INFlexibility, Weil-Petersson Distance, and Volumes of FIBered 3-Manifolds

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Abstract. A recent preprint of S. Kojima and G. McShane [KM] observes a beautiful explicit connection between Teichmüller translation distance and hyperbolic volume. It relies on a key estimate which we supply here: using geometric inflexibility of hyperbolic 3-manifolds, we show that for a closed surface, and \( \psi \in \text{Mod}(S) \) pseudo-Anosov, the double iteration \( Q(\psi^{-n}(X), \psi^n(X)) \) has convex core volume differing from \( 2n \text{vol}(M_\psi) \) by a uniform additive constant, where \( M_\psi \) is the hyperbolic mapping torus for \( \psi \). We combine this estimate with work of Schlenker, and a covering argument to obtain an explicit lower bound on Weil-Petersson translation distance of pseudo-Anosov \( \psi \in \text{Mod}(S) \) for general compact \( S \) of genus \( g \) with \( n \) boundary components: we have

\[
\text{vol}(M_\psi) \leq 3 \sqrt{\pi/2} (2g-2+n) \|\psi\|_{\text{WP}}.
\]

This gives the first explicit estimates on the Weil-Petersson systoles of moduli space in the closed and finite type case, and of the minimal distance between nodal surfaces in the completion of Teichmüller space. In the process, we recover the estimates of [KM] on Teichmüller translation distance via a Cauchy-Schwartz estimate (see [Lin]).

1. Introduction

Let \( S \) be a closed surface of genus \( g > 1 \). Let \( \psi : S \to S \) be a pseudo-Anosov, \( Q_n = Q(\psi^{-n}(X), \psi^n(X)) \) quasi-Fuchsian simultaneous uniformizations, and \( M_\psi \) the hyperbolic mapping torus for \( \psi \). Let core\((Q_n)\) denote the convex core of \( Q_n \).

We will prove the following theorem.

Theorem 1.1. The quantity

\[
|\text{vol}(\text{core}(Q_n)) - 2n \text{vol}(M_\psi)|
\]

is uniformly bounded.

The possibility of such a result was suggested in [Br3] as a means to give a direct proof of the main result comparing hyperbolic volume of \( M_\psi \) and Weil-Petersson translation distance of \( \psi \), directly from a similar comparison in the quasi-Fuchsian case [Br2]. A recent preprint of S. Kojima and G. McShane shows how this suggestion can be used to give sharper bounds between volume of \( M_\psi \) and normalized entropy, or the translation distance in the Teichmüller metric of \( \psi \).
In addition to supplying a proof of Theorem 1.1, we focus on volume implications for the Weil-Petersson metric on Teichmüller space. Indeed, by analyzing the renormalized volume, rather than the convex core volume, Jean-Marc Schlenker improved the upper bound in [Br3], for general compact $S$ with $\chi(S) < 0$.

**Theorem 1.2.** (Schlenker) Let $S$ be a closed surface of genus $g > 1$ and let $X, Y$ lie in $\text{Teich}(S)$. There is a constant $K_S > 0$ so that

$$\text{vol}(\text{core}(Q(X,Y))) \leq 3\sqrt{\pi(g-1)}d_{WP}(X,Y) + K_S.$$  

(See [Schlk]).

The Weil-Petersson translation length of $\psi$ as an automorphism of $\text{Teich}(S)$ is defined by taking the infimum

$$\|\psi\|_{WP} = \inf_{X \in \text{Teich}(S)} d_{WP}(X, \psi(X)).$$

Daskalopoulos and Wentworth [DW] showed this infimum is realized by $X$ in $\text{Teich}(S)$ when $\psi$ is pseudo-Anosov.

Combining Theorem 1.1, Theorem 1.2, and a covering trick, we obtain the following Theorem.

**Theorem 1.3.** Let $S$ be a compact surface with genus $g$ and $n$ boundary components $\chi(S) < 0$ and let $\psi \in \text{Mod}(S)$ be pseudo-Anosov. Then we have

$$\text{vol}(M_\psi) \leq 3\sqrt{\frac{\pi}{2}(2g - 2 + n)} \|\psi\|_{WP}.$$  

The case when $S$ is closed readily follows from Theorem 1.1 and Theorem 1.2. When $S$ has boundary, a branched covering argument allows us to recover the estimates from the closed case. We defer the proof to section 4.

If $S$ has genus $g$ with $n$ boundary components, then we take area($S$) to denote the Poincaré area of any $X \in \text{Teich}(S)$, namely

$$\text{area}(S) = 2\pi(2g - 2 + n).$$

By an application of the Cauchy-Schwartz inequality (see [Lin]), we have for each $X, Y$ in $\text{Teich}(S)$ the bound

$$d_{WP}(X,Y) \leq \sqrt{\text{area}(S)} d_T(X,Y)$$

from which we conclude

$$\|\psi\|_{WP} \leq \sqrt{\text{area}(S)} \|\psi\|_T.$$  

As it follows that

$$\text{vol}(M_\psi) \leq \frac{3}{2} \text{area}(S) \|\psi\|_T$$

we recover the Theorem of [KM] concerning volumes and Teichmüller translation distance for arbitrary $S$.

We note that the study of normalized entropy and dilatation has seen considerable interest of late, note in particular the papers of [ALM], and [FLM] which have greatly improved our understanding of fibered 3-manifolds of low dilatation. The
work of [KM] has been particularly important here. We will focus our attention primarily on implications for Teichmüller space with the Weil-Petersson metric.

**Weil-Petersson geometry.** Theorem 1.3 gives the first explicit estimates on the systole of moduli space with the Weil-Petersson metric. It was shown by Gabai, Meyerhoff and Milley [GMM], that the smallest volume closed orientable hyperbolic 3-manifold is the Weeks manifold $W$, obtained by $(5, 2)$ and $(5, 1)$ Dehn surgeries on the Whitehead link. An explicit formula for its volume is given by

$$\text{vol}(W) = \frac{3 \cdot 23^{3/2} \zeta(2)}{4\pi^4}.$$  

Applying Theorem 1.3, we conclude the following estimate on the Weil-Petersson systole of $M(S)$ for $S$ a closed surface:

**Theorem 1.4. (Weil-Petersson Systole - Closed Case)** Let $S$ be a closed surface with genus $g > 1$, and let $\gamma$ be a closed Weil-Petersson geodesic in the moduli space $M(S)$. Then we have

$$\frac{\text{vol}(W)}{3\sqrt{\pi(g-1)}} \leq \ell(\gamma).$$

Similarly, Cao and Meyerhoff [CM] show that the smallest volume orientable cusped hyperbolic 3-manifold is the figure eight knot complement which has volume $2\gamma_3$ where $\gamma_3$ is the volume of the regular ideal hyperbolic tetrahedron. An application of this bound yields a similar result for the Weil-Petersson systole of the moduli space of bordered surfaces.

**Theorem 1.5. (Weil-Petersson Systole - Punctured Case)** Let $S$ be a surface of genus $g$ with $n > 0$ boundary components and $\chi(S) < 0$, and let $\gamma$ be a closed Weil-Petersson geodesic in the moduli space $M(S)$. Then we have

$$\frac{\gamma_3}{3\sqrt{2\pi|\chi(S)|}} \leq \ell(\gamma).$$

It is remarkable that even to estimate the distance between nodal surfaces at infinity in the Weil-Petersson metric has been an elusive problem. Theorem 1.3 provides the first explicit means by which to do this, through a limiting process involving Dehn-twists iterates about a longitude-meridian pair $(\alpha, \beta)$ on the punctured torus. Specifically, let

$$\psi_n = \tau^n_\alpha \circ \tau^n_\beta$$

denote the composition of $n$-fold Dehn-twists about simple closed curves $\alpha$ and $\beta$ on $S$ with $i(\alpha, \beta) = 1$ and $S$ a one-holed torus. Then Theorem 1.3 gives

$$\text{vol}(M_{\psi_n}) \leq \|\psi_n\|_{WP}.$$  

The left hand side converges to $2\gamma_3$ or twice the volume of the ideal hyperbolic octahedron, while the right hand side converges to $2\ell_{\alpha, \beta}$ the length of geodesic joining the nodal punctured tori $N(\alpha)$ and $N(\beta)$ in the completion of the Weil-Petersson metric with the curves $\alpha$ and $\beta$ pinched to cusps.

Then we obtain:
Theorem 1.6. (Weil-Petersson Inradius) Let $\alpha$ and $\beta$ be simple closed curves on the one-holed torus $S$. The distance $\ell_{\alpha,\beta}$ in $\text{Teich}(S)$ between the nodal surfaces $N(\alpha)$ and $N(\beta)$ in the Weil-Petersson completion satisfies
\[
\frac{\gamma_8}{3\sqrt{\pi}} \leq \ell_{\alpha,\beta}.
\]

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2. Preliminaries

We review background for our results.

Weil-Petersson geometry. The above results give new explicit estimates on the geometry of Teichmüller space with the Weil-Petersson metric. The Weil-Petersson metric arises from the hyperbolic $L^2$-norm on the space of quadratic differentials $Q(X)$ on a Riemann surface $X$, given by
\[
\|\varphi\|_{\text{WP}}^2 = \int_X |\varphi|^2 \rho_X.
\]
Though known to be geodesically convex [Wol2] it is not complete [Wol1, Chu]. It has negative curvature [Ah1], but its curvatures are bounded away neither from 0 nor negative infinity. In [DW], Daskalopoulos and Wentworth showed that a pseudo-Anosov automorphism $\psi \in \text{Mod}(S)$ has an invariant axis along which $\psi$ translates. A primitive pseudo-Anosov element $\psi \in \text{Mod}(S)$, therefore, determines a closed Weil-Petersson geodesic in the moduli space of Riemann surfaces $\mathcal{M}(S)$.

The complex of curves. Let $S$ be a compact surface of genus $g$ with $n$ boundary components. The complex of curves $\mathcal{C}(S)$ is a $3g - 4$ dimensional complex, each vertex of which is associated to a simple closed curve on the surface $S$ up to isotopy, and so that $k$-simplices span collections of $k + 1$ vertices whose associated curves are disjoint. Masur and Minsky proved $\mathcal{C}^1(S)$ is a $\delta$-hyperbolic metric space with the distance $d_{\mathcal{C}}(\ldots)$ given by the edge metric.

Given $S$ there is an $L_S$ so that for each $X \in \text{Teich}(S)$ there is a $\gamma \in \mathcal{C}(S)$ so that $\ell_X(\gamma) < L_S$. By making such a choice of $\gamma$ for each $X$ we obtain a coarsely well-defined projection
\[
\pi_{\mathcal{C}} : \text{Teich}(S) \to \mathcal{C}^0(S).
\]
We refer to the distance between a point $X$ in Teichmüller space and a curve $\gamma$ in $\mathcal{C}^0(S)$ with the notation:
\[
d_{\mathcal{C}}(X, \gamma) = d_{\mathcal{C}}(\pi_{\mathcal{C}}(X), \gamma).
\]

Quasi-Fuchsian manifolds. Each pair $(X, Y) \in \text{Teich}(S) \times \text{Teich}(S)$ determines a quasi-Fuchsian simultaneous uniformization $Q(X, Y)$ with $X$ and $Y$ in its conformal
boundary. This is the quotient
\[ Q(X,Y) = \mathbb{H}^3 / \rho_{X,Y}(\pi_1(S)) \]
of a quasi-Fuchsian representation of the fundamental group
\[ \rho_{X,Y} : \pi_1(S) \to \text{PSL}_2(\mathbb{C}). \]

The quasi-Fuchsian representations sit as the interior of the space \( AH(S) \) of all hyperbolic 3-manifolds homotopy equivalent to \( S \), with the topology of convergence on generators of the fundamental group. For more information, see [Brm, BB1, BCM, Ag, CG].

A complete hyperbolic 3-manifold \( M \), marked by a homotopy equivalence \( f : S \to M \) determines a point in \( AH(S) \) up to isometry - we denote such a marked hyperbolic 3-manifold by the pair \( (f, M) \). Equipping \( M \) with a baseframe \( (M, \omega) \) determines a specific representation \( \rho : \pi_1(S) \to \text{PSL}_2(\mathbb{C}) \) and a Kleinian surface group \( \Gamma = \rho(\pi_1(S)) \).

The geometric topology on based hyperbolic 3-manifolds records geometric information: a sequence \( (M_n, \omega_n) \) converges to \( (M, \omega) \) if for each \( \varepsilon, R > 0 \) there is a \( N > 0 \), and for all \( n > N \) we have embeddings
\[ \varphi_n : (B_R(\omega), \omega) \to (M_n, \omega_n) \]
from the \( R \)-ball around \( \omega \) to \( M_n \) with 1-jet an isometry at \( \omega_n \) and bi-Lipschitz constant at most \( 1 + \varepsilon \) at all points of \( B_R(\omega) \).

The convergent sequence \( (f_n, M_n) \to (f_\infty, M_\infty) \) in \( AH(S) \) converges strongly if there are baseframes \( \omega_n \) in \( M_n \) and \( \omega_\infty \in M_\infty \) so that the resulting \( \rho_n \) converge to the resulting \( \rho_\infty \) on generators, and the manifolds \( (M_n, \omega_n) \) converge geometrically to \( (M_\infty, \omega_\infty) \).

**Convex core width.** Given \( M \in AH(S) \), let \( d_M(U, V) \) be the minimal distance between subsets \( U \) and \( V \) in \( M \). We prove the following in [BB2].

**Theorem 2.1.** Given \( \varepsilon, L > 0 \), there exist \( K_1 \) and \( K_2 \) so that if \( M \in AH(S) \) has \( \varepsilon \)-bounded geometry and \( \alpha \) and \( \beta \) in \( C^0(S) \) are simple closed curves with \( \ell_M(\alpha) \) and \( \ell_M(\beta) \) bounded by \( L \), then
\[ d_M(\alpha^*, \beta^*) \geq K_1 d_\varepsilon(\alpha, \beta) - K_2 \]

It is due to Bers that
\[ 2\ell_X(\gamma) \geq \ell_{Q(X,Y)}(\gamma). \]

Thus Theorem 2.1 serves to bound from below the width of the convex core of \( Q(X,Y) \) (the distance between its boundary components) in terms of the curve complex distance. Such convex core width estimates will be important to our application of the inflexibility theory outlined in the next section.
3. Geometric Inflexibility

To prove Theorem 1.1, our key tool will be the inflexibility theorem of [BB2].

**Theorem 3.1.** (Geometric Inflexibility) Let $M_0$ and $M_1$ be complete hyperbolic structures on a 3-manifold $M$ so that $M_1$ is a $K$-quasi-conformal deformation of $M_0$, $\pi_1(M)$ is finitely generated, and $M_0$ has no rank-one cusps.

There is a volume preserving $K^{3/2}$-bi-Lipschitz diffeomorphism

$$\Phi: M_0 \to M_1$$

whose pointwise bi-Lipschitz constant satisfies

$$\log \text{bilip}(\Phi, p) \leq C_1 e^{-C_2 d(p, \text{core}(M_0))}$$

for each $p \in M^{\geq \varepsilon}$, where $C_1$ and $C_2$ depend only on $K$, $\varepsilon$, and $\text{area}(\partial \text{core}(M_0))$.

The existence of a volume preserving, $K^{3/2}$ bi-Lipschitz diffeomorphism was established by Reimann [Rei], using work of Ahlfors [Ah2] and Thurston [Th1] (see McMullen [Mc] for a self-contained account). That the bi-Lipschitz constant decays exponentially fast with depth in the convex core at points in the thick part follows from comparing $L^2$ and pointwise bounds on harmonic strain fields arising from extending a Beltrami isotopy realizing the deformation. Exponential decay of the $L^2$ norm in the core can be converted to pointwise bounds via mean value estimates, building on work in the cone-manifold deformation theory of hyperbolic manifolds due to Hodgson and Kerckhoff [HK] and the second author [Brm].

Inflexibility was used in [BB2] to give a new, self-contained proof of Thurston’s Double Limit Theorem, the key tool in his proof of the existence of hyperbolic structures on 3-manifolds that fiber over the circle with pseudo-Anosov monodromy.

**Theorem 3.2.** (Thurston) Let $S$ be a closed hyperbolic surface. The sequence $Q(\psi^{-n}(X), \psi^n(X))$ converges algebraically and geometrically to $Q_\infty$, the infinite cyclic cover of the mapping torus $M_\psi$ corresponding to the fiber $S$.

The manifolds $Q_n = Q(\psi^{-n}(X), \psi^n(X))$ admit volume preserving, uniformly bi-Lipschitz Reimann maps

$$\phi_n: Q_n \to Q_{n+1}$$

as in Theorem 3.1. The key to obtaining Theorem 3.2 from Theorem 3.1 is an analysis of the growth rate of the convex core diameter in terms of the curve complex.

We will employ the the following key consequence of inflexibility [BB2, Prop. 9.7].

**Proposition 3.3.** Given $\varepsilon, R, L, C > 0$ there exist $B, C_1, C_2 > 0$ such that the following holds. Assume that $\mathcal{K}$ is a subset of $Q_N$ such that $\text{diam}(\mathcal{K}) < R$, $\text{inj}_p(\mathcal{K}) > \varepsilon$ for each $p \in \mathcal{K}$ and $\gamma \in \mathcal{C}^0(S)$ is represented by a closed curve in $\mathcal{K}$ of length at most $L$ satisfying

$$\min\{d_\varepsilon(\psi^{N+n}(Y), \gamma), d_\varepsilon(\psi^{-N-n}(X), \gamma)\} \geq K_\psi n + B$$
for all \( n \geq 0 \). Then we have
\[
\log \text{bilip}(\phi_{N+n}, p) \leq C_1 e^{-C_2 n}
\]
for \( p \) in \( \phi_{N+n-1} \circ \cdots \circ \phi_N(\mathcal{K}) \) and
\[
\frac{C_1}{1 - e^{-C_2}} < C.
\]

The simple closed curve \( \gamma \) serves to control the depth of the compact set \( \mathcal{K} \) in the convex core of \( Q_{N+n} \) as \( n \to \infty \) via inflexibility and Theorem 2.1: if \( \mathcal{K} \) starts out sufficiently deep, then the geometry freezes around it quickly enough that its depth grows linearly, resulting in the exponential convergence of the bi-Lipschitz constant.

**Double Iteration.** The pseudo-Anosov double \( \{ Q_n \} \) converges strongly to the doubly degenerate manifold \( Q_\infty \), invariant by the isometry
\[
\Psi: Q_\infty \to Q_\infty
\]
the isometric covering translation for \( Q_\infty \) over the mapping torus \( M_\psi \) for \( \psi \) (see [Th2, CT, Mc, BB2]). Likewise, McMullen showed the iteration \( Q(X, \psi^n(X)) \) also converges strongly to a limit \( Q_{X, \psi} \) in the Bers slice
\[
B_X = \{ Q(X, Y) : Y \in \text{Teich}(S) \}.
\]

Each element \( \tau \in \text{Mod}(S) \) acts on \( \text{AH}(S) \) by remarking, or precomposition of the representation by the corresponding automorphism of the fundamental group. This action is denoted by
\[
\tau(f, M) = (f \circ \tau^{-1}, M)
\]
Then the remarking of \( Q_{X, \psi} \) by \( \psi^{-n} \) produces a sequence
\[
\psi^{-n}(Q_{X, \psi}) = Q_{\psi^{-n}(X), \psi}
\]
converging strongly in \( \text{AH}(S) \) to \( Q_\infty \) (see [Mc]).

For reference, let \( F \) denote a homeomorphism
\[
F: S \times \mathbb{R} \to Q_\infty
\]
equipping the limit \( Q_\infty \) with a product structure, and assume the isometric covering transformation
\[
\Psi: Q_\infty \to Q_\infty
\]
in the homotopy class of \( \psi \) for the covering \( Q_\infty \to M_\psi \) preserves this product structure and acts by integer translation \( \Psi(S, t) = (S, t + 1) \) in the second factor. We denote by \( Q_\infty[a, b] \) the subset \( F(S \times [a, b]) \).

**Proposition 3.4.** Let \( \gamma \in \mathcal{C}^0(S) \) satisfy \( \ell_X(\gamma) < L_S \). Then there exists \( a > 0 \) and \( N_1 > 0 \) so that for each \( n > N_1 \) the compact subset \( Q_\infty[-a, a] \) contains \( \gamma^n \) and admits a marking preserving 2-bi-Lipschitz embedding
\[
\varphi_n: Q_\infty[-a, a] \to Q_{\psi^{-n}(X), \psi}.
\]
Proof: The Proposition follows from the observation that the geodesic representatives of $\psi^a(\gamma)$ lie arbitrarily deep in the convex core of $Q_{X,\underline{V}}$, and the fact that the isometric remarkings $\psi^{-n}(Q_{X,\underline{V}}) = Q_{\psi^{-n}(X),\underline{V}}$ converge to the fiber $Q_\infty$ (see [Mc, Thm. 3.11]). Choosing an interval $[-a,a]$ so that $Q_\infty[-a,a]$ contains $\gamma$, the marking preserving bi-Lipschitz embeddings

$$\varphi_n: Q_\infty[-a,a] \to Q_{\psi^{-n}(X),\psi}$$

are eventually 2-bi-Lipschitz, giving the desired $N_1$. \qed

We note that we may argue symmetrically for $Q_{\psi^{-1},\psi^n(X)}$.

4. THE PROOF

In this section we give the proof of Theorem 1.1.

Proof: After making an initial choice of $N$, we will find, for each $k$, a subset $\mathcal{X}_k$ of $Q_{N+k}$ accounting for all but a uniformly bounded amount of the volume of the core of $Q_{N+k}$. Fixing $k$, the volume preserving Reimann maps $\phi_{N+k+B}$ applied to $\mathcal{X}_k$ converge as $n \to \infty$ to a subset that contains roughly $2k$ copies of the fundamental domain for the action of $\psi$ on $Q_\infty$.

Step I. Choose constants. As the input for Theorem 3.3, let

$$R > 4 \text{diam}(Q_\infty[-a,a]),$$

and take $L > 4L_S$, and fix $\varepsilon < \varepsilon_\psi/4$, where

$$\varepsilon_\psi = \text{inj}(M_\psi) = \text{inj}(Q_\infty)$$

where $\text{inj}(M) = \inf_{p \in M} \text{inj}_p(M)$. Finally, taking $C = 2$, we take $B$, $C_1$ and $C_2$ satisfying the conclusion of Theorem 3.3. Recall that $\gamma \in \mathcal{E}_0(S)$ satisfies $\ell_X(\gamma) < L_S$. Then applying [BB2, Thm. 8.1] there is an $N_0 > 0$ so that

$$\min\{d(\psi^{-N_0+n}(X),\gamma), d(\psi^{N_0+n}(X),\gamma)\} \geq K_\psi n + B$$

for all $n \geq 0$.

Step II. Geometric convergence. Applying Proposition 3.4, there is an $N_1$ which we may take so that $N_1 > N_0$ so that for each $N > N_1$ there are 2-bi-Lipschitz embeddings

$$\varphi^-_n: Q_\infty[-a,a] \to \text{core}(Q_{\psi^{-n}(X),\psi})$$

$$\varphi^+_n: Q_\infty[-a,a] \to \text{core}(Q_{\psi^{-1},\psi^n(X)})$$

that are marking preserving. Applying strong convergence of

$$Q(Y,\psi^n(X)) \to Q_{Y,\psi}$$

and

$$Q(\psi^{-n}(X),Y) \to Q_{\psi^{-1},Y},$$

we take $N_2 > N_1$ so that for each $\delta > 0$, $r > 0$, and $N > N_2$, we have $k_0$ so that for $k > k_0$ there are diffeomorphisms

$$\varphi^-_{N,k}: Q_{\psi^{-N}(X),\psi} \to Q(\psi^{-N}(X),\psi^{N+2k}(X))$$

and

$$\varphi^+_{N,k}: Q_{\psi^{-1},\psi^n(X)} \to Q(\psi^{-N-2k}(X),\psi^n(X))$$
so that $\varphi_{N,k}^-$ has bi-Lipschitz constant satisfying $\log \text{bilip}(\varphi_{N,k}^-, p) < \delta$ for all points in the $D$-neighborhood of $\varphi_{N}^{-}(Q_{\infty}[-a,a])$, and likewise for $\varphi_{N,k}^+$.

It follows that if we fix $N$ satisfying $N > N_2$ for the remainder of the argument, we have that the images $\varphi_{N}^{-}(Q_{\infty}[-a,a])$ and $\varphi_{N}^{+}(Q_{\infty}[-a,a])$ determine product regions in $\text{core}(Q_{\psi^{-N}(X),\psi})$ and $\text{core}(Q_{\psi^{-1}(\psi^N(X))})$ whose complements contain one product region of volume bounded by $\gamma' > 0$.

Noting that the action by $\psi^k$ on $\text{AH}(S)$ gives

$$\psi^{-k}(Q(\psi^{-N}(X),\psi^{N+2k}(X))) = Q_{N+k}$$

and

$$\psi^k(Q(\psi^{-N-2k}(X),\psi^{N}(X))) = Q_{N+k},$$

we let, for each $k > 0$, the subsets $\mathcal{K}_k^-$ and $\mathcal{K}_k^+$ in $Q_{N+k}$ be given by

$$\psi^{-k}(\varphi_{N,k}^-(Q_{\infty}[-a,a])) \text{ and } \psi^k(\varphi_{N,k}^+(Q_{\infty}[-a,a]))$$

by following the embeddings of $Q_{\infty}[-a,a]$ from geometric convergence with the isometric remarkings $\psi^{-k}$ and $\psi^k$. Then geometric convergence implies that for each $k > k_0$ the component of $Q_{N+k} \setminus \mathcal{K}_k^-$ facing $\psi^{-N-k}(X)$ has intersection with the convex core bounded by $2\gamma'$ for $k$ large, and likewise for $\mathcal{K}_k^+$.

**Step III. Apply the Inflexibility Theorem (Theorem 3.3).** We take $B$ as in Theorem 3.3 given the above choices for $\varepsilon, L, R$ and $C$.

For our choice of $N$, we know $\mathcal{K}_k^+$ and $\mathcal{K}_k^-$ each have diameter at most $R$, injectivity radius at least $\varepsilon$, and as $L = 4L_S$, $\mathcal{K}_k^+$ and $\mathcal{K}_k^-$ contain representatives $\gamma_k^-$ of $\psi^{-k}(\gamma)$ and $\gamma_k^+$ of $\psi^k(\gamma)$ of length less than $L$. As $B$ is chosen as in the output of Theorem 3.3 and $N$ is chosen as above we have

$$\min\{d_\varphi(\psi^{-N-k-n}(X),\gamma_k^-), d_\varphi(\psi^{N+k+n}(X),\gamma_k^+)\} \geq K_\varepsilon n + B$$

is satisfied for all $n \geq 0$ and likewise for $\gamma_k^+$.

**Step IV: Follow the Reimann maps.** Let $\phi_N : Q_N \to Q_{N+1}$ denote the (marking preserving) Reimann map furnished by Theorem 3.3. Then the composition of Reimann maps

$$\Phi_n = \phi_{N+n} \circ \ldots \circ \phi_N : Q_N \to Q_{N+n+1}$$

is globally volume preserving.

Furthermore, since $\mathcal{K}_k^+$ and $\mathcal{K}_k^-$ each satisfy the hypotheses of Proposition 3.3 the compositions are uniformly bi-Lipschitz as $n \to \infty$. It follows that the limit $\Phi_\infty$ on $\mathcal{K}_k^+$ sends $\gamma_k^-$ to a curve of length at most $8L$ and likewise for $\mathcal{K}_k^-$ and $\gamma_k^+$. Since $\Phi_\infty$ is 2-bi-Lipschitz on $\mathcal{K}_k^-$ and $\mathcal{K}_k^+$, it follows that $\Phi_\infty(\mathcal{K}_k^-)$ has diameter $8R$, and contains a representative of $\psi^{-k}(\gamma)$ of length $8L_S$ and likewise for $\Phi_\infty(\mathcal{K}_k^+)$ and $\psi^k(\gamma)$.

It follows that if we take $k$ large enough, we may apply Theorem 2.1 to conclude that

$$d_{Q_{N+k}}(\gamma_k^-, \gamma_k^+) > 16R$$

which ensures that

$$\Phi_{n+k}(\mathcal{K}_k^-) \cap \Phi_{n+k}(\mathcal{K}_k^+) = \emptyset$$
for all $n \geq 0$. The complement of $Q_{N+k} \setminus \mathcal{X}_k^- \cup \mathcal{X}_k^+$ contains one subset $O_{N+k}$ with compact closure ‘between’ the product regions $\mathcal{X}_k^-$ and $\mathcal{X}_k^+$.

Letting

$$\mathcal{X}_k = \mathcal{X}_k^- \cup O_{N+k} \cup \mathcal{X}_k^+,$$

the images $\Phi_{k+n}(\mathcal{X}_k)$ satisfy

$$\text{vol}(\mathcal{X}_k) = \text{vol}(\Phi_{k+n}(\mathcal{X}_k))$$

since $\Phi_{k+n}$ is the composition of volume preserving maps.

There is thus a $d > 0$ depending only on $R$ and $L_S$ and $\varepsilon_{\psi}$ so that we have

$$\Phi_\infty(\mathcal{X}_k^-) \subset Q_\infty[-k-d, -k+d]$$

and likewise so that we have

$$\Phi_\infty(\mathcal{X}_k^+) \subset Q_\infty[k-d, k+d].$$

But strong convergence of $Q_{N+k+n}$ to $Q_\infty$ as $n \to \infty$ guarantees that for large $n$ there are nearly isometric marking-preserving embeddings

$$G_n : Q_\infty[-k-d, k+d] \to Q_{N+k+n}$$

that are surjective onto $\Phi_{k+n}(\mathcal{X}_k)$ for $n$ sufficiently large.

We conclude that

$$(2k - 2d) \text{vol}(M_\psi) \leq \text{vol}(\Phi_\infty(\mathcal{X}_k)) \leq (2k + 2d) \text{vol}(M_\psi)$$

and that

$$\text{vol}(\text{core}(Q_{N+k})) - 4\mathcal{V} \leq \text{vol}(\mathcal{X}_k) \leq \text{vol}(\text{core}(Q_{N+k}))$$

for all $k$ sufficiently large. Thus we conclude

$$|\text{vol}(\text{core}(Q_{N+k})) - 2(N+k) \text{vol}(M_\psi)| < 2(d+N) \text{vol}(M_\psi) + 4\mathcal{V}$$

completing the proof.

To complete the proof of Theorem 1.3, we conclude the section by addressing the case when $S$ has boundary.

**Proof of Theorem 1.3.** We now complete the proof of Theorem 1.3. It remains to treat the case when $S$ has boundary. We thank Ian Agol for suggesting such an argument applies in the setting of the Teichmüller metric; we employ a similar line of reasoning for the Weil-Petersson metric recovering the Teichmüller case as a consequence.

We note the following: for a surface $S_{g,n}$ with genus $g > 1$ and $n > 0$ boundary components, the natural forgetful map

$$\text{Teich}(S_{g,n}) \to \text{Teich}(S_{g,0})$$

obtained by filling in the $n$ punctures on a surface $X \in \text{Teich}(S_{g,n})$ is a contraction of Weil-Petersson metrics (see e.g. [ST]). Assuming an even number of punctures, we may branch at the punctures to obtain degree-$k$ covers $S_{k,g,n}$.

We consider the normalized Weil-Petersson distance $d_{\text{WP}}(\ldots)$ obtained by taking

$$d_{\text{WP}}(\ldots) = \frac{d_{\text{WP}}(\ldots)}{\sqrt{\text{area}(S)}}.$$
where \( \text{area}(S) = \text{area}(X) \) is the Poincaré area of any element \( X \in \text{Teich}(S) \) for \( S = S_{g,n}, \text{area}(S) = 2\pi(2g - 2 + n) \).

Then \( d_{WP}(\cdot, \cdot) \) is invariant under the passage to finite covers: the covering projection induces an isometry of normalized Weil-Petersson metrics. Given \( \psi \) pseudo-Anosov, let \( \|\psi\|_{WP} \) denote its translation length in the normalized Weil-Petersson metric.

Letting \( \psi = \psi_{g,n} \in \text{Mod}(S_{g,n}) \), then we let \( \tilde{\psi}_{kg,n} \) denote the lift to \( \text{Mod}(S_{kg,0}) \), obtained by filling in the punctures.

Then we have
\[
\|\psi_{g,n}\|_{WP} = \|\psi_{kg,n}\|_{WP} \geq \|\psi_{kg,0}\|_{WP} \cdot C_k
\]
where \( C_k = \sqrt{\text{area}(S_{kg,0})/\text{area}(S_{kg,n})} \to 1 \) as \( k \to \infty \). Applying Theorem 1.3 in the closed case we obtain
\[
\|\psi_{g,n}\|_{WP} \geq \frac{2 \text{vol}(M_{\psi_{kg,0}})}{3 \text{area}(S_{kg,0})}.
\]

As \( M_{\psi_{kg,0}} \) admits an order-\( k \) isometry corresponding to the \( k \)-fold branched covering, it covers a fibered orbifold with \( n \) order-\( k \) orbifold loci, which converges geometrically to the fibered 3-manifold \( M_{\psi_{kg,0}} \) as \( k \to \infty \). Likewise, \( S_{kg,0} \) covers an orbifold with \( n \) cone points with cone-angle \( 2\pi/k \), whose area is \( \text{area}(S_{kg,0})/k \), and this orbifold converges to the original punctured surface \( S_{g,n} \) as \( k \to \infty \).

Thus, dividing the top and the bottom by \( k \), the right hand side of the inequality tends to
\[
\frac{2 \text{vol}(M_{\psi_{kg,0}})}{3 \text{area}(S_{kg,n})}
\]
as \( k \to \infty \), the estimate holds.

Since any \( S_{g,n} \) with \( n > 0 \) is finitely covered by \( S'_{g',n'} \) with \( g' > 1 \) and \( n' \) even, we obtain the desired estimate. \( \square \)

5. Applications

We note the following applications to the Weil-Petersson geometry of Teichmüller space.

Restating the estimate from Theorem 1.3 we have
\[
(5.1) \quad \text{vol}(M_{\psi}) \leq 3 \sqrt{\frac{\pi}{2} |\chi(S)| \|\psi\|_{WP}}
\]
where \( \|\psi\|_{WP} \) is the translation distance of \( \psi \) in the Weil-Petersson metric on Teichmüller space. Thus the upper bound in [Br3] can be improved along the lines of (5.1).

When \( \alpha \) and \( \beta \) are a longitude and meridian pair on the punctured torus, the estimate of Theorem 1.6 gives a lower bound
\[
\frac{\gamma_8}{3\sqrt{\pi}} \leq \ell_{\alpha,\beta}.
\]
We remark that this estimate has implications for effective combinatorial models for \( \text{Teich}(S) \).
In particular, the main result of [Br2] guarantees the existence of $K_1, K_2$ depending only on $S$ so that

$$\frac{d_P(P_1, P_2)}{K_1} - K_2 \leq d_{WP}(N(P_1), N(P_2)) \leq K_1 d_P(P_1, P_2) + K_2.$$ 

Here, the distance $d_P$ is taken in the pants graph $P(S)$ whose vertices are associated to pants decompositions of $S$ and whose edges are associated to prescribed elementary moves (see [Br2], or [Br1] for an expository account) and $N(P)$ denotes the unique maximally noded Riemann surface in the boundary of Teichmüller space for which the curves in $P_i$ have been pinched to cusps.

To date, effective estimates on $K_1$ and $K_2$ have been elusive. Theorem 1.3 gives the following estimate in the case of the punctured torus $S$.

**Theorem 5.1.** Let $S$ be a one-holed torus. If $d_P(P_1, P_2) = 1$ then

$$\frac{\sqrt{8}}{3\sqrt{\pi}} \leq d_{WP}(N(P_1), N(P_2))$$

and if $d_P(P_1, P_2) > 1$ then we have

$$\frac{\sqrt{3}}{3\sqrt{2\pi}} \left( \frac{d_P(P_1, P_2)}{2} \right) \leq d_{WP}(N(P_1), N(P_2)).$$

**Proof.** The space $Teich(S)$ is naturally the unit disk $\Delta$, and edges of the usual Farey graph are geodesics in the Weil-Petersson (as well as Teichmüller) metric. Once $d_P(P_1, P_2)$ is at least 2, the Weil-Petersson geodesic $g$ joining $N(P_1)$ to $N(P_2)$ crosses a sequence edges $e_1 \ldots e_k$ in shortest path in the Farey graph joining $N(P_1)$ to $N(P_2)$. Each an interval in $g$ joining $e_i$ to $e_{i+2}$, has length at least the Weil-Petersson translation distance of $\psi$, the monodromy of the figure-8 knot complement. The lower bound then follows from Theorem 1.5. □

**References**


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