A FAMILY OF VARIETIES WITH EXACTLY ONE POINTLESS RATIONAL FIBER

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Abstract. We construct a concrete example of a 1-parameter family of smooth projective geometrically integral varieties over an open subscheme of $\mathbb{P}^1_{\mathbb{Q}}$ such that there is exactly one rational fiber with no rational points. This makes explicit a construction of Poonen.

1. Introduction

We present an explicit algebraic family where the geometry strikingly fails to control the arithmetic. Namely, we construct a family of smooth projective geometrically integral varieties over an open subscheme of $\mathbb{P}^1_{\mathbb{Q}}$ with the following curious arithmetic property: there is exactly one $\mathbb{Q}$-fiber with no rational points. This is the first example of its kind.

Theorem 1.1. *Define* $P_0(x) := (x^2-2)(3-x^2)$ *and* $P_\infty(x) := 2x^4+3x^2-1$. *Let* $\pi\colon X \to \mathbb{P}^1_{\mathbb{Q}}$ *be the Châtelet surface bundle over* $\mathbb{P}^1_{\mathbb{Q}}$ *given by*

$$y^2 + z^2 = a^2 P_0(x) + b^2 P_\infty(x),$$

where $a = 6u^2 - v^2$, $b = 12v^2$ *and* π *is projection onto* $(u : v)$. *Then* $\pi(X(\mathbb{Q})) = \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$.

Our proof makes explicit a non-effective construction of Poonen [Poo09, Prop. 7.2]. The non-effectivity in [Poo09, Prop. 7.2] stems from the use of higher genus curves and Faltings' theorem. We circumvent the use of higher genus curves by an appropriate choice of $P_\infty(x)$.

2. Background

This information can be found in [Poo09, §3,5, and 6]. We review it here for the reader's convenience.

Let $\mathcal{E}$ be a rank 3 vector sheaf on a k-variety B. A conic bundle C over B is the zero locus of a nowhere vanishing zero section $s \in \Gamma(\mathbb{P}\mathcal{E}, \mathrm{Sym}^2(\mathcal{E}))$. A diagonal conic bundle is a conic bundle where $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ and $s = \sum s_i, s_i \in \mathcal{L}_i^2$.

Now let $\alpha \in k^\times$, and let $P(x) \in k[x]$ be a separable polynomial of degree 3 or 4. Consider the diagonal conic bundle X given by $B = \mathbb{P}^1, \mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(2), s_1 = 1, s_2 = -\alpha, s_3 = -w^4 P(x/w)$. This smooth conic bundle contains the affine hypersurface $y^2 - \alpha z^2 = P(x) \subset \mathbb{A}^3$ as an open subscheme. We say that X is the Châtelet surface given by

$$y^2 - \alpha z^2 = P(x).$$

A Châtelet surface bundle is a flat proper morphism $V \to \mathbb{P}^1$ such that the generic fiber is a Châtelet surface. We can construct them in the following way. Let $P, Q \in k[x, w]$ be linearly independent homogeneous polynomials of degree 4 and let $\alpha \in k^\times$. Let V be the

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diagonal conic bundle over $\mathbb{P}^1_{(a:b)} \times \mathbb{P}^1_{(w:x)}$ given by $\mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(1,2), s_1 = 1, s_2 = -\alpha, s_3 = -(a^2 P + b^2 Q)$. By composing $V \to \mathbb{P}^1 \times \mathbb{P}^1$ with the projection onto the first factor, we realize V as a Châtelet surface bundle. We say that V is the Châtelet surface bundle given by

$$y^2 - \alpha z^2 = a^2 P(x) + b^2 Q(x),$$

where $P(x) = P(x,1)$ and $Q(x) = Q(x,1)$. Note we can also view a, b as relatively prime, homogeneous, degree d polynomials in u, v by pulling back by a suitable degree d map $\phi\colon \mathbb{P}^1_{(u:v)} \to \mathbb{P}^1_{(a:b)}$.

3. Proof of Theorem 1.1

By [Isk71], we know the Châtelet surface

$$y^2 + z^2 = (x^2 - 2)(3 - x^2)$$

has no $\mathbb{Q}$-rational points. Thus, $\pi(X(\mathbb{Q})) \subseteq \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$. Therefore, it remains to show that $X_{(u:1)}$, the Châtelet surface defined by

$$y^2 + z^2 = (6u^2 - 1)^2 P_0(x) + 12^2 P_\infty(x),$$

has a rational point for all $u \in \mathbb{Q}$.

If $P_{(u:1)} := (6u^2 - 1)^2 P_0(x) + 12^2 P_\infty(x)$ is irreducible, then by [CTSSD87a], [CTSSD87b] we know that $X_{(u:1)}$ satisfies the Hasse principle. Thus it suffices to show that $P_{(u:1)}$ is irreducible and $X_{(u:1)}(\mathbb{Q}_v) \neq \emptyset$ for all $u \in \mathbb{Q}$ and all places v of $\mathbb{Q}$.

3.1. **Irreducibility.** To prove irreducibility, we first need the following lemma.

Lemma 3.1. *Let k be a field such that* $\operatorname{char} k \neq 2$. *If $a, b, c \in k$ are such that $b^2 - 4ac$ and ac are not squares then $p(x) := ax^4 + bx^2 + c$ is irreducible in $k[x]$.*

Proof. By assumption, $b^2 - 4ac$ and ac are nonzero and therefore $p(x)$ has four distinct roots, $\pm\alpha, \pm\beta$. If $p(x)$ is reducible then it splits into two quadratic factors. Thus either α^2 or $\alpha\beta$ is in k. The former implies that $b^2 - 4ac$ is a square and the latter implies that ac is a square since $(\alpha\beta)^2 = c/a$. ☐

Instead of proving $P_{(u:1)}$ is irreducible, we prove the slightly more general statement, that

$$P_t(x) := (2x^4 + 3x^2 - 1) + t^2(x^2 - 2)(3 - x^2) = x^4(2 - t^2) + x^2(3 + 5t^2) + (-6t^2 - 1)$$

is irreducible. We will do this by using the criterion in Lemma 3.1.

Let us first check that $b^2 - 4ac$ is not a square. This is equivalent to proving that the affine curve $C\colon w^2 = t^4 + 74t^2 + 17$ has no rational points. The smooth projective model, $\overline{C}$, has 2 rational points at infinity so $\overline{C}$ is isomorphic to its Jacobian. A computation in `Magma` shows that $\operatorname{Jac}(C)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ [BCP97]. Therefore, the points at infinity are the only 2 rational points of $\overline{C}$ and thus C has no rational points.

Now we will show that $(-6t^2 - 1)(2 - t^2)$ is not a square for any t. As above, this is equivalent to determining whether $C'\colon w^2 = (-6t^2 - 1)(2 - t^2)$ has a rational point. Since 6 is not a square, this is equivalent to determining whether the smooth projective model, $\overline{C'}$, has a rational point. The curve $\overline{C'}$ is a genus 1 curve so it is either isomorphic to its Jacobian or has no rational points. A computation in `Magma` shows that $\operatorname{Jac}(C')(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ [BCP97]. Thus $\#C'(\mathbb{Q}) = 0$ or 2. If (t, w) is a rational point of C', then $(\pm t, \pm w)$ is also a rational point. Therefore, $\#C(\mathbb{Q}) = 2$ if and only if there is a point with $t = 0$ or $w = 0$ and one can easily check that this is not the case.

3.2. **Local Solvability.**

Lemma 3.2. *For any point $(u : v) \in \mathbb{P}^1_{\mathbb{Q}}$, the Châtelet surface $X_{(u:v)}$ has $\mathbb{R}$-points and $\mathbb{Q}_p$-points for every prime p.*

Proof. Recall that $a = 6u^2 - v^2$ and $b = 12v^2$. We will refer to $a^2P_0(x) + b^2P_\infty(x)$ both as $P_{(a:b)}$ and $P_{(u:v)}$.

$\mathbb{R}$-points: It suffices to show that given $(u : v)$ there exists an x such that

$$P_{(a:b)} = x^4(2b^2 - a^2) + x^2(3b^2 + 5a^2) + (-6a^2 - b^2)$$

is positive. If $2b^2 - a^2$ is positive, then any x sufficiently large will work. So assume $2b^2 - a^2$ is negative. Then $\alpha = \frac{-(3b^2+5a^2)}{2(2b^2-a^2)}$ is positive. We claim $P_{(a:b)}(\sqrt{\alpha})$ is positive.

$$\begin{aligned}
P_{(a:b)}(\sqrt{\alpha}) &= \alpha^2(2b^2 - a^2) + \alpha(3b^2 + 5a^2) + (-6a^2 - b^2) \\
&= \frac{(3b^2 + 5a^2)^2}{4(2b^2 - a^2)} + \frac{-(3b^2 + 5a^2)^2}{2(2b^2 - a^2)} + (-6a^2 - b^2) \\
&= \frac{1}{4(2b^2 - a^2)} \left(4(2b^2 - a^2)(-6a^2 - b^2) - (3b^2 + 5a^2)^2\right) \\
&= \frac{1}{4(2b^2 - a^2)} \left(-17b^4 - 74a^2b^2 - a^4\right)
\end{aligned}$$

Since $2b^2 - a^2$ is negative by assumption and $-17b^4 - 74a^2b^2 - a^4$ is always negative, we have our result.

$\mathbb{Q}_p$-points:

- **$p \geq 5$:** Without loss of generality, let a and b be relatively prime integers. Let $\overline{X}_{(a:b)}$ denote the reduction of $X_{(a:b)}$ modulo p. Since $P_{(a:b)}$ is degree at most 4, there is some $x \in \mathbb{F}_p$ such that $P_{(a:b)}(x)$ is nonzero. For this value of x, the equation $y^2 + z^2 = P_{(a:b)}(x)$ defines a smooth rational affine curve which has at least 1 rational point. Using Hensel's lemma, we can lift this point to a $\mathbb{Q}_p$-point.
- **$p = 3$:** From the equations for a and b, one can check that for any $(u : v) \in \mathbb{P}^1_{\mathbb{Q}}$, $v_3(b/a)$ is positive. Since $\mathbb{Q}_3(\sqrt{-1})/\mathbb{Q}_3$ is an unramified extension, it suffices to show that given a, b as above, there exists an x such that $P_{(a:b)}(x)$ has even valuation. Since $v_3(b/a)$ is positive, $v_3(2b^2 - a^2) = 2v_3(a)$. Therefore, if $x = 3^{-n}$, for n sufficiently large, the valuation of $P_{(a:b)}(x)$ is $-4n + 2v_3(a)$ which is even.
- **$p = 2$:** From the equations for a and b, one can check that for any $(u : v) \in \mathbb{P}^1_{\mathbb{Q}}$, $v_2(b/a)$ is at least 2. Let $x = 0$ and $y = a$. Then we need to find a solution to $z^2 = a^2(-7 + (b/a)^2)$. Since $v_2(b/a) > 1$, $-7 + (b/a)^2 \equiv 1^2 \mod 8$. By Hensel's lemma, we can lift this to a solution in $\mathbb{Q}_2$.

□

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References

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). MR 1484478 ↑3.1

[CTSSD87a] Jean-Louis Colliot-Thélène, Jean-Jacques Sansuc, and Peter Swinnerton-Dyer, *Intersections of two quadrics and Châtelet surfaces. I*, J. Reine Angew. Math. **373** (1987), 37–107. MR **870307 (88m:**11045a) ↑3

[CTSSD87b] ———, *Intersections of two quadrics and Châtelet surfaces. II*, J. Reine Angew. Math. **374** (1987), 72–168. MR **876222 (88m:**11045b) ↑3

[Isk71] V. A. Iskovskih, *A counterexample to the Hasse principle for systems of two quadratic forms in five variables*, Mat. Zametki **10** (1971), 253–257 (Russian). MR 0286743 (44 #3952) ↑3

[Poo09] Bjorn Poonen, *Existence of rational points on smooth projective varieties*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 3, 529–543. MR 2505440 ↑1, 2

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