \nu$ we have either $i(\mu, \alpha) > 0$ or $\alpha$ is peripheral in $S$.

Generalizing, we make the following definition.

**Definition 3.1.** The measurable lamination $\nu$ relatively fills $S$ if for each component $\nu' \subset E(\nu)$, $\nu'$ fills the subsurface component of $S - P(\nu)$ in which it lies.

We define $\mathcal{E} \mathcal{L}(S) \subset \mathcal{P} \mathcal{L}(S)/\|\|$ to be the subset of laminations that relatively fill $S$. Each measurable $\nu$ has an implicit partition $\hat{P}(\nu)$: this is the minimal partition containing $P(\nu)$ so that $E(\nu) \sqcup \hat{P}(\nu)$ is a lamination that relatively fills $S$. There is a natural projection
\[
\mathcal{P} \mathcal{L}(S)/\|\| \to \mathcal{E} \mathcal{L}(S)
\]
given by $\nu \mapsto E(\nu) \sqcup \hat{P}(\nu)$; let $\tilde{\nu} = E(\nu) \sqcup \hat{P}(\nu)$ (see figure 1).

![Figure 1. Adding the implicit partition $\hat{P}(\nu)$](image)

In this section we prove the following:

**Theorem 3.2.** The map $\mathcal{E}$ is a surjection onto $\mathcal{E} \mathcal{L}(S)$.

We first prove $\mathcal{E}$ is well-defined as a map to $\mathcal{E} \mathcal{L}(S)$.

**Lemma 3.3.** For any $M \in \partial B_Y$, the end-invariant $\mathcal{E}(M)$ relatively fills $S$.

**Proof:** Let $(f: S \to M)$ be the implicit marking for $M$, and let $\mathcal{E}(M) = P \sqcup E$ be the decomposition of $\mathcal{E}(M)$ into its sets of parabolics $P$ and infinite minimal components $E$. If $\mathcal{E}(M)$ does not relatively fill $S$, then for some connected sub-lamination $\nu \subset E$ lying in a connected component $T$ of $S - P$, there is a simple closed curve $\gamma \in \mathcal{S}(T)$ in the implicit partition for $\nu$ that is non-peripheral in $T$. It follows that $\gamma$ is not parabolic in $M$ and is therefore realizable (see [Th1, §9.7], [CEG, Thm. 5.3.11]).
Let \( t_n c_n \to \mu \), be a sequence of weighted simple closed curves converging to a measured lamination \( \mu \) with support \( \nu = |\mu| \) so that \( \iota(\gamma, c_n) = 0 \). There is a sequence of pleated surfaces \( (g_n, X_n) \in \mathcal{PS}_{np}(f|_T) \) realizing \( \gamma \cup c_n \). Since \( (g_n, X_n) \) all realize \( \gamma \), a subsequence converges to \( (g, X) \in \mathcal{PS}_{np}(f|_T) \) by theorem 2.1. By theorem 2.2, the limit realizes \( \nu \), a contradiction. Thus \( \gamma \) either intersects \( \nu \) or lies in \( P \), so \( \nu \) relatively fills \( S \).

(A similar argument appears in [Br2, Thm. 4.7]).

**Proof:** (of theorem 3.2). Let \( \nu \in E\mathcal{L}(S) \). Then there is a measured lamination \( \mu \in \mathcal{ML}(S) \) so that \( |\mu| = \nu \). Let \( \Pi = P(\nu) \), let \( E(\nu) = \nu_1 \cup \ldots \cup \nu_k \), and let

\[
S - \Pi = S_1 \cup \ldots \cup S_k \cup T_1 \cup \ldots \cup T_s
\]

denote the collection of subsurfaces of \( S \) determined up to isotopy as the complement of small pairwise embedded open annular neighborhoods of the curves in \( \Pi \), so that \( \nu_j \) lies in \( G\mathcal{L}(S_j) \), \( j = 1, \ldots, k \). Let \( \mu_j \subset \mu \) denote the measured sub-lamination so that \( |\mu_j| = \nu_j \).

For each \( j \), let \( \{c_{j,n}\} \subset S \) be simple closed curves in \( S(S_j) \) so that for positive real weights \( t_{j,n} \) we have \( t_{j,n} c_{j,n} \to \mu_j \) as \( j \to \infty \). Letting \( \mu_{\Pi} \subset \mu \) be the measure determined by \( \mu \) on \( \Pi \) (i.e. \( |\mu_{\Pi}| = \Pi \)), the unions

\[
\xi_n = \mu_{\Pi} \bigcup \left( t_{j,n} c_{j,n} \right)
\]

are measured laminations so that \( \xi_n \to \mu \) in \( \mathcal{ML}(S) \).

A maximal partition \( P \) of \( S \) determines Fenchel-Nielsen length and twist coordinates

\[
(\text{length}_\gamma(X), \text{twist}_\gamma(X)) \in \mathbb{R}_+^P \times \mathbb{R}^P
\]

for \( X \in \text{Teich}(S) \), where \( \gamma \in P \) (see e.g. [IT]). Given a subset \( P \subset \mathcal{P} \), the pinching deformation along \( P \) is the family of Riemann surfaces \( X_t \in \text{Teich}(S) \), \( t \to 0 \), determined by setting the coordinates

\[
\text{length}_\gamma(X_t) = t \text{length}_\gamma(X)
\]

for each \( \gamma \in P \) and leaving all other coordinates unchanged. Then the pinching deformation along \( P \) determines a path \( Q(X_t, Y) \) in \( By \) that converges to a limit \( M \in \partial By \) with \( E(M) = P \) (see [Ab], [Mc6, Thm. 9.5]).

Let \( M_n \in \partial By \) be obtained from the quasi-Fuchsian manifold \( Q(X, Y) \) by performing the pinching deformation along the collection

\[
P_n = |\xi_n| = \Pi \bigcup \left( c_{j,n} \right)
\]

on \( X \). For given \( r \), and for each \( M_n \) let \( W_n \in \text{Teich}(T_r) \) denote the corresponding conformal boundary component of \( M_n \). With respect to a fixed maximal partition \( \mathcal{P}_T \) of \( \cup_r T_r \), the Fenchel-Nielsen coordinates for \( W_n \) are the limiting Fenchel-Nielsen coordinates for \( X_t \) along \( \mathcal{P}_T \cap T_r \). Hence, they do not depend on \( n \) and \( W_n \) is constant; we set \( W_n = W \).

We have

\[
\text{length}_{M_n}(\xi_n) = 0
\]

for all \( n \). By continuity of \( \text{length} \) ([Br1, Thm. 7.1], we have

\[
\text{length}_{M}(\mu) = 0.
\]
Since $μ ∈ M_L(S)_+$, it follows that each component of $μ$ is non-realizable in $M$. Thus $ν = |μ|$ is a sub-lamination of $E(M)$.

Let $f: S → M$ denote the implicit marking on $M$, and let $π_1(T_r)$ denote the subgroup of $π_1(S)$ induced by inclusion $T_r ⊂ S$ after choosing a basepoint in $T_r$. Since $E(M)$ relatively fills $S$ by lemma 3.3, to see that $ν = E(M)$ it suffices to show that the cover $M(r)$ of $M$ corresponding to $f_∗(π_1(T_r))$ is quasi-Fuchsian (every lamination is realizable in a quasi-Fuchsian manifold, see [Th1, Prop. 8.7.7] [CEG, Thm. 5.3.11]).

Let $f_n: S → M_n$ denote the implicit markings on $M_n$. For fixed $r$, the cover of $M_n$ corresponding to $(f_n)_∗(π_1(T_r))$ is a quasi-Fuchsian manifold $Q(W, Z_n) ∈ QF(T_r)$. The cover $Y_r$ of $Y$ corresponding to $π_1(T_r)$ (which is no longer of finite type) admits a holomorphic inclusion into $Z_n$, which is a contraction of the Poincaré metric by the Schwarz lemma. Thus, there is a pair of simple closed curves $α$ and $β$ in $S(T_r)$ that bind $T_r$ (i.e. $i(α, γ) + i(β, γ) > 0$ for any $γ ∈ S(T_r)$) and have uniformly bounded length in $Z_n$. Such a bound guarantees that $Z_n$ range in a compact subset of $Teich(T_r)$ (see e.g. [Th4, Prop. 2.4] [Ker]) so $Q(W, Z_n)$ converges to a quasi-Fuchsian manifold $Q(W, Z_∞)$. Thus $M(r)$ is quasi-Fuchsian, since it is the limit of $Q(W, Z_n)$.

It follows that $ν = E(M)$, and the theorem is proven.

\[\text{END-INVARIANTS FOR HYPERBOLIC 3-MANIFOLDS 11}\]

4. Lower-semi-continuity

From now on, we view $E$ as a map from quasi-isometry classes $[M] ∈ ∂B_Y/qi$ to the quotient $E_L(S)$ of $PL(S)$ under the projection $[μ] → [\hat{μ}]$. In this section we investigate the behavior of $E$ in the quotient topologies on domain and range.

**Theorem 4.1.** Let $\dim_C(\text{Teich}(S)) > 1$. Then the mapping $E$ is strictly lower-semi-continuous in the quotient topologies.

Again, ‘lower-semi-continuity’ has the interpretation:

\[\text{(4.1) Given } [M_n] → [M] \text{ any limit } E_∞ \text{ of } \{E([M_n])\} \text{ satisfies } E_∞ ⊂ E([M]),\]

and strict lower-semi-continuity means there exists $M_n → M$ for which the final containment is proper. As remarked, when $\dim_C(\text{Teich}(S)) = 1$, $E$ is a homeomorphism [Min3].

**Proof:** We first find a point of discontinuity for $E$ (to prove strict lower-semi-continuity). Since $\dim_C(\text{Teich}(S)) > 1$ we can find a pair of distinct isotopy classes $γ$ and $δ$ in $S$ with $i(γ, δ) = 0$. Let $P ⊂ S$ be a maximal partition containing $δ$ and $γ$. Adjust the Fenchel-Nielsen coordinates of $X ∈ \text{Teich}(S)$ along $P$ so that $X_{m,n} ∈ \text{Teich}(S)$ has Fenchel-Nielsen coordinates

\[
\text{length}_b(X_{m,n}) = 1/m \quad \text{and} \quad \text{length}_i(X_{m,n}) = 1/n
\]

and all other coordinates equal to those of $X$. Then, as above, the sequence $\{Q(X_{m,n}, Y)\}_{m=1}^∞$ converges to a limit $M_n$ for which $E(M_n) = γ$. Likewise, the sequence $\{M_n\}_{m=1}^∞$ converges to a limit $M$ such that $E(M) = δ ∪ γ$.

Just as a weakly convergent sequence of measures with constant support cannot converge to a measure with larger support, there is no sequence of transverse measures (weights) on the simple closed curve $γ$ that converges in $M_L(S)$ to a
transverse measure on $\gamma \sqcup \delta$. Hence the quasi-isometry class of $M$ is a point of discontinuity of $E$ as a map to $EL(S)$ with the quotient topology.

To see that the map $E$ is lower-semi-continuous in the sense of line 4.1, note that for any convergent sequence $M_n \to M$ in $\partial BY$, and any convergent sequence of measured laminations $\mu_n \to \mu$ with $|\mu_n| = E(M_n)$, we have

$$\text{length}_{M_n} (\mu_n) = 0$$

for each $n$. Continuity of length implies that $\text{length}_M (\mu) = 0$, and we conclude

$$|\mu| \subset E(M).$$

Spinning maximal cusps. We briefly give another example of discontinuity of $E$ in the quotient topologies. We do this to motivate a new topology on the range, which we introduce in the next section.

Let $C \subset S$ be a maximal partition. Then the maximal cusp $M(C) \in \partial BY$ is the unique point for which $\alpha$ is parabolic for each $\alpha \in C$. It is determined up to isometry by the collection $C$ (see, e.g. [Bers2], [Mc2]).

As above, assume $\dim_C(\text{Teich}(S)) > 1$, let $C_0$ be a maximal partition for $S$, and let $\gamma \sqcup \delta \subset S$ be isotopy classes of disjoint simple closed curves so that $i(\alpha, \gamma)$ and $i(\alpha, \delta)$ are non-zero for each $\alpha \in C_0$.

Let $\tau_\gamma$ and $\tau_\delta$ be Dehn-twists about $\gamma$ and $\delta$ respectively, and let

$$C_n = \tau_\gamma^{n_2} \circ \tau_\delta^n (C_0),$$

where $n \in \mathbb{N}$. Consider any limit $M$ of the sequence of maximal cusps $\{M(C_n)\}_{n=0}^\infty$.

![Figure 2. Spinning maximal cusps. The Hausdorff limit of $C_n = \tau_\gamma^{n_2} \circ \tau_\delta^n (C_0)$ contains both $\gamma$ and $\delta$ as measurable sub-laminations.](image)

Notice that

1. Any sequence $\mu_n \in M\mathcal{L}(S)$ of measures (weights) on $C_n$ has projective classes $[\mu_n] \in \mathcal{P}\mathcal{L}(S)$ converging to $[1 \cdot \gamma]$. Thus theorem 4.1 guarantees only that $\gamma$ is parabolic in $M$. 

2. One expects that both classes $\gamma$ and $\delta$ are parabolic in $M$.\footnote{This follows, for example, from the techniques of [KT] and [Br2] and a study of the \textit{geometric limit} of $M(C_n)$; we develop a point of view more closely aligned with the present techniques.}

The topology on $\mathcal{PL}(S)$ is insensitive to all but the maximal growth rate of transverse measure. Our goal in the next section will be to formulate a topology on $\mathcal{EL}(S)$ called the \textit{end-invariant topology} that is sensitive to different orders of convergence. Proving continuity of $\mathcal{E}$ in the end-invariant topology, we capture more geometric information about general limits $M$.

5. Continuity in the end-invariant topology

\textbf{Definition 5.1.} The end-invariant topology on $\mathcal{EL}(S)$ is the topology of convergence for which $\nu_n \rightarrow \nu$ if for any Hausdorff limit $\lambda_H$ of any subsequence $\nu_n$, the maximal measurable sub-lamination $\eta \subset \lambda_H$ is a sub-lamination of $\nu$.

Continuity in the end-invariant topology relies on uniform estimates for the shapes of train tracks in 3-manifolds.

\textbf{Definition 5.2.} A train track $\tau$ in a hyperbolic surface $X \in \text{Teich}(S)$ is an embedded 1-complex in $X$ whose edges (branches) are $C^1$ arcs meeting at vertices (switches) so that each switch $v$ has a neighborhood $U \subset X$ for which $\tau \cap U$ is a collection of $C^1$ arcs passing through with a common tangent line at $v$. We require in addition that the double of each component of $X - \tau$ along the interiors of the branches in its boundary has negative Euler characteristic.

A train-path $r$ is a monotone $C^1$ immersion $r: \mathbb{R} \rightarrow X$ ($r$ is “bi-infinite”) or $r: S^1 \rightarrow X$ ($r$ is “closed”) with image in $\tau$. A train track $\tau$ on $X$ carries a geodesic lamination $\lambda$ if there is a $C^1$ map $p: X \rightarrow X$ that is homotopic to the identity and non-singular on the tangent spaces to the leaves of $\lambda$ so that $p$ sends each leaf of $\lambda$ to a train-path for $\tau$. We say $\tau$ \textit{minimally carries} $\lambda$ if for each branch $b$ of $\tau$, there is a train-path corresponding to a leaf of $\lambda$ that traverses $b$.

A train track $\tau^*$ in a marked hyperbolic manifold $(f: S \rightarrow M) \in \mathcal{AH}(S)$ is a train track $\tau$ on a hyperbolic surface $(h: S \rightarrow X) \in \text{Teich}(S)$, together with a marking-preserving smooth map $g: X \rightarrow M$ so that $g(\tau) = \tau^*$. The surface $X$ serves to mark the train track $\tau^*$ with homotopy information: we say $\tau^*$ carries $\lambda$ if $\tau$ does.

To make a train-track $\tau$ carry more laminations, we may \textit{enlarge} $\tau$ by adding branches. For our purposes, we enlarge $\tau$ by adding branches $b$ each endpoint of which either terminates in a switch of $\tau$ or attaches to a simple closed curve component of $\tau$.

Finally, a train track $\tau$ in $X$ (or in $M$) is $\epsilon$-\textit{nearly-straight} if each train path $r$ is $C^2$ with geodesic curvature less than $\epsilon$. An important property of nearly-straight train tracks is the following: for any $\epsilon_0 \in (0, 1)$ there is a “tracking constant” $C_{\text{tr}} > 1$ so that for any $\epsilon \in (0, \epsilon_0)$ if $\tau$ is an $\epsilon$-nearly-straight train track in $X$ (resp. $M$), any train path $r$ lifts to an embedding $\tilde{r}: \mathbb{R} \rightarrow \mathbb{P}^{\mathbb{R}}$ into the projective unit tangent bundle $\mathbb{P}^{\mathbb{R}}$ of $\mathbb{H}^2$ (resp. $\mathbb{P}^{\mathbb{H}^2}$) that is smoothly homotopic to a complete geodesic by an isotopy that moves each point a distance less than $C_{\text{tr}} \epsilon$. Assume $\epsilon_0 = 1/2$ and let $C_{\text{tr}}$ be the corresponding tracking constant.

When a closed train-path on an $\epsilon$-nearly-straight train track is straightened to its geodesic representative, its arc-length does not decrease too much: there is a
continuous contraction bound $K$: $[0, 1) \to [1, \infty)$ with $K(\epsilon) \to 1$ as $\epsilon \to 0$ so that any arc $\alpha \in \mathbb{H}^n$ of geodesic curvature less than $\epsilon$ satisfies
\begin{equation}
\ell(\alpha^*) \geq \frac{1}{K(\epsilon)} \ell(\alpha)
\end{equation}
where $\alpha^*$ is the geodesic representative of $\alpha$ rel-endpoints (see [Br1, §4] or [Min1] for more on nearly-straight train tracks)

We employ these ideas to prove the following:

**Theorem 5.3.** The mapping $\mathcal{E}$ is a continuous surjection from the quotient topology on $\partial B_\nu/\partial \mathcal{L}_\nu$ to $\mathcal{E} \mathcal{L}(S)$ with the end-invariant topology.

**Proof:** We have shown surjectivity in theorem 3.2. It remains to show continuity in the end invariant topology.

Let $\mathcal{M}_n \to \mathcal{M}$ in $\partial \mathcal{B}_\nu$. After passing to a subsequence, let $\mathcal{E}(\mathcal{M}_n) = \mathcal{E}_n$ tend to $\lambda_H$ in the Hausdorff topology. For each $n$, let $P_{j,n} \subset \Delta$ be as constructed in the proof of theorem 3.2 so that $P_{j,n} \to \mathcal{E}_n$ in the Hausdorff topology as $j \to \infty$.

Arguing as in the proof of lemma 3.3, theorem 2.1 implies that given any compact set $K \subset \mathcal{M}_n$, there is a $J$ so that for all $j > J$ no curve in $P_{j,n}$ has a geodesic representative intersecting $K$.

Let $\nu$ be any connected, measurable sub-lamination of $\lambda_H$. Suppose that $\nu$ is realizable in $\mathcal{M}$ by a pleated surface $g: X \to \mathcal{M}$. Let $K \subset \mathcal{M}$ be a compact set containing the radius 1 neighborhood $\mathcal{N}_1(g(\nu))$ of $g(\nu)$, the locally-isometric image of the geodesics in $\nu$ under $g$. By algebraic convergence, there are smooth, marking-preserving homotopy equivalences $q_n: \mathcal{M} \to \mathcal{M}_n$ that tend $C^\infty$ to a local isometry on $K$. It follows that for any $\delta > 0$, each geodesic leaf $l \subset \nu$ has image $q_n(g(l))$ with geodesic curvature less than $\delta$ for $n \gg 0$.

Therefore we may diagonalize as follows: there is a sequence $j_n \to \infty$ so that $P_{j_n,n} = P_n$ converges to $\lambda_H$ in the Hausdorff topology, and so that no curve in $P_n$ has geodesic representative intersecting the compact sets $q_n(K)$ for $n \gg 0$.

After passing to a further subsequence, there are curves $c_n \in P_n$ that converge in the Hausdorff topology to a lamination $\lambda'$ so that $\nu \subset \lambda'$. Applying the construction of nearly-straight train tracks in [Br1, Lem. 5.2, Cor. 5.3], there is a uniform $C$ depending only on $S$ and the injectivity radius along the image $g(\nu)$ of $\nu$ in $\mathcal{M}$ for which the following holds: for any $\epsilon > 0$

1. there exists an $\epsilon$-nearly-straight train track $\tau \subset \mathcal{M}$ carrying $\nu$, and
2. $\tau$ admits an enlargement $\tau_n$ that minimally carries $c_n$ with a $C\epsilon$-nearly-straight realization $\tau_n^*$ in $\mathcal{M}_n$ for $n \gg 0$.

Choosing $\epsilon$ and $\delta$ sufficiently small, then, for $n \gg 0$, both the image $q_n(g(\nu))$ and the train track $\tau_n^*$ lie close to the realization of $\nu$ in $\mathcal{M}_n$ and hence close to each other: precisely, $q_n(g(\nu))$ lies within $C_\epsilon(C\epsilon + \delta)$ of $\tau_n^*$, since $\tau_n^*$ carries $\nu$. As $\tau_n^*$ also carries $c_n$, and $\tau_n^*$ is nearly-straight, $c_n$ is realizable in $\mathcal{M}_n$ with geodesic representative $c_n^*$. Indeed, $c_n^*$ lies within $C_\epsilon C\epsilon$ of $\tau_n^*$ and thus within $C_\epsilon(2C\epsilon + \delta)$ of $q_n(g(\nu))$. We have a contradiction, since either $c_n$ is non-realizable, or its geodesic representative $c_n^*$ lies outside $q_n(K)$ for all $n$ sufficiently large.

The contradiction implies that $\nu$ is not realizable in $\mathcal{M}$, and hence $\nu \subset \mathcal{E}(\mathcal{M})$. 


6. Convergence in Bers’ compactification

The above methods bear on the question of how the divergent surfaces $X_n \in \text{Teich}(S)$ for which $Q(X_n, Y) \rightarrow M \in \partial B_Y$ and the quotient manifolds $M_n = Q(X_n, Y)$ determine the end invariant $\mathcal{E}(M)$ of their limit in Bers’ boundary.

A direct consequence of theorem 4.1 is the following:

**Theorem 6.1.** Let $X_n \rightarrow [\mu]$ in Thurston’s boundary $\mathcal{P}\mathcal{L}(S)$ for $\text{Teich}(S)$. Then for any limit $M \in \partial B_Y$ of $\{Q(X_n, Y)\}$, we have $|\mu| \subset \mathcal{E}(M)$.

**Proof:** In [Th5], Thurston constructs measured laminations $\mu_n$ so that $\mu_n \rightarrow \mu$ in $\mathcal{ML}(S)$, and $\text{length}_{X_n}(\mu_n) \rightarrow 0$. The theorem follows from an application of theorem 4.1.

As with maximal cusps, however, the support $|\mu|$ of the limit lamination $[\mu] \in \mathcal{P}\mathcal{L}(S)$ is often a small piece of $\mathcal{E}(M)$. We now formulate a construction to obtain partitions $\Pi(M_n)$ of $S$ using the limiting geometry of $M_n$ so that $\Pi(M_n)$ converge to $\mathcal{E}(M)$ in the end-invariant topology. We remark that various such constructions are possible, requiring various levels of detail. We present a simple one.

**Constructing partitions.** By a theorem of Bers (see [Bus, Thm. 5.2.6]), there is a uniform constant $B > 0$ depending only on $S$ so that any given $X \in \text{Teich}(S)$ admits a maximal partition $\Pi$ all of whose elements $\gamma$ satisfy

$$\text{length}_{X}(\gamma) < B.$$ 

Consider a sequence $M_n = Q(X_n, Y)$ converging to $M \in \partial B_Y$, and consider the set $\mathcal{B}_n \subset S$ consisting of curves of length less than $B$ on $X_n$. For each $n$, let $\beta_n^1$ denote an element of $\mathcal{B}_n$ that minimizes the ratio

$$\frac{\text{length}_{M_n}(\beta)}{\text{length}_{Y}(\beta)}$$

over all elements $\beta \in \mathcal{B}_n$. Continuing inductively, let $\beta_n^k$ be an element of

$$\mathcal{B}_n \cap S(S - \beta_n^1 \sqcup \ldots \sqcup \beta_n^{k-1})$$

that minimizes the above ratio.

Let $k_0$ denote the maximal $k$ for which the ratio

$$\frac{\text{length}_{M_n}(\beta_n^k)}{\text{length}_{Y}(\beta_n^k)} \rightarrow 0,$$ 

and let

$$\Pi(M_n) = \beta_n^1 \sqcup \ldots \sqcup \beta_n^{k_0}.$$ 

Then we have the following.

**Theorem 6.2.** Let $X_n \rightarrow \infty$ in $\text{Teich}(S)$ determine quasi-Fuchsian manifolds $M_n = Q(X_n, Y) \rightarrow M$ in $\partial B_Y$. Then the partitions $\Pi(M_n)$ converge to $\mathcal{E}(M)$ in the end-invariant topology.

**Proof:** Consider a Hausdorff limit $\lambda_H$ of $\Pi(M_n)$. If $\alpha \in S$ is an isolated simple closed curve in $\lambda_H$, then $\alpha$ lies in infinitely many $\Pi(M_n)$ so we have

$$\inf \{\text{length}_{M_n}(\alpha)\} = 0.$$ 

Hence $\alpha \subset \mathcal{E}(M)$, by theorem 2.3.
For any other measurable sublamination $\nu \subset \lambda_H$ there is a sequence $c_n \in \Pi(M_n)$ so that $\text{length}_Y(c_n) \to \infty$ and $\nu$ lies in the Hausdorff limit of $c_n$ after passing to a subsequence. Assume $\nu$ is realizable in $M$. As in the proof of theorem 5.3, there is an $\epsilon$-nearly-straight train track $\tau \subset M$ carrying $\nu$, and a uniform $C > 1$ so that $\tau$ admits enlargements $\tau_n$ minimally carrying $c_n$ with $C\epsilon$-nearly-straight realizations $\tau_n^*$ in $M_n$, for $n \gg 0$.

Given a branch $b$ of $\tau_n$, let $m_b(c_n)$ be the weight $c_n$ assigns to $b$; i.e. the number of times $c_n$ traverses $b$. Then by [Br1, Cor. 5.3] given any $b \in \tau$, the weight $m_b(c_n)$ grows without bound. Since the total length $\ell_{\tau_n^*}(c_n)$ of the train-path homotopic to $c_n$ on $\tau_n^*$ satisfies

$$\text{length}_{M_n}(c_n) \geq \frac{1}{K(C\epsilon)} \ell_{\nu_n^*}(c_n),$$

where $K(C\epsilon)$ is the contraction bound of equation 5.2 of §5 (see also [Br1, §4]), it follows that $\text{length}_{M_n}(c_n)$ diverges.

Since, however, we have

$$\text{length}_{M_n}(c_n) \leq 2\text{length}_{X_n}(c_n),$$

by [Bers2, Thm. 3] or [Mc1, Prop. 6.4], it follows that $\text{length}_{M_n}(c_n) < 2B$, contradicting the divergence of $\text{length}_{M_n}(c_n)$. Thus $\nu$ is non-realizable, and therefore $\nu$ lies in $\mathcal{E}(M)$.

\[\Box\]

**Convergence to the boundary in $B^{-}\mathcal{Y}$.** We unify these two perspectives on $\mathcal{E}(M)$ as follows. Given $M \subset \partial B^{-}\mathcal{Y}$, the conformal boundary $\partial M - Y$ is a (possibly empty) union $X$ of hyperbolic surfaces. Given any sequence $M_n \subset \overline{B^{-}\mathcal{Y}}$ converging to $M$, let $X_n = \partial M_n - Y$. We construct partitions $\Pi(M_n)$ of $X_n$, exactly as above: Choose pairwise disjoint curves $\beta_n^1, \ldots, \beta_n^{k_n}$ from the set $\mathcal{B}_n \subset \mathcal{S}(X_n)$ of curves of length less than $B$ on $X_n$ so that each $\beta_n^k$ minimizes the ratio

$$\frac{\text{length}_{M_n}(\beta)}{\text{length}_Y(\beta)}$$

over all $\beta \in \mathcal{B}_n \cap \mathcal{S}(X_n - \beta_n^1 \cup \ldots \cup \beta_n^{k-1})$ and so that we have

$$\frac{\text{length}_{M_n}(\beta_n^{k_n})}{\text{length}_Y(\beta_n^{k_n})} \to 0.$$

Then the resulting union $\mathcal{E}(M_n) \sqcup \Pi(M_n)$ is a geodesic lamination on $S$.

**Corollary 6.3.** The laminations $\mathcal{E}(M_n) \sqcup \Pi(M_n)$ converge to $\mathcal{E}(M)$ in the end-invariant topology.

**Proof:** Pass to a subsequence so that $\mathcal{E}(M_n) \sqcup \Pi(M_n)$ converges to $\lambda_H$ in the Hausdorff topology. Then for any connected measurable sub-lamination $\nu \subset \lambda_H$, there is a further subsequence so that $\nu$ lies either in the Hausdorff limit of the partition $\Pi(M_n)$ or the laminations $\mathcal{E}(M_n)$. It follows from theorems 2.3 and the proof of theorem 6.2 that $\nu$ lies in $\mathcal{E}(M)$.

\[\Box\]
7. The failure of the Hausdorff topology to predict the end-invariant

In this section we address the questions of whether the $\mathcal{E}$ can have a continuous inverse in the end-invariant topology, and whether limiting values of $\mathcal{E}$ give a complete description of the end-invariant.

The inverse $\mathcal{E}^{-1}$ is known to be well defined on points $|\mu|$ of $PL(S)/|\cdot|$ for which $|\mu|$ is a collection of simple closed curves; each $M$ for which $\mathcal{E}(M) = |\mu|$ is quasi-isometrically unique ($M$ is a geometrically finite cusp).\footnote{\textsuperscript{4}} In the end-invariant topology, there are abundant discontinuities of $\mathcal{E}^{-1}$ on this set arising from approximation by maximal cusps. For example, given a single simple closed curve $\gamma \in S$ and an $M$ for which $\mathcal{E}(M) = \gamma$, there are maximal cusps $M(C_n)$ converging to $M$ by the main result of [Mc2]. By theorem 5.3 any Hausdorff limit of $C_n$ has $\gamma$ as its unique measurable sub-lamination. In the end-invariant topology, however, any measurable lamination $\lambda$ containing $\gamma$ is a limit of $C_n$, and when $\dim_{C}(\text{Teich}(S)) > 1$ there are infinitely many such $\lambda$. In this case, then, $\gamma$ is necessarily a point of discontinuity for $\mathcal{E}^{-1}$ in the end-invariant topology.

In the setting of convergent maximal cusps $M(C_n) \to M$, where $\mathcal{E}(M(C_n))$ cannot be enlarged, it is natural to ask whether the maximal measurable sub-lamination $\nu$ of any Hausdorff limit of $\{C_n\}$ gives a complete picture of the end-invariant $\mathcal{E}(M)$. If $C_n$ converges in the Hausdorff topology to a lamination that does not relatively fill (such examples are easy to arrange), lemma 3.3 shows that at the very least one must enlarge $\nu$ to the lamination $\hat{\nu}$ (by adding any missing curves in its implicit partition) to hope for the equality $\hat{\nu} = \mathcal{E}(M)$.

We conclude this paper with an example that shows that adding the implicit partition for $\nu$ is not in general enough to obtain this equality: new parabolics can arise that are neither contained nor implicit in $\nu$.

**Theorem 7.1. Implicit Cusps** Let $\dim_{C}(\text{Teich}(S)) > 1$, and let $\gamma$ lie in $S$. Then for any $\alpha$ in $S(S - \gamma)$, there are maximal partitions $C_n \to \lambda_H$ in the Hausdorff topology and associated maximal cusps $M(C_n) \to M$ in $\partial B_Y$ for which:

1. $\gamma$ is the maximal measurable sub-lamination of $\lambda_H$, and
2. $\alpha$ lies in $\mathcal{E}(M)$.

**Proof:** By the assumption that $\dim_{C}(\text{Teich}(S)) > 1$, there are infinitely many $\alpha$ satisfying the hypotheses.

We construct the sequence of maximal partitions $C_n$ as follows. Let $\varphi \in \text{Mod}(S)$ be a mapping class so that

1. $\varphi$ fixes $\alpha$,
2. $\varphi$ restricts to a pseudo-Anosov mapping class on the closure of the component $T$ of $S - \alpha$ containing $\gamma$,
3. $\varphi$ is the identity otherwise

(see [FLP, Exp. 9], [Th3], [Br2]). Let $\tau_\gamma \in \text{Mod}(S)$ be a Dehn twist about the curve $\gamma$. Let $P_0$ be a maximal partition, all of whose elements cross $\alpha$. Let $\varphi^k(P_0) = P_k$. By assigning weight 1 to each element of $P_k$ we obtain a sequence $\{|P_k|\} \subset PL(S)$, that converges to a limit $|\mu_{\infty}|$ after passing to a subsequence.

Let $\mu^u \in M\mathcal{L}(S)$ denote the unstable lamination for the pseudo-Anosov restriction of $\varphi$ to $T$; i.e. $\mu^u$ is the unique measured lamination for which $\varphi(\mu^u) = c\mu^u$.
with $c > 1$. Noting that

$$i(\mu^u, \varphi^k(P_0)) = i(\varphi^{-k}(\mu^u), P_0) = \frac{i(\mu^u, P_0)}{c^k},$$

it follows from continuity of $i(\cdot, \cdot)$ (see [Bon1, Prop. 4.5]) that $i(\mu^u, \mu_\infty) = 0$.

Let $\lambda$ be a Hausdorff limit of a subsequence of $P_k$. If $\alpha$ separates $S$, then let $T' = S - T$. Then $\varphi(\beta) = \beta$ for each $\beta \in \mathcal{S}(T')$, so $i(\beta, P_k)$ does not depend on $k$ (and is therefore bounded). Thus, $\lambda$ contains no measurable sub-lamination $\eta$ for which $\eta = |\mu'|$ and $\mu' \in \mathcal{ML}(T')$.

Hence, either $[\mu_\infty] = [\mu^u]$ or $\alpha$ is a sub-lamination of $\mu_\infty$. We wish to avoid this possibility, so we adjust each $P_k$ by the power $m_k \in \mathcal{CI}$ of an $\alpha$-Dehn twist $\tau_\alpha$ for which the total length of

$$P_k' = \tau_\alpha^{m_k}(P_k)$$

on $Y$ is minimized. It follows that the curves in $P_k'$ and $\alpha$ realized as geodesics on $Y$ intersect with angle uniformly bounded away from 0.

For any $\beta \in \mathcal{S}(S - \alpha)$ we have $i(\beta, P_k') = i(\beta, P_k)$, so the above intersection number arguments apply to $P_k'$: after passing to perhaps further subsequences, we have $[P_k'] \to [\mu^u]$ in $\mathcal{PL}(S)$ and $P_k'$ converge as geodesic laminations to a Hausdorff limit $\lambda'$ with maximal measurable sublamination $|\mu^u|$ (see figure 3).

![Figure 3. An implicit cusp: $\mathcal{E}(M) = \gamma \sqcup \alpha$, but $\alpha$ does not lie in $\lambda_H$.](image)

Now consider the action of the Dehn twist $\tau_\gamma$ on $\lambda'$. Since $i(\mu^u, \gamma) > 0$ and every leaf of $|\mu^u|$ is dense in $|\mu^u|$, every leaf of $|\mu^u|$ crosses $\gamma$ infinitely in each direction. Each leaf of $\lambda'$ is either a leaf of $|\mu^u|$ or asymptotic to leaves of $|\mu^u|$ in each direction, so every leaf of $\lambda'$ crosses $\gamma$ infinitely often in each direction. The Hausdorff limit $\lambda_H$ of $\{\tau_\gamma^n(\lambda')\}_{n=1}^\infty$ consists of $\gamma$ together with a finite number of pairwise disjoint bi-infinite geodesics that spiral into $\gamma$ from either side (figure 3).

Thus, $\gamma$ is the only measurable sub-lamination of $\lambda_H$, and $\lambda_H$ crosses the simple closed curve $\alpha$ transversely (again, as geodesics on $Y$). Diagonalizing, for each $n$ we
choose \( k_n \) so that \( \tau^n_\gamma(P'_{k_n}) \) converges to \( \lambda_H \) in the Hausdorff topology as \( n \to \infty \). Let

\[
C_n = \tau^n_\gamma(P'_{k_n}).
\]

We claim that by enlarging \( k_n \) further we may guarantee that the maximal cusps \( M_n = M(C_n) \in \partial B_Y \) satisfy

\[
\text{length}_{M_n}(\alpha) < \frac{1}{n}.
\]

(7.3)

To see this, note that if we let \( k \) tend to \( \infty \) with \( n \) fixed, the maximal cusps \( \{M(\tau^n_\gamma(P'_{k}))\}_{k=1}^\infty \) converge up to subsequence to a limit \( M_\infty(n) \in \partial B_Y \) with the property that

\[
|\tau^n_\gamma(\mu^n)| \subset E(M_\infty(n)).
\]

Since for each \( n \) the implicit partition \( \tilde{P}(|\tau^n_\gamma(\mu^n)|) \) of \( |\tau^n_\gamma(\mu^n)| \) is the single simple closed curve \( \alpha \), lemma 3.3 guarantees that \( \alpha \) lies in \( E(M_\infty(n)) \). Thus, \( \alpha \) is parabolic in \( M_\infty(n) \), so the claim (inequality 7.3) follows by continuity of length (theorem 2.3).

Applying theorem 2.3 once again, we have that \( \alpha \) is parabolic in \( M \).

\[ \blacksquare \]

A concluding remark: The reader familiar with geometric or Gromov-Hausdorff convergence of hyperbolic manifolds will recognize the similarity of the above example to the main example of [KT, §3] and others like it (cf. [Br2]). In the case above, the geometric limit \( M_G \) covered by \( M \) has a degenerate end that forces an implicit cusp at \( \alpha \), as well as a rank-two cusp with core-curve \( \gamma \). The parabolic \( \alpha \) lifts to \( M \) while the cusp at \( \gamma \) provides an obstruction to lifting the degenerate end. It would seem that a complete understanding of how values of \( E \) vary on Bers boundary depends, like many issues in the deformation theory, on developing a better understanding of the full spectrum of possible geometric limits of sequences \( \{M_n\} \subset \partial B_Y \).

References


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