J. Fixed Point Theory Appl. 11 (2012) 225–243
DOI 10.1007/s11784-012-0090-3
Published online October 18, 2012

Journal of Fixed Point Theory and Applications

Monotonicity properties of harmonic maps into NPC spaces

Georgios Daskalopoulos and Chikako Mese

To Dick Palais on his 80th birthday

Abstract. In this paper, we study monotonicity properties of harmonic maps into general NPC spaces. In addition, we introduce the notion of Alexandrov tangent maps and state a criterion for uniqueness.

Mathematics Subject Classification. 58E20.

Keywords. Harmonic maps, monotonicity, Alexandrov tangent map.

1. Introduction

In this article, we present results about harmonic maps into NPC (non-positively curved) spaces. The theory of harmonic maps to singular spaces originated with the work of Gromov and Schoen [GS] and was subsequently extended in [KS1], [KS2] and [Jo]. The main regularity result in [GS] and [KS1] is to show that harmonic maps are Lipschitz continuous. In [GS], this is achieved by proving the monotonicity property of the order function (sometimes also called the frequency function) associated with a harmonic map. Recall that for a harmonic function $u : \Omega \to \mathbb{R}$, the value of the order function $\mathrm{Ord}^u(x)$ is the order with which u attains its value $u(x)$ at x. In [KS1], a different proof of the Lipschitz continuity is given that is independent of the order function.

In Section 2, we outline the approach of [GS] and sketch the proof of the monotonicity of the order and the Lipschitz continuity property for harmonic maps (cf. Theorems 2 and 3). Moreover, by following [KS1] and [KS2], we discuss blowup maps and tangent maps for harmonic maps into arbitrary NPC spaces (cf. Theorem 8 and Definition 9).

In Section 3, we discuss other quantities associated with harmonic maps into NPC spaces that satisfy monotonicity formulas (cf. Lemma 11). In the case when the target space is a tree, these formulas are due to Caffarelli–Lin [CL]. We also give a characterization of homogeneous maps in terms of these quantities (cf. Lemma 12).

In Section 4, we introduce the notion of Alexandrov tangent maps, which is the main contribution of this article. Given an arbitrary NPC space Y and a point $P_0 \in Y$, there is the well-known notion of Alexandrov tangent cone $T_{P_0}Y$. For a harmonic map $u : \Omega \to Y$ with $u(x_0) = P_0$, the tangent map $u_* : \mathbb{R}^n \to Y_*$ maps into a space Y_* that may be different from $T_{P_0}Y$. In this section, we show that the Alexandrov tangent map v_* is a homogeneous map of degree $\alpha = \mathrm{Ord}^u(x_0)$ and furthermore it induces the same metric as u_* (cf. Proposition 14 and Theorem 15). We also give a criterion for when the notions of tangent map and Alexandrov tangent map agree (cf. Proposition 17).

Finally in Section 5, we describe results of Caffarelli and Lin on uniqueness of tangent maps (cf. [L]). We prove these results for Alexandrov tangent maps into arbitrary NPC spaces whose tangent cones are locally compact rather than trees, which is the case of the original work of Caffarelli and Lin. The main results are Proposition 20, where it is shown that condition (16) is sufficient for uniqueness of tangent maps, and Example 22, where it is shown that uniqueness is equivalent to condition (16) for the case of trees. We also discuss special cases where condition (16) is satisfied.

2. Definitions

Recall that a metric space (Y, d) is called NPC space if

(i) The space (Y, d) is a length space. That is, for any two points P and Q in Y, there exists a rectifiable curve c so that the length of c is equal to $d(P, Q)$. We call such distance-realizing curve a geodesic.

(ii) For any three points $P, R, Q \in Y$, let $c : [0, l] \to Y$ be the arclength parameterized geodesic from Q to R and let $Q_t = c(tl)$ for $t \in [0, 1]$. Then

$$d^2(P, Q_t) \leq (1 - t)d^2(P, Q) + td^2(P, R) - t(1 - t)d^2(Q, R).$$

Let Y be an NPC space, $Q, R \in Y$ and $t \in [0, 1]$. In what follows, we use the notation

$$(1 - t)Q + tR \tag{1}$$

to denote the point that is at distance $td(Q, R)$ from Q and at distance $(1 - t)d(Q, R)$ from R. Since (ii) implies the uniqueness of geodesic between Q and R, this point is well defined.

We next define the notion of energy of a map to a metric space (Y, d). Let Ω be a smooth bounded n-dimensional Riemannian domain. A map $f : \Omega \to Y$ is said to be an L^2-map (or that $f \in L^2(\Omega, Y)$) if for some (and hence all) $P \in Y$, we have

$$\int_\Omega d^2(f(x), P)d\mu < \infty.$$

For $f \in L^2(\Omega, Y)$, define $e_\epsilon^f : \Omega \to \mathbb{R}$ by

$$e_\epsilon^f(x) = \begin{cases} \displaystyle\int_{y\in\partial B_\epsilon(x)} \frac{d^2(f(x), f(y))}{\epsilon^2} \, d\Sigma & \text{for } x \in \Omega - N_\epsilon(\partial\Omega), \\ 0 & \text{for } x \in N_\epsilon(\partial\Omega), \end{cases}$$

where $N_\epsilon(\partial\Omega) = \{x \in \Omega : d(x, \partial\Omega) < \epsilon\}$. Define a family of functionals

$$E_\epsilon^f : C_c(\Omega) \to \mathbb{R}$$

by setting

$$E_\epsilon^f(\varphi) = \int_\Omega \varphi e_\epsilon^f \, d\mu.$$

We say that f has finite energy (or that $f \in W^{1,2}(\Omega, Y)$) if

$$E^f := \sup_{\varphi \in C_c(\Omega), 0 \leq \varphi \leq 1} \limsup_{\epsilon \to 0} E_\epsilon^f(\varphi) < \infty.$$

It can be shown (cf. [KS1]) that if f has finite energy, then the measures $e_\epsilon^f(x)d\mu$ converge to a measure absolutely continuous with respect to the Lebesgue measure. Therefore, there exists a function $e^f(x)$, called the energy density, such that $e_\epsilon^f(x)d\mu \rightharpoonup e^f(x)d\mu$. In analogy to the case of real-valued functions, we write $|\nabla f|^2(x)$ in place of $e^f(x)$. In particular,

$$E^f = \int_\Omega |\nabla f|^2 d\mu.$$

If $f \in W^{1,2}(\Omega, Y)$, then there exists a well-defined notion of a trace of f, denoted $\mathrm{Tr}(f)$, which is an element of $L^2(\partial\Omega, Y)$. Two maps $f, g \in W^{1,2}(\Omega, Y)$ have the same trace (i.e., $\mathrm{Tr}(f) = \mathrm{Tr}(g)$) if and only if $d(f, g) \in W_0^{1,2}(\Omega)$. For details we refer the reader to [KS1].

In what follows, given $x \in \Omega$ and a finite energy map $f : \Omega \to Y$, we will use the following notation:

$$E_x^f(r) := \int_{B_r(x)} |\nabla f|^2 d\mu \quad \text{and} \quad I_x^f(r) := \int_{\partial B_r(x)} d^2(f, f(x)) \, d\Sigma.$$

Sometimes we may omit the subscripts x or/and f from the above notation if they are clear from the context.

Definition 1. A $W^{1,2}$-map $u : \Omega \to Y$ to an NPC space (Y, d) is said to be *harmonic* or an *energy minimizer* if for any $x \in \Omega$ and any geodesic ball $B_r(x) \subset \Omega$, the restriction $f\big|_{B_r(x)}$ is energy minimizing among all $W^{1,2}$-maps with the same trace.

The following regularity theorem is due to Gromov and Schoen [GS] and Korevaar and Schoen [KS1].

Theorem 2. *A harmonic map $u : \Omega \to Y$ to an NPC space (Y, d) is locally Lipschitz continuous with the local Lipschitz constant dependent only on the energy of u, the dimension of Ω, the domain metric g and the distance to the boundary of Ω.*

The key to proving Theorem 2 is the existence of an *order function*, also called a *frequency function* (cf. [GS, Part I, Section 2]).

Theorem 3. *Let $u : \Omega \to Y$ be a harmonic map to an NPC space (Y, d). There exists a constant $c > 0$ depending only on the domain metric g (in particular, $c = 0$ when Ω is Euclidean) such that*

$$r \mapsto \mathrm{Ord}^u(x, r) := e^{cr^2} \frac{r\ E^u_x(r)}{I^u_x(r)}$$

is nondecreasing for any $x \in \Omega$.

As a nonincreasing limit of continuous functions,

$$\mathrm{Ord}^u(x) := \lim_{r \to 0} \mathrm{Ord}^u(x, r)$$

is an upper semicontinuous function. By following the proof of [GS, Theorem 2.3], we have that $\mathrm{Ord}^u(x) \geq 1$. The value $\mathrm{Ord}^u(x)$ is called the *order* of u at x. Note that for a harmonic function $u : \Omega \to \mathbb{R}$, $\mathrm{Ord}^u(x)$ is the order with which u attains its value $u(x)$ at x. Alternatively, it is the degree of the dominant homogeneous harmonic polynomial which approximates $u - u(x)$ near x.

The proof of Theorem 3 is based on the following two lemmas which are related to the first variation formulas for target and domain variations of classical harmonic maps between Riemannian manifolds.

Lemma 4 (cf. [GS]**).** *If $u : (B_2(0), g) \to Y$ is a harmonic map into an NPC space (Y, d), then*

$$2\ E^u(r) \leq \int_{\partial B_r(0)} \frac{\partial}{\partial r} d^2(u, P_0)\, d\mu$$

for any $P_0 \in Y$.

Proof. This is proved by using the fact that $E^u(r) \leq E^{u_\eta}(r)$, where $\eta \in C_c(B_r(0))$ and

$$u_\eta(x) = (1 - \eta)u(x) + \eta P_0.$$

For details, follow the proof in [GS] where this lemma is proved for the special case when Y is an NPC Riemannian simplicial complex. We can use [KS1, Theorem 2.3.2] to justify various steps for the general case when Y is simply an NPC space. □

Lemma 5 (cf. [GS]**).** *If $u : (B_2(0), g) \to Y$ is a harmonic map into an NPC space (Y, d), then*

$$\frac{d}{dr} E^u(r) = \frac{n - 2 + O(r^2)}{r} E^u(r) + 2 \int_{\partial B_r(0)} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma,$$

where $|O(r^2)| \leq Cr^2$ with C depending only on the domain metric g. In particular, if the domain metric is Euclidean, then we can set $O(r^2) = 0$.

Proof. This is proved by a standard computation using the fact that

$$\frac{d}{dt}E^{u_t}(r)\Big|_{r=0} = 0,$$

where $u_t = u \circ F_t$. Here, F_t is a smooth one-parameter family of diffeomorphism of the domain with F_0 the identity map and $F_t = F_0$ in a neighborhood of $\partial B_r(0)$. By using [KS1, Theorem 2.3.2] to justify various steps, we can follow the arguments in [GS]. □

Proof of Theorem 3. Recall the standard identity (cf. [GS, p. 193])

$$\frac{d}{dr}I^u(r) = \int_{\partial B_r(x_0)} \frac{\partial}{\partial r} d^2(v, u(0))\, d\Sigma + \frac{n-1}{r} I^u(r) + O(r^2) I^u(r), \tag{2}$$

where $O(r^2)$ is as in Lemma 5. Combining this with Lemma 5, we obtain

$$\begin{aligned}
&\frac{\frac{d}{dr}E^u(r)}{E^u(r)} - \frac{\frac{d}{dr}I^u(r)}{I^u(r)} + \frac{1}{r} \\
&\qquad = (E^u(r)I^u(r))^{-1}\Bigg(I^u(r)\int_{\partial B_r(0)} \left|\frac{\partial u}{\partial r}\right|^2 d\Sigma \\
&\qquad\qquad\qquad - E^u(r)\int_{\partial B_r(0)} \frac{\partial}{\partial r} d^2(u, u(0))\, d\Sigma \Bigg) + O(r).
\end{aligned}$$

Thus, Lemma 4 and the Hölder inequality imply the monotonicity formula

$$\begin{aligned}
&\frac{d}{dr}\left(e^{cr^2} \frac{rE^u(r)}{I^u(r)} \right) \\
&\quad \geq \frac{\left(I^u(r) \int_{\partial B_r(0)} \left|\frac{\partial u}{\partial r}\right|^2 - \left(\int_{\partial B_r(0)} \frac{\partial}{\partial r} d^2(u, u(0))\, d\Sigma \right)^2 \right)}{E^u(r)I^u(r)} \\
&\quad \geq \frac{\left(\int_{\partial B_r(0)} \left|\frac{\partial u}{\partial r}\right|^2 d\Sigma - \int_{\partial B_r(0)} \left|\frac{\partial}{\partial r} d(u, u(0))\right|^2 d\Sigma \right)}{E^u(r)} \\
&\quad \geq 0
\end{aligned} \tag{3}$$

for some constant c depending only on the second derivative estimates of the domain metric g. Here, we are also using the triangle inequality

$$d(u(x + \epsilon\partial_r), u(x)) \geq d(u(x + \epsilon\partial_r), u(0)) - d(u(x), u(0)), \tag{4}$$

where ∂_r is the radial coordinate vector (of the normal coordinates centered at $x = 0$) and $x + \epsilon\partial_r$ denotes the point obtained by flowing for time ϵ along ∂_r starting at x. This implies

$$\left|\frac{\partial u}{\partial r}\right|^2 \geq \left|\frac{\partial}{\partial r} d(u, u(0))\right|^2. \tag{5}$$

□

For any finite energy map $u : \Omega \to Y$ to an NPC space (Y, d), $x \in \Omega$ and $\alpha > 0$, we set

$$\mathcal{I}_x^u(r) := \frac{I_x^u(r)}{r^{n-1+2\alpha}} \quad \text{and} \quad \mathcal{E}_x^u(r) := \frac{E_x^u(r)}{r^{n-2+2\alpha}}. \tag{6}$$

Analogously with $E_x^u(r)$ and $I_x^u(r)$, we may omit the subscripts x or/and u from the above notation.

The following corollary, also due to Gromov and Schoen (cf. [GS, Part I, Section 2]), gives the monotonicity properties of the functions $\mathcal{I}_x^u(r)$ and $\mathcal{E}_x^u(r)$ which plays a central role in this paper.

Corollary 6. *Let $u : \Omega \to Y$ be a harmonic map to an NPC space (Y, d) and $x \in \Omega$. There exists a constant $c > 0$ depending only on the domain metric g (in particular, $c = 0$ when Ω is Euclidean) such that*

$$\frac{d}{dr}\left(e^{cr^2}\mathcal{I}(r)\right) \geq 0 \quad \textit{and} \quad \frac{d}{dr}\left(e^{cr^2}\mathcal{E}(r)\right) \geq 0.$$

We now describe homogeneous approximations for harmonic maps into NPC spaces. We begin with defining a notion of homogeneity in this setting.

Definition 7. A map $v : \overline{B_1(0)} \subset \mathbb{R}^n \to T$ into an NPC space (T, δ) is said to be *homogeneous of order* α if, for every $x \in B_1(0)$, $v(x)$ is contained in a geodesic between $v(0)$ and $v\left(\frac{x}{|x|}\right)$ and

$$\delta(v(x), v(0)) = |x|^\alpha \delta\left(v\left(\frac{x}{|x|}\right), v(0)\right).$$

Let $u : \Omega \to (Y, d)$ be a harmonic map from a Riemannian domain into an NPC space and $x \in \Omega$. First, identify $x = 0$ via normal coordinates. By rescaling the domain if necessary, we can assume that $B_2(0) \subset \Omega$. Set

$$\mu(\sigma) = \sqrt{\frac{I^u(\sigma)}{\sigma^{n-1}}}.$$

By Corollary 6, we have

$$\lim_{\sigma \to 0} \mu(\sigma) = \sigma^{-\alpha}\sqrt{\mathcal{I}^u(r)} = 0. \tag{7}$$

For $\sigma > 0$, the metric space $(Y, \mu^{-1}(\sigma)d)$ is still an NPC space. Since we are in normal coordinates, $g_0 := g(0)$ is the Euclidean metric given by δ_{ij} in local coordinates. We will denote the volume form of $B_r(0)$ and of $\partial B_r(0)$ with respect to g_0 by μ_0 and Σ_0, respectively. Define the metric g_σ by setting $g_\sigma(x) = g(\sigma x)$ and let

$$u_\sigma : (B_1(0), g_\sigma) \to (Y, \mu^{-1}(\sigma)d =: d_\sigma)$$

by setting

$$u_\sigma(x) = u(\sigma x).$$

The one-parameter family of harmonic maps $\{u_\sigma\}$ is called the *blowup maps of* u. The normalization by $\mu(\sigma)$ gives the property that

$$I^{u_\sigma}(1) = 1 \tag{8}$$

and

$$\lim_{\sigma\to 0}\frac{E^{u_\sigma}(1)}{I^{u_\sigma}(1)}=\frac{\sigma E^u(\sigma)}{I^u(\sigma)}=\alpha. \tag{9}$$

Furthermore, the monotonicity formula in Theorem 3 implies

$$E^{u_\sigma}(1)\le 2\alpha \quad \text{for } \sigma>0 \text{ sufficiently small.}$$

In particular, by Theorem 2, the family $\{u_\sigma\}$ is uniformly Lipschitz in $B_\sigma(0)$ for any $\sigma\in(0,1)$. By [KS1, Proposition 3.7] and [GS, Lemma 3.2], we obtain the following theorem.

Theorem 8. *Let $u:(B_2(0),g)\to(Y,d)$ be a harmonic map into an NPC space (Y,d) and let $\{u_\sigma\}$ be the blowup maps of u. Given any sequence $\sigma_i\to 0$, there exists a subsequence (denoted again by σ_i by a slight abuse of notation) such that u_{σ_i} converges locally uniformly in the pullback sense (cf.* [KS2, Definition 3.3]*) to a map $u_*:B_1(0)\to Y_*$ into an NPC space (Y_*,d_*). The map u_* is harmonic and homogeneous of order α.*

Definition 9. The map u_* defined in Theorem 8 is called a *tangent map of u* at $x=0$.

In general, the geometry of the space (Y_*,d_*) may be very different from the local geometry of (Y,d).

3. Deviation from homogeneity

In this section, we introduce and study quantities that indicate the deviation of a map from being a homogeneous map. For the special case when Y is a tree, these formulas were first studied by Caffarelli and Lin [CL]. Throughout this section, we denote by C a generic constant depending on the domain metric g.

Definition 10. Assume $u:(B_1(0),g)\to Y$ is a harmonic map and the order of u at 0 is equal to α. Define

$$\begin{aligned}
F^u(r) &:= E^u(r)-\frac{\alpha}{r}I^u(r),\\
\Delta^u(r) &:= \int_{\partial B_r(0)}\frac{\partial}{\partial r}d^2(u,u(0))\,d\Sigma-2E^u(r),\\
R^u(r) &:= \int_{\partial B_r(0)}\left(r\frac{\partial}{\partial r}d(u,u(0))-\alpha d(u,u(0))\right)^2 d\Sigma.
\end{aligned}$$

Furthermore, we define

$$\begin{aligned}
\mathcal{F}^u(r) &:= \frac{F^u(r)}{r^{n-2+2\alpha}}=\mathcal{E}^u(r)-\frac{\alpha}{r}\mathcal{I}^u(r),\\
\mathcal{D}^u(r) &:= \frac{\alpha\Delta^u(r)}{r^{n-2+2\alpha}},\\
\mathcal{R}^u(r) &:= \frac{2R^u(u)}{r^{n-1+2\alpha}}.
\end{aligned}$$

Remark 1. We note the following positivity properties of the quantities in Definition 10. First, clearly we have

$$R^u(r),\ \mathcal{R}^u(r) \geq 0.$$

Additionally, by Lemma 4,

$$\Delta^u(r),\ \mathcal{D}^u(r) \geq 0.$$

Finally, if g is the Euclidean metric g_0, then by Theorem 3,

$$F^u(r),\ \mathcal{F}^u(r) \geq 0.$$

We now record the following relationship between $F^u(r)$, $\Delta^u(r)$, $R^u(r)$ and $I^u(r)$.

Lemma 11. *If $u : (B_2(0), g) \to Y$ is a harmonic map into an NPC space (Y, d) with* $\mathrm{Ord}^u(0) = \alpha$*, then*

$$\frac{2\mathcal{F}^u(r)}{r} + \frac{\mathcal{D}^u(r)}{r} = \frac{d}{dr}\mathcal{I}^u(r) + O(r)\mathcal{I}^u(r) \tag{10}$$

and

$$0 \leq \frac{\mathcal{R}^u(r)}{r} + \frac{\mathcal{D}^u(r)}{r} \leq \frac{d}{dr}\mathcal{F}^u(r) + O(r)(\mathcal{E}^u(r) + \mathcal{I}^u(r)). \tag{11}$$

Here, we have that $|O(r)| \leq Cr$. If g is the Euclidean metric g_0, then we can set $O(r) = 0$.

Proof. By (2), we have

$$\begin{aligned}
&\frac{d}{dr}\left(\frac{I^u(r)}{r^{n-1+2\alpha}}\right)\\
&\quad= \frac{1}{r^{n-1+2\alpha}}\int_{\partial B_r(0)} \frac{\partial}{\partial r} d^2(u, u(0))\, d\Sigma - \frac{2\alpha + O(r^2)}{r}\frac{I^u(r)}{r^{n-1+2\alpha}}\\
&\quad= \frac{1}{r^{n-1+2\alpha}}\left(2E^u(r) + D^u(r)\right) - \frac{2\alpha + O(r^2)}{r}\frac{I^u(r)}{r^{n-1+2\alpha}}\\
&\quad= \frac{2}{r}\left(\frac{E^u(r)}{r^{n-2+2\alpha}} - \alpha\frac{I^u(r)}{r^{n-1+2\alpha}}\right) + \frac{1}{r}\left(\frac{D^u(r)}{r^{n-2+2\alpha}}\right) - O(r)\frac{I^u(r)}{r^{n-1+2\alpha}}.
\end{aligned}$$

This proves (10). To prove (11), we use Lemma 5 to compute

$$\begin{aligned}
\frac{d}{dr}\left(\frac{E^u(r)}{r^{n-2+2\alpha}}\right) &= \frac{\frac{d}{dr}E^u(r)}{r^{n-2+2\alpha}} - \frac{(n-2+2\alpha)E^u(r)}{r^{n-1+2\alpha}}\\
&= \frac{2}{r^{n+2\alpha}}\left(r^2\int_{\partial B_r(0)}\left|\frac{\partial u}{\partial r}\right|^2 d\Sigma - r\alpha E^u(r)\right) - O(r)\frac{E^u(r)}{r^{n-2+2\alpha}}.
\end{aligned}$$

By combining with (2),

$$
\begin{aligned}
&\frac{d}{dr}\left(\frac{F^u(r)}{r^{n-2+2\alpha}}\right)\\
&= \frac{d}{dr}\left(\frac{E^u(r)}{r^{n-2+2\alpha}} - \alpha\frac{I^u(r)}{r^{n-1+2\alpha}}\right)\\
&= \frac{2}{r^{n+2\alpha}}\left(r^2\int_{\partial B_r(0)}\left|\frac{\partial u}{\partial r}\right|^2 d\Sigma - r\alpha E^u(r)\right.\\
&\qquad\qquad \left. -\frac{r\alpha}{2}\int_{\partial B_r(0)}\frac{\partial}{\partial r}d^2(u,u(0))\,d\Sigma + \alpha^2 I^u(r)\right)\\
&\quad - O(r)\left(\frac{E^u(r)}{r^{n-2+2\alpha}} + \frac{I^u(r)}{r^{n-1+2\alpha}}\right)\\
&= \frac{2}{r^{n+2\alpha}}\left(r^2\int_{\partial B_r(0)}\left|\frac{\partial u}{\partial r}\right|^2 d\Sigma - r\alpha\int_{\partial B_r(0)}\frac{\partial}{\partial r}d^2(u,u(0))\,d\Sigma + \alpha^2 I^u(r)\right)\\
&\quad + \frac{\alpha\Delta^u(r)}{r^{n-1+2\alpha}} - O(r)\left(\frac{E^u(r)}{r^{n-2+2\alpha}} + \frac{I^u(r)}{r^{n-1+2\alpha}}\right)\\
&\geq \frac{2}{r^{n+2\alpha}}\left(\int_{\partial B_r(0)} r^2\left(\frac{\partial}{\partial r}d(u,u(0))\right)^2\right.\\
&\qquad\qquad \left. - 2r\alpha d(u,u(0))\frac{\partial}{\partial r}d(u,u(0)) + \alpha^2 d^2(u,u(0))\,d\Sigma\right)\\
&\quad + \frac{\alpha\Delta^u(r)}{r^{n-1+2\alpha}} - Cr\left(\frac{E^u(r)}{r^{n-2+2\alpha}} + \frac{I^u(r)}{r^{n-1+2\alpha}}\right) \qquad \text{by (5)}\\
&= \frac{2}{r^{n+2\alpha}}\int_{\partial B_r(0)}\left(r\frac{\partial}{\partial r}d(u,u(0)) - \alpha d(u,u(0))\right)^2 d\Sigma + \frac{\alpha\Delta^u(r)}{r^{n-1+2\alpha}}\\
&\quad - Cr\left(\frac{E^u(r)}{r^{n-2+2\alpha}} + \frac{I^u(r)}{r^{n-1+2\alpha}}\right).
\end{aligned}
$$

The fact that the first two terms on the right-hand side are both positive proves (11). The assertion of the lemma regarding dependence of the constant C on the domain metric follows immediately from Lemma 5. □

Lemma 12. *Assume that $u : (B_1(0), g_0) \to (Y, d)$ is a harmonic map with* $\mathrm{Ord}^u(0) = \alpha$. *The following statements are equivalent:*

- (i) *u is homogeneous;*
- (ii) *$F^u(r) = 0$ for all $r \in (0,1)$;*
- (iii) *$R^u(r) = 0$ and $\Delta^u(r) = 0$ for all $r \in (0,1)$.*

Proof. The direction (ii) ⇒ (i) follows as in [GS, Lemma 3.2] by replacing the NPC Riemannian simplicial complex with an arbitrary NPC space Y and justifying the various steps using [KS1]. To show (i) ⇒ (ii), note that

$r\frac{\partial}{\partial r}d(u,u(0)) = \alpha d(u,u(0))$ by definition of homogeneity. Thus,

$$\begin{aligned}
&\frac{d}{dr}\mathcal{I}^u(r) \\
&= \frac{d}{dr}\left(\frac{I^u(r)}{r^{n-1+2\alpha}}\right) \\
&= \frac{2}{r^{n+2\alpha}}\left(r\int_{\partial B_r(0)} d(u,u(0))\frac{\partial}{\partial r}d(u,u(0))\,d\Sigma - \alpha\int_{\partial B_r(0)} d^2(u,u(0))\,d\Sigma\right) \\
&= 0.
\end{aligned}$$

The positivity properties (cf. Remark 1) and Lemma 11 (10) (noting that $O(r) = 0$ since the domain metric is Euclidean) imply that $F^u(r) = 0$. Thus, we have shown that (i) is equivalent to (ii). The implication (ii) $\Rightarrow$ (iii) follows immediately from Lemma 11 (11). To show (iii) $\Rightarrow$ (ii), note that $\Delta^u(r) = 0$ implies an equality in the first inequality of (3) (where Lemma 4 is applied). Furthermore, $R^u(r) = 0$ implies $\frac{\partial}{\partial r}d(u,u(0)) = \frac{\alpha}{r}d(u,u(0))$ which in turn implies an equality in the second (i.e., where the Hölder inequality is applied). Here, note that $c = 0$ and that there is no $O(r)$ term in (3) since the domain metric g_0 is Euclidean. Thus, $\frac{d}{dr}\operatorname{Ord}^u(r) = 0$ which implies $\operatorname{Ord}^u(r) = \alpha$ which in turn implies $F^u(r) = 0$. Thus, we have shown that (ii) is equivalent to (iii). □

4. Alexandrov tangent maps

We now recall the notion of the Alexandrov tangent cone of an NPC metric space (Y,d). Let $\Gamma_{P_0}Y$ be the set of all arclength parameterized geodesic rays $\gamma : [0,\infty) \to Y$ emanating from the point $P_0 = \gamma(0)$. For $\gamma_1,\gamma_2 \in \Gamma_{P_0}Y$, define $\theta(\gamma_1,\gamma_2)$ to be the angle between the two geodesics at P_0; i.e.,

$$\theta(\gamma_1,\gamma_2) = \lim_{t,s\to 0}\arccos\left(\frac{t^2+s^2-d^2(\gamma_1(t),\gamma_2(s))}{2ts}\right).$$

Define an equivalence class in $\Gamma_{P_0}Y$ by letting

$$\gamma_1 \sim \gamma_2 \quad \text{if and only if} \quad \theta(\gamma_1,\gamma_2) = 0.$$

Let $S_{P_0}Y$ be the metric completion of the space of equivalence classes $[\gamma]$ of $\Gamma_{P_0}Y$ with respect to the distance function $\Theta(\cdot,\cdot)$ defined by

$$\Theta([\gamma_1],[\gamma_2]) = \theta(\gamma_1,\gamma_2).$$

The tangent cone of Y at P_0 is the space

$$T_{P_0}Y = [0,\infty) \times S_{P_0}Y/\sim',$$

where $\sim'$ identifies all points of the form $(0,[\gamma])$ as the vertex $\mathcal{P}_0$. We define a distance function $\delta(\cdot,\cdot)$ on $T_{P_0}Y$ by

$$\delta((\rho_1,[\gamma_1]),(\rho_2,[\gamma_2])) = \rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\Theta([\gamma_1],[\gamma_2]).$$

There is a natural projection map $\log_{P_0} : Y \to T_{P_0}Y$ defined by setting

$$\log_{P_0}(P) = (d(P,P_0),[\gamma^P]),$$

where γ^P is a geodesic ray emanating from P_0 and that goes through P.

Let u_{σ_i} be a sequence of blowup maps converging locally uniformly in the pullback sense to u_*. Since $\log_{P_0}$ is distance nonincreasing, $\log_{P_0} \circ u_{\sigma_i}$ is a sequence of maps with a uniform Lipschitz bound. Thus, there exists a subsequence of $\log_{P_0} \circ u_{\sigma_i}$ converging locally uniformly in the pullback sense to a map $v_* : \mathbb{R}^n \to (T_*, \delta_*)$ into an NPC space (cf. [KS1, Proposition 3.7]).

Definition 13. The map $v_* : \mathbb{R}^n \to (T_*, \delta_*)$ as above is said to be an Alexandrov tangent map of u at 0.

Proposition 14. *Let $u : B_2(0) \to Y$ be a harmonic map into an NPC space (Y, d) with $\mathrm{Ord}^u(0) = \alpha$. Then any Alexandrov tangent map $v_* : \mathbb{R}^n \to (T_*, \delta_*)$ of u at 0 is homogeneous of degree α.*

Proof. Let $\sigma_i \to 0$ such that u_{σ_i} and $\log_{P_0} \circ u_{\sigma_i}$ converge locally uniformly in the pullback sense to a tangent map u_* and v_*, respectively. Since $\log_{P_0}$ preserves radial directions, we have that $\delta_*(\log_{P_0} \circ u_{\sigma_i}, \mathcal{P}_0) = d_*(u_{\sigma_i}, P_0)$ which then implies $\delta_*(v_*, v_*(0)) = d_*(u_*, u_*(0))$. Thus, the homogeneity of u_* implies that for $x \in \partial B_1(0)$ and $t \in [0, 1]$,

$$\begin{aligned} \delta_*(v_*(tx), v_*(0)) &= d_*(u_*(tx), u_*(0)) \\ &= t^\alpha d_*(u_*(x), u_*(0)) \\ &= t^\alpha \delta_*(v_*(x), v_*(0)). \end{aligned} \tag{12}$$

Furthermore,

$$\begin{aligned} \delta_*(v_*(tx), v_*(x)) &= \lim_{i\to\infty} \delta_{\sigma_i}(v_{\sigma_i}(tx), v_{\sigma_i}(x)) \\ &\le \lim_{i\to\infty} d_{\sigma_i}(u_{\sigma_i}(tx), u_{\sigma_i}(x)) \\ &= d_*(u_*(tx), u_*(x)) \\ &= (1 - t^\alpha) d_*(u_*(x), u_*(0)) \\ &= (1 - t^\alpha) d_*(v_*(x), v_*(0)). \end{aligned} \tag{13}$$

Equations (12) and (13) imply that $v_*(tx)$ is the point on the geodesic from $v_*(0)$ to $v_*(x)$ at distance $t^\alpha d_*(v_*(x), v_*(0))$ from $v_*(0)$. In turn, this implies the homogeneity of v_*. □

Remark 2. In the above, we have not answered whether Alexandrov tangent maps are energy minimizers. This is an important question.

Let $u : B_2(0) \to Y$ be a finite energy map into an NPC space (Y, d). Recall the construction in [KS1] of a continuous, symmetric, bilinear, nonnegative tensorial operator

$$\pi : \Gamma(\overline{TB_2(0)}) \times \Gamma(\overline{TB_2(0)}) \to L^1(B_2(0)),$$

where $\Gamma(\overline{TB_2(0)})$ is the space of Lipschitz vector fields on $B_2(0)$ defined by

$$\pi(Z, W) := \frac{1}{4}|u^*(Z + W)|^2 - \frac{1}{4}|u^*(Z - W)|^2,$$

where $|u^*(Z)|^2$ is the directional energy density function (cf. [KS1, Section 1.8]). This generalizes the notion of the pullback metric for maps into a Riemannian manifold, and hence we shall refer to π also as the pullback metric for u.

Theorem 15. *Let $u : B_2(0) \to Y$ be a harmonic map into an NPC space (Y, d) and $\sigma_i \to 0$ such that u_{σ_i} converges locally uniformly in the pullback sense to a tangent map u_* and $\log_{P_0} \circ u_{\sigma_i}$ converges locally uniformly in the pullback sense to an Alexandrov tangent map v_*. Then the pullback metric for u_* and the pullback metric for v_* agree (in L^1).*

Proof. By Lemma 12 and Proposition 14, we have that $\triangle^{u_*}(r) = 0 = \triangle^{v_*}(r)$. Therefore, using the fact that $\delta_*(v_*, v_*(0)) = d_*(u_*, u_*(0))$, we obtain

$$\begin{aligned} 2E^{v_*}(r) &= \int_{\partial B_r(0)} \frac{\partial}{\partial r} {\delta_*}^2(v_*, v_*(0))\, d\Sigma_0 \\ &= \int_{\partial B_r(0)} \frac{\partial}{\partial r} {d_*}^2(u_*, u_*(0))\, d\Sigma_0 \\ &= 2E^{u_*}(r). \end{aligned}$$

By the distance-nonincreasing property of $\log_{P_0}$, we have that

$$\delta_*(v_*(x), v_*(\xi)) \le d_*(u_*(x), u_*(\xi)) \quad \text{for all } x, \xi \in B_1(0)$$

which in turn implies that $|{v_*}^*(Z)|^2 \le |{u_*}^*(Z)|^2$ for any $Z \in \Gamma(\overline{B_1(0)})$. Combined with the equality $E^{v_*}(r) = E^{u_*}(r)$ and formula (1.10v) of [KS1], we conclude that the directional energy density functions of v_* and u_* agree. The assertion now follows from the definition of the pullback metric. $\square$

Note that Theorem 15 does not necessarily imply that the pullback pseudodistance functions of the maps u_* and v_* agree. In general, the pullback pseudodistance functions of two maps may not agree even if their pullback metrics agree. To illustrate this, we consider the following example.

Example 16. A 4-pod is a tree with one vertex and 4 edges. Let K be a 4-pod embedded (not isometrically) as the x and y axes in the Euclidean plane $\mathbb{R}^2$ and let $u : B_1(0) \subset \mathbb{R}^2 \to K \subset \mathbb{R}^2$ be the projection map associated with the vertical foliation of the holomorphic quadratic differential $z^2|dz|^2$. Define a "folding" $f : K \to \mathbb{R}$ by setting $f(x, 0) = x$ if $x \ge 0$, $f(0, y) = y$ if $y \ge 0$, $f(x, 0) = -x$ if $x < 0$ and $f(0, y) = -y$ if $y < 0$. The maps u and $v = f \circ u$ have the same pullback metric but not the same pullback pseudodistance.

The following lemma gives a criterion when a tangent map and an Alexandrov tangent map have the same pullback pseudodistance.

Proposition 17. *Let $u : B_2(0) \to Y$ be a harmonic map into an NPC space (Y, d) and $\sigma_i \to 0$ such that u_{σ_i} converges in the pullback sense to u_* and $\log_{P_0} \circ u_{\sigma_i}$ converges uniformly to v_* on compact sets. If $\log_{P_0} : Y \to T_{P_0}Y$ has an inverse map $\exp_{P_0} : T_{P_0}Y \to Y$ such that its Lipschitz constant on the restriction to $B_r(\mathcal{P}_0) \subset T_{P_0}Y$ is bounded by $1 + o(1)$, where $o(1) \to 0$ as $r \to 0$, then the pullback pseudodistance under u_* is equal to that of the v_*.*

Proof. The assumption implies

$$\begin{aligned}\delta_*(\log_{P_0}\circ u_{\sigma_i}(x),\log_{P_0}\circ u_\sigma(x')) &\le d_*(u_{\sigma_i}(x),u_{\sigma_i}(x'))\\ &\le (1+o(1))\delta_*(\log_{P_0}\circ u_{\sigma_i}(x),\log_{P_0}\circ u_\sigma(x'))\end{aligned}$$

for $x, x' \in B_1(0)$, where $o(1) \to 0$ as $i \to \infty$. Taking the limit as $i \to \infty$, we obtain

$$\delta_*(v_*(x),v_*(x')) = d_*(u_*(x),u_*(x')).$$

□

Example 18. For a DM-complex (Y, G) (differentiable manifold-complex as defined in [DM1], i.e., a locally finite complex with a smooth metric $G\big|_M$ defined on each embedded differentiable manifold M (called a DM)), let $P_0 \in Y$ and $\mathcal{M}_{P_0}$ denote the set of DMs M containing P_0. If C denotes the tangent cone of Y at the point P_0 as defined in [Fe, Subsection 3.1.21], then clearly C is an unbounded cell complex and $T_{P_0}Y$ is isometric to $(C, G(P_0))$, where $G(P_0)$ is the metric defined by the value of G at P_0. We can define the exponential map

$$\exp^Y_{P_0} : T_{P_0}Y \to \bigcup_{M\in\mathcal{M}_{P_0}} M \subset Y$$

by piecing together the exponential maps defined on each $M \in \mathcal{M}_{P_0}$. This is equivalent to the exponential map defined from Alexandrov tangent cone point of view defined above; i.e., given a unit speed geodesic γ and $t \in [0,\infty)$, $\exp^Y_{P_0}(\gamma, t) = \gamma(t)$. By the smoothness of DMs, the Lipschitz constant on the restriction to $B_r(\mathcal{P}_0) \subset T_{P_0}Y$ is bounded by $1 + O(r)$. It follows by Proposition 17 that DM-complex Alexandrov tangent maps are the same as tangent maps.

5. The uniqueness condition

We will now assume that the tangent cone $(T_{P_0}Y, \delta)$ is locally compact. This includes some cases when Y is not locally compact; for example, the case of the Weil–Petersson completion of Teichmüller space. Consequently, given a harmonic map $u : (B_2(0), g) \to (Y, d)$ with $u(0) = P_0$, we can apply the Arzela–Ascoli theorem to assert the existence of a subsequence of $\log_{P_0}\circ u_{\sigma_i}$ that converges locally uniformly to a map $v_* : B_1(0) \to (T_{P_0}Y, \delta)$. In other words, we can assume that the target space for any Alexandrov tangent map is the tangent cone. We continue to denote $v_\sigma := \log_{P_0}\circ u_\sigma$ and define

$$C_\sigma(r) := \frac{1}{r^{n-1+2\alpha}}\left(\int_{\partial B_r(0)}\frac{\partial}{\partial r}\delta^2(v_\sigma, v_*)\,d\Sigma_0 - \frac{2\alpha}{r}\int_{\partial B_r(0)}\delta^2(v_\sigma, v_*)\,d\Sigma_0\right), \tag{14}$$

$$\mathcal{A}_\sigma(r) := \int_{\partial B_1(0)}\delta^2(v_\sigma, v_*)\,d\Sigma_0 - \frac{1}{r^{n-1+2\alpha}}\int_{\partial B_r(0)}\delta^2(v_\sigma, v_*)\,d\Sigma_0. \tag{15}$$

Lemma 19. *With the notation as above,*

$$\mathcal{A}_\sigma(r) = \int_r^1 C_\sigma(\varrho)d\varrho.$$

Proof. We compute

$$
\begin{aligned}
&\frac{d}{dr}\left(\frac{1}{r^{n-1+2\alpha}}\int_{\partial B_r(0)}\delta^2(v_\sigma,v_*)\,d\Sigma_0\right)\\
&\quad=\frac{1}{r^{n-1+2\alpha}}\int_{\partial B_r(0)}\frac{\partial}{\partial r}\delta^2(v_\sigma,v_*)\,d\Sigma_0-\frac{2\alpha}{r^{n+2\alpha}}\int_{\partial B_r(0)}\delta^2(v_\sigma,v_*)\,d\Sigma_0\\
&\quad=C_\sigma(r).
\end{aligned}
$$

Integrating the above differential equality, we obtain

$$
\begin{aligned}
\mathcal{A}_\sigma(r)&=\int_{\partial B_1(0)}\delta^2(v_\sigma,v_*)\,d\Sigma_0-\frac{1}{r^{n-1+2\alpha}}\int_{\partial B_r(0)}\delta^2(v_\sigma,v_*)\,d\Sigma_0\\
&=\int_r^1 C_\sigma(\varrho)\,d\varrho. \qquad \square
\end{aligned}
$$

Proposition 20. *Let $u : B_2(0)\to Y$ be a harmonic map into an NPC space (Y,d) with $u(0)=P_0$ and $T_{P_0}Y$ locally compact. If*

$$\liminf_{\sigma\to 0}\mathcal{A}_\sigma(r)\geq 0 \quad \text{uniformly for all } 0<r\leq 1, \tag{16}$$

then we have the following statements.

(i) *The Alexandrov tangent map of u at 0 is unique.*
(ii) *If $\log_{P_0} : Y\to T_{P_0}Y$ has an inverse map $\exp_{P_0} : T_{P_0}Y\to Y$ whose Lipschitz constant restricted to $B_r(\mathcal{P}_0)\subset T_{P_0}Y$ is bounded by $1+o(1)$, where $o(1)\to 0$ as $r\to 0$, then the pullback distant function of any tangent map of u at 0 is unique. In other words, if $u_* : \mathbb{R}^n\to(Y_*,d_*)$ and $\bar u_* : \mathbb{R}^n\to(\bar Y_*,\bar d_*)$ are tangent maps of u at 0, then*

$$d_*(u_*(\cdot),u_*(\cdot))=\bar d_*(\bar u_*(\cdot),\bar u_*(\cdot)).$$

Proof. Suppose that there exist sequences $\sigma_i\to 0$ and $s_j\to 0$ such that $\log_{P_0}\circ u_{\sigma_i}$ and $\log_{P_0}\circ u_{s_j}$ converge locally uniformly in the pullback sense to Alexandrov tangent maps $v_* : \mathbb{R}^n\to(T_{P_0}Y,\delta)$ and $\bar v_* : \mathbb{R}^n\to(T_{P_0}Y,\delta)$, respectively. For any s_j, we then have

$$
\begin{aligned}
&\int_{\partial B_1(0)}\delta^2\left(\sqrt{\frac{I^u(s_j)/s_j^{n-1+2\alpha}}{I^u(\sigma_i)/\sigma_i^{n-1+2\alpha}}}\log_{P_0}\circ u_{s_j}(x),v_*(x)\right)d\Sigma_0(x)\\
&\quad=\left(\frac{s_j}{\sigma_i}\right)^{-2\alpha}\int_{\partial B_1(0)}\delta^2\left(\frac{\mu^{-1}(\sigma_i)}{\mu^{-1}(s_j)}\log_{P_0}\circ u_{s_j}(x),\left(\frac{s_j}{\sigma_i}\right)^{\alpha}v_*(x)\right)d\Sigma_0(x)\\
&\quad=\left(\frac{s_j}{\sigma_i}\right)^{-2\alpha}\int_{\partial B_1(0)}\delta^2\left(\mu^{-1}(\sigma_i)\log_{P_0}\circ u(s_jx),\left(\frac{s_j}{\sigma_i}\right)^{\alpha}v_*(x)\right)d\Sigma_0(x)\\
&\quad=\left(\frac{s_j}{\sigma_i}\right)^{-2\alpha}\int_{\partial B_1(0)}\delta^2\left(\log_{P_0}\circ u_{\sigma_i}\left(\frac{s_j}{\sigma_i}x\right),v_*\left(\frac{s_j}{\sigma_i}x\right)\right)d\Sigma_0(x).
\end{aligned}
\tag{17}
$$

For a fixed $\sigma_i > 0$, we let $r_j = \frac{s_j}{\sigma_i}$ in the right-hand side of (17) to obtain

$$\begin{aligned}
&\int_{\partial B_1(0)} \delta^2 \left(\sqrt{\frac{I^u(s_j)/s_j^{n-1+2\alpha}}{I^u(\sigma_i)/\sigma_i^{n-1+2\alpha}}} \log_{P_0} \circ u_{s_j}, v_* \right) d\Sigma_0 \\
&\quad = r_j^{-2\alpha} \int_{\partial B_1(0)} \delta^2(\log_{P_0} \circ u_{\sigma_i}(r_j x), v_*(r_j x))\, d\Sigma_0 \\
&\quad = \frac{1}{r_j^{n-1+2\alpha}} \int_{\partial B_{r_j}(0)} \delta^2(\log_{P_0} \circ u_{\sigma_i}, v_*)\, d\Sigma_0 \\
&\quad = \int_{\partial B_1(0)} \delta^2(\log_{P_0} \circ u_{\sigma_i}, v_*)\, d\Sigma_0 - \mathcal{A}_{\sigma_i}(r_j).
\end{aligned} \tag{18}$$

We now claim that

$$\mathcal{I}_* = \lim_{r \to 0} \mathcal{I}(r) \neq 0. \tag{19}$$

Indeed, assume that $\mathcal{I}_* = 0$. Then for a fixed σ_i,

$$\sqrt{\frac{I^u(s_j)/s_j^{n-1+2\alpha}}{I^u(\sigma_i)/\sigma_i^{n-1+2\alpha}}} \log_{P_0} \circ u_{s_j}$$

converges to the vertex $\mathcal{P}_0$ of the Alexandrov tangent cone. For $\delta > 0$, we can take s_j sufficiently small in (18) to obtain

$$\int_{\partial B_1(0)} \delta^2(\mathcal{P}_0, v_*)\, d\Sigma_0 + \mathcal{A}_{\sigma_i}(r_j) < \int_{\partial B_1(0)} \delta^2(\log_{P_0} \circ u_{\sigma_i}, v_*)\, d\Sigma_0 + \delta.$$

Then take $\sigma_i \to 0$ and use (16) to obtain

$$\int_{\partial B_1(0)} \delta^2(\mathcal{P}_0, v_*)\, d\Sigma_0 \leq \delta.$$

Since $\delta > 0$ can be chosen arbitrarily small, this contradicts the fact that v_* is nonconstant and completes the proof of (19). We can now let $\sigma_i \to 0$, $s_j \to 0$ in (18) and apply (19) to conclude

$$\int_{\partial B_1(0)} \delta^2(\bar{v}_*, v_*)\, d\Sigma_0 = 0$$

which immediately implies assertion (i) of the lemma. Assertion (ii) follows immediately from Proposition 17 and (i). □

Remark 3. It is an important question to ask for what NPC spaces Y condition (16) holds. In the example below, we will prove that (16) holds when Y is a smooth manifold. It thus follows from a deep regularity result of Gromov and Schoen that (16) also holds for Euclidean buildings provided $\mathrm{Ord}^u(0) = 1$ (cf. [GS, Theorem 5.4]). Under the same assumption on the order, it follows that (16) holds also for DM-complexes (cf. [DM1]).

Example 21. Let $u : (B_2(0), g_0) \to (Y, h)$ be a harmonic map into a smooth NPC Riemannian manifold with $u(0) = P_0$ and let $v_* : (B_1(0), g_0) \to T_{P_0}Y \approx \mathbb{R}^n$ be its tangent map at 0. Let y be normal coordinates centered at P_0 and let $h_\sigma(y) := h(\mu(\sigma)y)$. The map v_σ is harmonic with respect to the metric

h_σ on the target. Thus, $\triangle v_\sigma = O(\mu(\sigma))$ since the Christoffel symbols of h_σ satisfy $\Gamma^i_{jk}(v_\sigma) = O(\mu(\sigma))$, and we obtain

$$\begin{aligned}\triangle|v_\sigma - v_*|^2 &= 2|\nabla(v_\sigma - v_*)|^2 + 2(v_\sigma - v_*)\cdot\triangle(v_\sigma - v_*)\\ &= 2|\nabla(v_\sigma - v_*)|^2 + 2(v_\sigma - v_*)\cdot\triangle v_\sigma\\ &\geq 2|\nabla(v_\sigma - v_*)|^2 - 2c\mu(\sigma)|v_\sigma - v_*|\\ &= 2|\nabla v_*|^2 + 2|\nabla v_\sigma|^2 - 4\nabla v_*\cdot\nabla v_\sigma - 2c\mu(\sigma)|v_\sigma - v_*|.\end{aligned}$$

Furthermore,

$$|v_\sigma - v_*|^2 = |v_\sigma|^2 + |v_*|^2 - 2v_\sigma\cdot v_*.$$

The homogeneity of v_* and Lemma 12 imply

$$\rho\int_{B_\rho(0)}|\nabla v_*|^2 d\mu_0 - \alpha\int_{\partial B_\rho(0)}|v_*|^2\,d\Sigma_0 = 0$$

and

$$\frac{\partial v_*}{\partial r} = \frac{\alpha}{r}v_*.$$

The last equality along with the Stoke's theorem implies

$$\begin{aligned}-\int_{B_\rho(0)}\nabla v_*\cdot\nabla v_\sigma d\mu_0 &= \int_{B_\rho(0)}\mathrm{div}(v_\sigma\nabla v_*)\,d\mu_0\\ &= \int_{\partial B_\rho(0)} v_\sigma\frac{\partial v_*}{\partial r}\,d\Sigma_0\\ &= \frac{\alpha}{\rho}\int_{\partial B_\rho(0)} v_\sigma\cdot v_*\,d\Sigma_0.\end{aligned}$$

Therefore,

$$\begin{aligned}&C_\sigma(\rho)\\ &= \frac{1}{\rho^{n-1+2\alpha}}\left(\int_{B_\rho(0)}\frac{\partial}{\partial r}\delta^2(v_\sigma,v_*)\,d\mu_0 - \frac{2\alpha}{\rho}\int_{\partial B_\rho(0)}\delta^2(v_\sigma,v_*)\,d\Sigma_0\right)\\ &= \frac{1}{\rho^{n-1+2\alpha}}\left(\int_{B_\rho(0)}\triangle\delta^2(v_\sigma,v_*)\,d\mu_0 - \frac{2\alpha}{\rho}\int_{\partial B_\rho(0)}\delta^2(v_\sigma,v_*)\,d\Sigma_0\right)\\ &\geq \frac{2}{\rho^{n-1+2\alpha}}\left(\int_{B_\rho(0)}|\nabla v_\sigma|^2 - 2c\mu(\sigma)|v_\sigma - v_*|\,d\mu_0 - \frac{\alpha}{\rho}\int_{\partial B_\rho(0)}|v_\sigma|^2\,d\Sigma_0\right)\\ &= \frac{2}{\rho}\;\frac{I^{v_\sigma}(\rho)}{\rho^{n-1+2\alpha}}\left(\frac{\rho E^{v_\sigma}(\rho)}{I^{v_\sigma}(\rho)} - \alpha\right) - \frac{2c\mu(\sigma)}{\rho^{n-1+2\alpha}}\int_{B_\rho(0)}|v_\sigma - v_*|\,d\mu_0\\ &\geq -\frac{C\mu(\sigma)}{\rho^{-1+2\alpha}},\end{aligned}$$

where we have used Theorem 3 for the last inequality. By (7) and Lemma 19, we obtain

$$\liminf_{\sigma\to 0}\mathcal{A}_\sigma(r)\geq 0\quad\text{for all } 0<r\leq 1.$$

Example 22 (Caffarelli–Lin [L]). In this example, we consider a harmonic map $u : (B_2(0), g_0) \to T$ into an $\mathbb{R}$-tree. Since we are studying local properties, it suffices by [Su] to assume that T is a k-pod with vertex $u(0) = P_0$. (A k-pod is a tree with k edges joined at one vertex.) Thus, the notion of tangent maps and Alexandrov tangent maps agree. Furthermore, we can identify the tangent cone at P_0 with T, hence we can assume that $u = v$ and that the target space for the tangent map is T. We will show that condition (16) is equivalent to uniqueness of tangent maps. In view of Proposition 20, it suffices to show that (16) is a necessary condition for uniqueness. Indeed, let $u_* : B_1(0) \to T$ be the unique tangent map. For a flat F_0 (i.e., $\mathbb{R}$ isometrically embedded in T), let U be a connected component of $u_\sigma^{-1}(F_0) \cap u_*^{-1}(F_0)$. By identifying F_0 with $\mathbb{R}$, we can compute in U

$$\begin{aligned}\triangle d^2(u_\sigma, u_*) &= \triangle|u_\sigma - u_*|^2 = 2|\nabla(u_\sigma - u_*)|^2 + 2(u_\sigma - u_*)\triangle(u_\sigma - u_*) \\ &= 2|\nabla u_\sigma|^2 + 2|\nabla u_*|^2 - 4\nabla u_\sigma \cdot \nabla u_*\end{aligned}$$

and

$$d^2(u_\sigma, u_*) = |u_\sigma - u_*|^2 = |u_\sigma|^2 + |u_*|^2 - 2u_\sigma \cdot u_*.$$

Furthermore,

$$\begin{aligned}&-4\int_{B_\rho(0)\cap U} \nabla u_\sigma \cdot \nabla u_* d\mu_0 \\ &\quad= -4\int_{\partial B_\rho(0)\cap U} u_\sigma \cdot \frac{\partial u_*}{d\rho} d\Sigma_0 - 4\int_{B_\rho(0)\cap \partial U} u_\sigma \cdot \frac{\partial u_*}{\partial \nu_U} d\Sigma_0 \\ &\quad= -\frac{4\alpha}{\rho}\int_{\partial B_\rho(0)\cap U} u_\sigma \cdot u_* d\Sigma_0 - 4\int_{B_\rho(0)\cap \partial U} u_\sigma \cdot \frac{\partial u_*}{\partial \nu_U} d\Sigma_0,\end{aligned}$$

where ν_U is the outward pointing normal along ∂U. Thus,

$$\begin{aligned}&\int_{B_\rho(0)\cap U} \triangle d^2(u_\sigma, u_*)\, d\mu_0 - \frac{2\alpha}{\rho}\int_{\partial B_\rho(0)\cap U} d^2(u_\sigma, u_*)\, d\Sigma_0 \\ &\quad= 2\int_{B_\rho(0)\cap U} |\nabla u_\sigma|^2\, d\mu_0 - \frac{2\alpha}{\rho}\int_{\partial B_\rho(0)\cap U} |u_\sigma|^2\, d\Sigma_0 \\ &\qquad+ 2\int_{B_\rho(0)\cap U} |\nabla u_*|^2\, d\mu_0 - \frac{2\alpha}{\rho}\int_{\partial B_\rho(0)\cap U} |u_*|^2\, d\Sigma_0 \\ &\qquad- 4\int_{B_\rho(0)\cap \partial U} u_\sigma \cdot \frac{\partial u_*}{\partial \nu_U}\, d\Sigma_0.\end{aligned}$$

By applying the divergence theorem on U and using the homogeneity of u_*, we obtain

$$\begin{aligned}&\int_{\partial B_\rho(0)\cap U} \frac{\partial}{\partial r} d^2(u_\sigma, u_*)\, d\Sigma_0 - \frac{2\alpha}{\rho}\int_{\partial B_\rho(0)\cap U} d^2(u_\sigma, u_*)\, d\Sigma_0 \\ &\quad= 2\int_{B_\rho(0)\cap U} |\nabla u_\sigma|^2\, d\mu_0 - \frac{2\alpha}{\rho}\int_{\partial B_\rho(0)\cap U} |u_\sigma|^2\, d\Sigma_0 \\ &\qquad- 4\int_{B_\rho(0)\cap \partial U} u_\sigma \cdot \frac{\partial u_*}{\partial \nu_U}\, d\Sigma_0 - \int_{B_\rho(0)\cap \partial U} \frac{\partial}{\partial \nu_U} d^2(u_\sigma, u_*)\, d\Sigma_0.\end{aligned}$$

Summing over such U and denoting the union of the boundaries of all such U by S_σ, we obtain for a.e. $\rho \in (r, 1)$

$$\begin{aligned}
&\int_{\partial B_\rho(0)} \frac{\partial}{\partial r} d^2(u_\sigma, u_*)\, d\Sigma_0 - \frac{2\alpha}{\rho} \int_{\partial B_\rho(0)} d^2(u_\sigma, u_*)\, d\Sigma_0 \\
&\quad \geq 2 \int_{B_\rho(0)} |\nabla u_\sigma|^2\, d\mu_0 - \frac{2\alpha}{\rho} \int_{\partial B_\rho(0)} |u_\sigma|^2\, d\Sigma_0 \\
&\qquad - 8 \int_{B_\rho(0)\cap S_\sigma} d(u_\sigma, P_0)\, |\nabla u_*|\, d\Sigma_0 \\
&\qquad - 4 \int_{B_\rho(0)\cap S_\sigma} d(u_\sigma, u_*)(|\nabla u_\sigma| + |\nabla u_*|)\, d\Sigma_0.
\end{aligned}$$

Note that

$$\sup_{B_\rho(0)\cap S_\sigma} d(u, P_0)|\nabla u_*| \leq \sup_{B_\rho(0)} d(u_\sigma, u_*)(|\nabla u| + |\nabla u_*|)$$

since $u_\sigma(x) = P_0$ or $u_*(x) = P_0$ for $x \in S_\sigma$. Hence

$$\begin{aligned}
&\int_{\partial B_\rho(0)} \frac{\partial}{\partial r} d^2(u_\sigma, u_*)\, d\Sigma_0 - \frac{2\alpha}{\rho} \int_{\partial B_\rho(0)} d^2(u_\sigma, u_*)\, d\Sigma_0 \\
&\quad \geq 2E^{u_\sigma}(\rho) - \frac{2\alpha}{\rho} I^{u_\sigma}(\rho) \\
&\qquad - 12\, \mathcal{H}^{n-1}(B_\rho(0) \cap S_\sigma) \sup_{B_\rho(0)} d(u_\sigma, u_*)(|\nabla u_\sigma| + |\nabla u_*|) \\
&\quad \geq \frac{2I^{u_\sigma}(\rho)}{\rho} \left(\frac{\rho E^{u_\sigma}(\rho)}{I^{u_\sigma}(\rho)} - \alpha \right) \\
&\qquad - C\mathcal{H}^{n-1}(B_\rho(0) \cap S_\sigma) \sup_{B_\rho(0)} d(u_\sigma, u_*) \\
&\quad \geq -C\mathcal{H}^{n-1}(B_\rho(0) \cap S_\sigma) \sup_{B_\rho(0)} d(u_\sigma, u_*) \quad \text{(by Theorem 3)}.
\end{aligned}$$

Since $\mathcal{H}^{n-1}(S) < \infty$, where S is defined analogously to S_σ with u replaced by u_σ, we have

$$\limsup_{\sigma \to 0} \frac{\mathcal{H}^{n-1}(S_\sigma \cap B_\rho(0))}{\rho^{n-1}} = \limsup_{\sigma \to 0} \frac{\mathcal{H}^{n-1}(S \cap B_{\sigma\rho}(0))}{(\sigma\rho)^{n-1}} < \infty.$$

Thus, for a.e. $\rho \in (r, 1)$, we obtain

$$\begin{aligned}
C_\sigma(\rho) &= \frac{-1}{\rho^{n-1+2\alpha}} \left(\int_{B_\rho(0)} \frac{\partial}{\partial r} d^2(u, u_*)\, d\mu - \frac{2\alpha}{\rho} \int_{\partial B_\rho(0)} d^2(u, u_*)\, d\mu \right) \\
&\geq -\frac{C}{\rho^{2\alpha}} \sup_{B_1(0)} d(u_\sigma, u_*).
\end{aligned}$$

By the assumption on the uniqueness of the tangent map and by Lemma 19, we obtain

$$\liminf_{\sigma \to 0} \mathcal{A}_\sigma(r) \geq 0 \quad \text{for all } 0 < r \leq 1$$

as desired.

We end by making the following conjecture, which we have shown in the course of the proof of Proposition 20 to be implied by (16).

Conjecture 23. *If* $u : (B_2(0), g) \to Y$ *is a harmonic map into an NPC space* (Y, d) *with* $\mathrm{Ord}^u(0) = \alpha$*, then*

$$\lim_{r\to 0} \mathcal{I}^u(r) = \lim_{r\to 0} \frac{I^u(r)}{r^{n-1+2\alpha}} \neq 0. \tag{20}$$

Acknowledgements

The first author was supported by a research grant from the Simons Foundation. The second author was supported by research grant NSF DMS-1105599. The authors would like to thank F.-H. Lin for communicating the results [CL] and [L] to them.

References

[CL] L. Caffarelli and F.-H. Lin, *Analysis on the junctions of domain walls.* Discrete Contin. Dyn. Syst. **28** (2010), 915–929.

[DM1] G. Daskalopoulos and C. Mese, *On the singular set of harmonic maps into DM-complexes.* Preprint.

[Fe] H. Federer, *Geometric Measure Theory.* Die Grundlagen der Math. Wissenschaften 153, Springer-Verlag, Berlin, 1969.

[GS] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one.* Publ. Math. Inst. Hautes Études Sci. **76** (1992), 165–246.

[Jo] J. Jost, *Nonpositive curvature: Geometric and analytic aspects.* Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag, Basel, 1997.

[KS1] N. Korevaar and R. Schoen, *Sobolev spaces and harmonic maps for metric space targets.* Comm. Anal. Geom. **1** (1993), 561–659.

[KS2] N. Korevaar and R. Schoen, *Global existence theorem for harmonic maps to non-locally compact spaces.* Comm. Anal. Geom. **5** (1997), 333–387.

[L] F.-H Lin, *private communication.*

[Su] X. Sun, *Regularity of harmonic maps to trees.* Amer. J. Math. **125** (2003), 737–771.

Georgios Daskalopoulos
Department of Mathematics
Brown University
Providence, RI 02912
USA
e-mail: `daskal@math.brown.edu`

Chikako Mese
Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218
USA
e-mail: `cmese@math.jhu.edu`