Functions of Two Variables

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1 Introduction

Calculus is the study of functions. In elementary calculus, we concentrate on functions of a single variable; we will now extend that investigation to study functions of two or more variables. One-variable calculus makes extensive use of graphs in order to visualize properties of functions of one variable by relating them to geometric properties of plane curves. In an analogous way, we will construct graphs of functions of two variables as surfaces in three-dimensional space. Graphs of single-variable functions in the plane can be handled adequately by a graphing calculator; the corresponding technique for the study of graphs of two-variable functions is interactive computer graphics on a workstation. We will exploit the power and versatility of interactive graphics in many ways to provide experience with visualization of the geometric aspects of calculus of several variables, as a complement to the algebraic and analytical approaches, which are better suited for classroom presentation.

2 Graphs of Functions over Rectangular Domains

As in the case of functions of one variable, each function of two variables has a domain , the set of values for which the function is defined. The domain of a function f of a single variable x is a subset of the real line, usually one or more intervals or rays, or the whole real line. For a function f of two variables x and y, the domain will be a subset of the coordinate plane. In contrast with the one-variable situation, the domain of a function of two variables can be much more complicated. To begin with we will look at rectangular domains given by $a \leq x \leq b$ and $c \leq y \leq d$ for constants a, b, c, and d.

The graph of a function f of a single variable x is defined to be the collection of pairs (x, y) where x is in the domain and $y = f(x)$. Similarly the graph of a function f of two variables (x, y) is defined to be the collection of triples (x, y, z) where (x, y) is in the domain of f and $z = f(x, y)$.

Graphs of functions over rectangular domains **START : caption**

The solidified graph of
$x^2 + y^2$. **END : caption**

For function of a single variable defined over an interval, we may approximate the graph by a polygon in the plane, obtained by choosing a subdivision of the domain into a finite number of subintervals, and by connecting the points on the graph lying over the subdivision points. In two-variable calculus, we may approximate the graph of a function defined over a rectangular domain by a polyhedral surface with verices lying over a partition of the rectangle into subrectangles determined by the subdivisions of the intervals $a \leq x \leq b$ and $c \leq y \leq d$. Each small rectangle is broken into two right triangles by including the diagonal from the lower left to the upper right corner. Once we know the location of the three vertices of such a triangle, we can fill in the triangle to obtain the polyhedral approximation to the graph of the function over that triangular region.

There are three windows, one showing the domain, one showing the graph, and one showing the controls.

Choosing a point in the domain with the middle mouse button automatically selects the corresponding point on the graph of the function. The coordinates of the point on the graph are indicated in a small window. Hitting the spacebar in the graph window will toggle the image between wire-frame and filled-in mode.

In the control panel, it is possible to toggle on and off the coordinate axes in the graph window. It is also possible to toggle on and off the domain of the function, as a subset of the horizontal (x,y) coordinate plane in three-space. It is possible to change the beginning and end points of the domain of x and the domain of y, and to change the resolution of each of these intervals. (You may want to experiment with this to see how it affects the graph.) It is also possible to type in new functions in the control window, keeping the same domain and the same view in the graphing window.

The window labelled Surface z = f(x,y) shows the graph of the function as a red surface in three-space. More precisely, this window shows the polyhedral approximation to the graph of the function evaluated over the subdivision points of the rectangular domain, where each subrectangle is divided into two triangles.

Notice that in this demonstration the type-ins for the beginning and end points of the intervals have the label R, and you my input any real numbers. The resolution sliders have the label Z, and the only allowable inputs are integers, greater than 1 in this case.

You may have noticed that the type-ins and sliders defining the domain have labels

R

and

Z

. This means the values are, respectively, real and integer values. You can choose any real value in the type-ins, but the sliders restrict you to integers.

In the first demonstration, we display the graph of an arbitrary function defined over a rectangular domain in the plane. The default example is the graph of

$f(x, y) = x^2 + y^2$ defined over the rectangle

$-1 \leq x \leq 1$,

$-1 \leq y \leq 1$, with each interval divided into ten subintervals.

Exercise 1

Investigate graphs of various functions $f(x, y)$ over the default domain, for example,

$f(x, y) = x^2 - y^2$ or

$f(x, y) = x^2 - y^3$. What will be the intersection of the graph with the (x, y)-plane? What can you say in general about the graph of

$f(x, y) = x^m - y^n$ for various pairs of positive integers m and n?

Exercise 2

Describe the shape of the graph of $f(x, y) = x^2 + cy^2$ for different choices of the constant c.

Exercise 3

For which values of the constant c will the graph of

$f(x, y) = x^2 + cxy + y^2$ intersect the (x,y)-coordinate plane in a single point?

Exercise 4

Describe the "geographical features" of various standard surfaces, like the graph of

$f(x, y) = -x^4 + x^2 - y^2$ or of

$f(x, y) = x^3 - x - y^2$. What can you say about the graph of

$f(x, y) = x^3 - 3xy^2$?

Optional Material

Functions of Two Variables: Two-Color View

Showing the surface colored differently on the top and the bottom can make it easier to see which parts of the surface are in front of other parts. An extra demonstration includes this functionality.

Graphs of Functions over Rectangular Domains: Two-Color View **START : caption**

The solidified graph of
$x^2 + y^2$ with two colors of red, one for the top side and for the bottom side. **END : caption**

The functionality of this demonstration is almost identical to that of the previous one. The one difference is that here we color the top of the surface differently than we do the bottom, allowing a greater ability to differentiate the two in the display. The way we are coloring the surface differently on the top and bottom is by displaying two differently-colored copies of the surface (in bright and dark red), one above the other. For certain surfaces, especially at higher resolution, this might result in an interference of colors. This can be fixed by adjusting the slider labelled Height Difference Between Surfs to bring the two surfaces farther apart.

This demonstration shows the graph of a function over a rectangular domain. The top and bottom of the surface are darker and lighter, respectively, by default, and you can reverse them. For more information on functions over rectangular domains, see the first demo , because this demo is very similar.

Optional Material

Graphs of Two Functions

Another extension of this first demonstration is the abillity to define two functions over the same domain and to display their graphs simultaneously. We can use this to see where two surfaces intersect, for example
$f(x, y) = 1 - x^2$ and
$g(x, y) = 1 - y^2$ defined over the default domain.

Graphs of Two Functions in One Window **START : caption**

A pair of rotated hyperboloids. **END : caption**

This demonstration shows two user-defined functions z=f(x,y) and z=g(x,y) over the same domain in the same window. The function $f(x, y)$ is displayed in red; the function $g(x, y)$ is displayed in green. The functionality of this demonstration is identical to the one that shows just one function graph, except for the added checkboxes that toggle the display of each function.

To see the intersection better, you should solidify the graph by hitting the spacebar in the Surface window.

Exercise 5

Observe how the functions $f(x, y) = x^2$ and $g(x, y) = x^2 - y^4$ intersect. Now, leaving g the same, change f to $f(x, y) = -y^4$ and observe the intersection.

Exercise 6

Let f be any function, say $f(x, y) = x^3 - 3xy^2$ and let g be constant. What do the intersection curves look like for various values of g? Try other functions for f.

Optional Material

Techniques for Graphing Functions of Two Variables

The computer is an invaluable tool for studying the graphs of functions. We can use Fnord and similar programs to graph the functions for us so we can concentrate on their properties without having to draw the graphs ourselves. It is, however, extremely useful to be able to graph a function on paper without the aid of a computer. It is only by drawing the graph that we really get inside a function and learn to understand its behavior.

Although Fnord makes graphing look easy, it often is not. There are many techniques that we can use to facilitate this task. Here we discuss some of them.

The way Fnord graphs a function is by calculating its value at various evenly spaced points in the domain. It then connects these points with lines. It is always a good idea to begin a graph in this manner, though it is tedious and impractical to calcultate the value of the function at as many points as Fnord does.

Another thing we can do is to view the surface as a collection of curves running parallel to the x-axis or parallel to the y-axis. The equations of these curves are given by setting either x or y equal to a constant. We are then left with a set of functions of one variable which are much easier to graph.

We may then give these curves width by extending them horizontally so they look like stripes. This will give a fairly good approximtion of the function graph if enough stripes are drawn.

Another technique is to find a box that contains every point of the graph over the domain. The top of this box is at a height equal to the maximum of the funciton over the domain and the bottom is at a height equal to the minimum. The sides of the box are determined by the limits of the domain. Once we draw this bounding box, we know that every point of the graph will fall inside of it.

Graphing Techniques: Functions of Two Variables

This demonstration is similar to the first demonstration in this lab. It has some additional functionality pertaining to how a graph may be drawn.

The Graph By X and Graph By Y

CheckBoxes toggle the display of the union of curves in the x and y directions, respectively. The

Stripes CheckBox will extend the chosen set of curves into stripes. The Do Bounding Box CheckBox will toggle the display of a box in which every point of the function graph is contained.

This demonstration graphs a function of two variables using the techniques of the bounding box and curves in the x and y directions.

We may also define the graphs of functions over regions other than rectangles. For functions with discs as domains, it is often convenient to describe the domain in polar coordinates and the graph in "cylindrical coordinates".

Lab1b contains a discussion of all the topics contained in this lab, but in cylindrical, instead of rectangular, coordinates. If you are unfamiliar with polar coordinates, you are advised to go through lab1b in entirety. Even if you have studied polar and/or cylindrical coordinates previously, you still might want to take advantage of the links, like this one, which appear after each section of this lab, to the corresponding section of Lab1b. It is recommended that you try out some of the cylindrical coordinate demos to see the advantages and disadvantages of that system.

3 Domains

As mentioned earlier, in the case of a function of two variables, the domain can be a very complicated sebset of the plane. In the case where the function f is given by a formula, we may define the
natural domain
of f to be the collection of all points (x, y) where the formula makes sense.

Example 1
The natural domain of the function
$f(x) = 1/x$ is all real numbers x except 0. The natural domain of
$f(x, y) = \frac{1}{(x^2+y^2)}$ is all pairs (x, y) except $(0, 0)$. The natural domain of
$f(x, y) = \frac{1}{xy}$ is all pairs with neither x nor y equal to zero.

Example 2
For
$f(x) = \sqrt{1 - |x|}$, the natural domain is
$-1 \leq x \leq 1$. The natural domain of
$f(x, y) = \sqrt{1 - |x + y| - |x - y|}$ is the square region given
$-1 \leq x \leq 1$ and
$-1 \leq y \leq 1$, while the natural domain of
$f(x, y) = \sqrt{(1 - x^2 - y^2)}$ is the disc of radius 1 about the origin, given by all (x, y) such that $(x^2 + y^2) \leq 1$.

Example 3
The natural domain of $f(x) = \log x$ is the open ray consisting of all x greater than 0 while the natural domain of $f(x, y) = \log(y - x)$ is the set of points where $y \geq x$, i.e. the open half-plane above the line $y = x$.

Exercise 7

Describe the natural domains of the following functions:

a)

$f(x, y) = \sqrt{1 - x^2}$

b)

$f(x, y) = \sqrt{(1 - x^2)(1 - y^2)}$

c)

$f(x, y) = \sqrt{1 - x^2 - y^2}$

d)

$f(x, y) = \sqrt{r^2 - x^2 - y^2}$, for various values of r

Exercise 8

Describe the natural domains of $\sqrt{-x^4 + x^2 - y^2}$ and $\sqrt{x^3 - x - y^2}$.

What are the natural domains of $\ln(-x^4 + x^2 - y^2)$ and $\ln(x^3 - x - y^2)$?

Much of one-variable calculus concerns the local

properties of functions, in an interval neighborhood of a point, so it is appropriate to restrict the domain of a function to an interval of the form

$a \le x \le b$. Similarly, in the study of two-variable functions we begin by restricting the domain to a rectangular neighborhood of the form

$a \le x \le b$,

$c \le y \le d$.

4 Slice Curves

If we restrict the domain of a function of two variables by holding one of the variables fixed, then we obtain a one-variable function. For example if we set y = 0, then we restrict the domain of

$f(x, y) = x^2 + y^2$ to the x-axis, and we get the quadratic function

$f(x, 0) = x^2$. We can graph this one-variable function in a vertical plane by displaying all points (x,0,z) with (x,0) in the domain and z = f(x,0). This graph is called the slice curve above the line y =0. More generally, for any function and any choice of

y_0 , we can define the slice curve over the line

$y = y_0$ to be the points

$(x, y_0, f(x, y_0))$, where

(x, y_0) lies in the domain of the function. These points will all lie in the vertical (slicing) plane defined by setting

$y = y_0$.

Similarly by holding x fixed at a value
x_0 , we obtain the slice curve over
$x = x_0$, a curve in the vertical plane
$x = x_0$ defined by all points
$(x_0, y, f(x_0, y))$ for points
(x_0, y) in the domain of f.
Slice curves for rectangular domains **START : caption**

The graph of
$x^2 + y^2$ with the x-slice plane. Left: The graph of the intersection curve between the surface and the x-slice plane. Middle: The surface with the slice plane. Right: The domain with crosshairs at a point selected with the mouse. **END : caption**

In the demonstration, we may now choose the position of
(x_0, y_0) in the domain window and we also see the lines
$x = x_0$ and
$y = y_0$.

In the surface window, the two slice curves are indicated on the graph, and we may toggle on and off the slice planes
$x = x_0$ or
$y = y_0$.

In two separate windows we see the slice curve
$(x, y_0, f(x, y_0))$ in the (x,z)-coordinate plane, and the slice curve
$(x_0, y, f(x_0, y))$ in the (y,z)- coordinate plane.

The point
(x_0, y_0) can be selected by clicking the middle mouse button in the domain window.

This demo has a slider labeled
c - a constant
. The letter
c can be used in the equations of functions to represent the value of the slider. You can observe surfaces at various values of
c simply by manipulating the slider, without having to type a new function definition. The various surfaces obtained by changing only the value of
c make up a one-parameter family of surfaces.

When you select a point
(x_0, y_0) by clicking with the middle mouse button in the domain, the pair of yellow crosshairs on the domain highlights the horizontal curve
$y = y_0$ and the vertical curve
$x = x_0$. The corresponding curves
$(x, y_0, f(x, y_0))$ and

$(x_0, y, f(x_0, y))$ are highlighted on the surface in the graph window. Two additional windows display the

y_0 -slice curve as a graph in the

(x, z) coordinate plane and the

x_0 -slice curve as a graph in the

(y, z) coordinate plane. Clicking on the checkboxes in the control panel causes the slicing planes themselves to be displayed in the surface window.

This demonstration shows an x -slice curve and a y -slice curve on a function graph as well as a graph of each slice curve in its vertical plane.

Later on, we will see a more general demonstration where you can choose the angle at which to slice the surface at a selected point.

Exercise 9

Describe the slice curves of these functions:

a)

$f(x, y) = x^2 - y^2$

b)

$f(x, y) = x^2 - y^3$

c)

$f(x, y) = x^3 y - 2xy^2 - 2xy$

d)

$f(x, y) = \sin(2\pi xy)$ (You can type *pi* for π)

In which cases will all slice curves

$x = x_0$ be congruent?

Exercise 10

What is the general nature of a function for which all the

$x = x_0$ slice curves are congruent? If all the

$x = x_0$ slice curves are congruent, will it necessarily be true that all the

$y = y_0$ slice curves are congruent?

Exercise 11

For which

(x_0, y_0) will both slice curves of

$f(x, y) = x^4 + xy^2$ both be convex upward? Same question for

$f(x, y) = x^2 - xy^2$.

Exercise 12

For which (x_0, y_0) will the slice curves of $f(x, y) = x^3 + xy$ be always increasing or always decreasing? What can be said about a function f for which the $y = y_0$ slice curves are all straight lines?

Exercise 13

For which values of (x,y) does the function $f(x, y) = -x^4 + 2x^2 - y^2$ (Twin Peaks) have a maximum? Explain why a function of two variables has a maximum only if both slice functions have maxima. Is this necessary condition sufficient?

Optional Material

Hint

Try the function $f(x, y) = \frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2+0.001)}$. The 0.001 is there because without it the function would not be defined at (0,0). (Why not?)

You may also want to examine the slice curves of functions in cylindrical coordinates.

5 Contours and Level Sets

Slicing by vertical planes is a straightforward process–we merely restrict the domain of the function to a line in the (x,y)- coordinate plane and compute the value of the function at those points. It is more difficult, and ultimately more useful, to compute the intersection of the graph of a function with horizontal planes z = k, a constant. In order to find such an intersection, we must find all (x,y) in the domain such that f(x,y) =k. Sometimes this is just an algebraic exercise, but more frequently it is difficult to find this locus explicitly. This is one place where the real power of the computer comes in to allow us to investigate the most important properties of function graphs.

The slice of the function graph at level z = k is called the level set of f at height k and the set of points (x,y) in the domain for which f(x,y) = k is called the contour of f at level k . The level set at height k is thus the graph of the function f restricted to the contour of f at level k.

Example 4

For

$f(x, y) = x^2 + y^2$ the contour at level 1 is the unit circle in the (x,y) plane and the level set is the unit circle in the plane z = 1. More generally, the contour at level k is the circle of radius

$\sqrt{k}$ if $0 < k < 1$, and if $k = 0$, the contour consists of a single point, (0,0). For $k < 0$, the contour is the empty set (and so is the level set at height k) because f(x,y) is never negative.

Optional Material

Note

If we consider the natural domain of the function

$f(x, y) = x^2 + y^2$, then the contour will be a circle for all k ¿ 0. If we restrict f to the standard square domain, then the contour for

$k > \sqrt{2}$ will be empty, the contour for k = 2 will be the four corners of the square, and the contour for

$1 < k < \sqrt{2}$ will be four circular arcs, the intersection of the contour for the natural domain and the restricted square domain.

Contours and Level Sets **START : caption**

The graph of

$x^2 + y^3 - y/2 - 1/2$ with the contour level set drawn in yellow at k = 0.5. Left: The domain and k-contour. Right: The surface with k-level set. **END : caption**

In the demonstration, the control panel includes a toggle for turning on and off the plane z =k. In the domain window, the contour is indicates, and in the graph window, the level curve is displayed on the surface.

The

k - height of level set

slider determines the height of the contour. The level set and contour are displayed in yellow on the gray surface and domain, respectively. You can view the horizontal slicing plane

$z = k$ in the graph window by clicking on the

Do z = k Plane

(Checkbox)

This demonstration shows contours and level sets of functions of two variables.

It is useful to visualize the plane z = k as being the "water level" and the level set z = k as being the "shore line" where the water level meets the graph of the function. We think of the graph as being made of some porous material so that the water level shows up most often as a collection of "shorelines". In the case of the paraboloid described above, the level curves are the shorelines of circular pools that shrink down to a point and disappear as the level k sinks below the value 0.

In general we expect that the level sets of a function graph will be curves or families of curves, but at certain critical levels, the situation can be quite different. The contour might contain one or more isolated points, or the contour might consist of two or more curves intersecting at a point.

Exercise 14

Consider the contours of the function
$f(x, y) = x^2 - y^2$. There is one critical contour at level 0, where the graph of the function intersects the (x,y)-coordinate plane in a pair of intersecting straight lines. What are the other contours, for the natural domain of the function and for the standard square domain?

Exercise 15

Describe the contours of Twin Peaks (
$f(x, y) = -x^4 + 2x^2 - y^2$), and tell what happens to the contours as a result of a "geological shear" producing the family of functions
$f(x, y) = -x^4 + 2x^2 - y^2 + cx$ (obtained by adding the term cx to the formula for f, for different values of c).

Exercise 16

Do the same for Crater Lake ($f(x, y) = -(x^2 + y^2)^2 + 2(x^2 + y^2)$).

Exercise 17

Describe the contours of the function
$f(x, y) = x^2 - y^3$. More generally consider the contours of the family of surfaces
$f(x, y) = x^2 - y^3 + cy$ for various values of c. What are the critical levels when c = -1?

Exercise 18

Try the function $f(x, y) = \sin(10xy)$.

You may also want to look at level sets and contours in cylindrical coordinates .

6 Tangent Planes to Graphs

If the function

$f(x, y_0)$ that describes the

$y = y_0$ slice curve is a differentiable function of one variable, x , then we may compute its derivative using ordinary one-variable calculus and evaluate this derivative function at a point,

x_0 . As expected, this gives the slope of the tangent line to the curve at

x_0 . Because this process is just as valid for finding the derivative of an

$x = x_0$ slice curve as a function of y (i.e. neither is a complete derivative of the graph at the point

$(x_0, y_0))$, this slope is called the

the *partial* derivative of f with respect to x at

(x_0, y_0)

and it is denoted by

$f_x(x_0, y_0)$. In one-variable calculus, the equation of the line through the point

$(x_0, f(x_0))$ with slope m is given by

$y = f(x_0) + m(x - x_0)$, so the equation of the tangent line, with slope

$f'(x_0)$ is given by

$y = f(x_0) + f'(x_0)(x - x_0)$. We may analogously describe the tangent line to the

$y = y_0$ slice curve in the vertical plane

$y = y_0$ as the graph of the function

$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. Similarly we may then describe the tangent line to the

$x = x_0$ slice curve in the vertical plane

$x = x_0$ as the graph of the function

$z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$, where

$f_y(x_0, y_0)$ is the slope of the

$x = x_0$ slice curve at

(x_0, y_0) , called

the partial derivative of f with respect to y at

(x_0, y_0)

.

Since two lines determine a plane, as long as these two tangent lines are not parallel, we can find the plane containing both of them. This is called the Tangent Plane to the Graph of f at

$(x_0, y_0, f(x_0, y_0))$. One of the main tasks of the theory of differential calculus of two variables is to show that under certain hypotheses, the plane we have described contains the tangent lines to *all possible* smooth curves in the surface through this point.

Tangent lines to slice curves and tangent planes to surfaces **START : caption**

Left: The intersection curve between the y-slice plane and the surface $x^2 + y^3$ with the tangent line drawn in green. Right: The graph of $x^2 + y^3$ with the tangent plane at a selected point drawn in green. **END : caption**

In this demostration, we indicate the tangent lines to the slice curves at the chosen point

(x_0, y_0) , both in the slice curve windows and on the graph of the function. We may choose the length h of a segment in the tangent line to the

$y = y_0$ slice curve, above the interval

$x_0 - h \leq x \leq x_0 + h$, and do the same for the

$x = x_0$ slice curve above the interval

$y_0 - h \leq y \leq y_0 + h$.

We display as well the parallelogram determined by the segments in the tangent lines to the slice curves at

$(x_0, y_0, f(x_0, y_0))$. This parallelogram is the graph of the function

$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ over the rectangular domain

$x_0 - h \leq x \leq x_0 + h$,

$y_0 - h \leq y \leq y_0 + h$. The plane of this parallelogram is the only candidate for the tangent plane to the graph of f at

$(x_0, y_0, f(x_0, y_0))$.

The demonstration includes on the control panel a toggle for turning on and off this parallelogram determined by the two tangent segments at the chosen point.

When you select a point

(x_0, y_0) by clicking with the middle mouse button in the domain, the pair of yellow crosshairs on the domain highlights the horizontal curve

$y = y_0$ and the vertical curve

$x = x_0$. The corresponding curves

$(x, y_0, f(x, y_0))$ and

$(x_0, y, f(x_0, y))$ are highlighted on the surface in the graph window. The tangent lines to the

x_0 -slice and

y_0 -slice curves at the image of the point

(x_0, y_0) are displayed in the

X-Curve

and

Y-Curve

windows as well as on the slice curves shown on the graph in the surface window.

Clicking on the

Do Tangent Plane

checkbox causes the tangent plane determined by the two tangent lines to be displayed in the surface window instead of the lines themselves.

The tangent line to the

y_0 -slice curve at the value

x_0 appears as a segment defined over a subinterval, from

$x_0 - e$ to

$x_0 + e$, where

e is controlled by a slider. The same number

e determines the size of the portion of the tangent line to the

y_0 -slice curve at the value

y_0 . Clicking on the checkboxes in the control panel causes the slicing planes to be displayed in the surface window.

The tangent plane is displayed as a parallelogram in the plane determined by the two tangent segments. When you toggle solid surface mode the parallelogram will be filled in, which can make it easier to see how the tangent plane intersects the function graph. The intersection can give you information about special critical points of a graph, such as maxima and saddle points, a topic we will consider in detail in later labs .

This demonstration shows the tangent lines to two slice curves through a point on a function graph and the tangent plane to the graph at the point. This demo is based on the slice curves demo .

Exercise 19

Consider the function

$f(x, y) = x^2 + y^2$. For which points

(x, y) in the plane will the slopes of the tangent lines to both slice curves be positive? When will both be zero? What about the function

$f(x, y) = x^2 - y^2$?

Exercise 20

Consider the function

$f(x, y) = -x^4 + 2x^2 - y^2$ (Twin Peaks). For which points will the slopes of the tangent lines to both slice curves be zero?

You may also want to look at tangent lines to graphs in cylindrical coordinates .

Optional Material

7 Continuity and Differentiability

Continuity and *Differentiability* are concepts of fundamental importance in calculus of functions of one variable. They play a comparable role in two-variable calculus. They are best visualized using cylindrical coordinates, just one example of the benefits of this corrdinate system.

We say that a function f is
continuous
at a point
(x_0, y_0) if, for any
$\epsilon > 0$, the value of
$|f(x, y) - f(x_0, y_0)|$ can be made less than ϵ by making the distance between (x, y)
and
(x_0, y_0) sufficiently small, less than some
$\delta > 0$. We can investigate whether or not a function is continuous at a point by graphing over a disc domain of radius r centered at
(x_0, y_0) not only f but two constant functions,
$z = f(x_0, y_0) - \epsilon$ and
$z = f(x_0, y_0) + \epsilon$.
By looking straight down the positive z-axis, we can then see if the surface lies below the graph of
$z = f(x_0, y_0) + \epsilon$ for
$r < \delta$; by looking straight up the negative z-axis, we can see if the surface
$z = f(x_0, y_0) - \epsilon$ lies below the graph of f for
$r < \delta$. If we can choose a
δ for any given
ϵ , then we have proven that the function is continuous at that point.
Continuity Demonstration

This is a demonstration of the epsilon-delta definition of continuity of a function of two variables. You can choose a point in the domain and an epsilon and then show that there exists a satisfactory delta.

You can choose a point in the Domain window with the middle mouse button. At the corresponding point on the function graph (displayed in red) in the Surface window, two blue discs representing planes will be drawn, one on either side of the graph at a distance of epsilon. As epsilon decreases, these planes will approach contact with the graph. Delta is manifested as the radius of the discs.

The function is continuous if for any epsilon greater than zero, you can find a delta greater than zero, such that the discs *do not* intersect the function graph.To see wheher they do, you can press 'z' in the surface window, rotating the view so that you are looking straight down on the top of the surface parallel to the z-axis. The blue region should look like a disc (i.e. you should see the entire disc). If it does not, the upper disc is intersecting the surface and you have to make delta smaller. You can do the same thing for the lower disc by pressing "Z" (shift-z) to look up at the bottom of the surface parallel to the z-axis. The blue regions in each direction should look like discs.

This demonstration provides some additional functionality to make the visualization easier. The Zoom window contains a close up view of a small neighborhood of the

Optional Material

Here is a more complete discussion of continuity, including examples.

Continuity is a property of functions from one space to another. In the case of a function $f : R- > R$ from the real line to the real line, we say that f is continuous at x_0 if for all x near x_0, the value $f(x)$ is near $f(x_0)$.

To get an idea of what we mean by continuity, we consider some examples where this property fails. The simplest is called a jump discontinuity. Let $f : R- > R$ be defined by $f(x) = 1$ if x not equal to 0 and $f(0) = 0$. Then for x not equal to 0, we have $|f(x) - f(0)| = 1$ so it is impossible to make this difference smaller than 1 no matter how near we stay to the point $x = 0$.

In this example, it is possible to redefine the value of f(0) so as to make the function continuous. Letting $f(0) = 1$ leads to the constant function $f(x) = 1$ for all x, and this will be continuous no matter what definition we choose to use.

A more complex kind of discontinuity occurs in the signum function, defined by the condition $\text{sgn }(x) = -1$ if $x < 0$, $\text{sgn }(x) = 1$ if $x > 0$, and $\text{sgn }(0) = 0$. As in the previous case, if x is not equal to zero, then $|\text{sgn }(x) - \text{sgn }(0)| = 1$, so it is impossible to make this difference small by restricting the domain of x near 0. Moreover it is impossible to change the definition of $f(0)$ so as to make the function continuous at $x = 0$. If x is any positive number, then either $|f(x) - f(0)|$ or $|f(-x) - f(0)|$ will be greater than 1, so once again we cannot make the difference small by restricting x to lie in a small interval about zero.

In the first example above, the right-hand limit and the left-hand limit both exist and are equal. Only if $f(0)$ equals this common value will the function be continuous at 0. In the second example, the right- and left-hand limits both exist, but they are unequal so it is not possible to define $f(0)$ to be their common value. In general,we say that a function of one variable is continuous at a point if the right- and left-hand limits both exist and are equal to the value of the function at the point.

There are other ways for a function to be discontinuous at a point. One occurs where one or both of the one-sided limits fail to exist. For the function $f(x) = \frac{1}{x^2}$ when x is not equal to zero, there is no (finite) value of $f(0)$ that will make the function continuous at 0, since $|f(x) - f(0)|$ will become arbitrarily large as x gets nearer and nearer to 0.

Even if f is a bounded function, it is possible that the right-hand limit will not exist at 0. The standard example of this phenomenon occurs for the function $f(x) = \sin \frac{1}{x}$ for x not equal to zero. This function has value 1 for infinitely many values of x arbitrarily close to $x = 0$, at all $x = \frac{1}{(pi/2+2\pi k)}$ for positive integers k, and there are also infinitely many $x = \frac{1}{(3\pi/2+2\pi k)}$ for which $f(x) = -1$. Thus no matter how we define $f(0)$, there are values of x arbitrarily close to 0 for which $|f(x) - f(0)| > 1$.

This last example is very important since it provides a number of counterexamples to otherwise plausible conjectures. Consider the function $f(x) = x \sin \frac{1}{x}$ for x not equal to 0 and $f(0) = 0$. It is not hard to show that this function is continuous at 0, since $|f(x) - f(0)| = |x \sin \frac{1}{x} - 0| = |x||\sin \frac{1}{x}| < |x|$ and the limit of $|x|$ as x approaches 0 is 0. However this function is not differentiable at 0 since $\frac{|f(x)-f(0)|}{|x-0|<} = |\sin \frac{1}{x}|$ does not approach a limit as x approaches 0.

The function $f(x) = x^2 \sin \frac{1}{x}$ for x not equal to 0 and $f(0) = 0$ is differentiable at 0, since $\frac{|f(x)-f(0)|}{|x-0|} = |x sin \frac{1}{x}|$ which approaches 0 as x approaches 0. The function is differentiable for all other x just by the chain rule:

This lab introduced graphs of functions of two variables. It described three techniques for analyaing these graphs: looking at slice curves, looking at level sets and contours, and looking at tangent planes. It also introduced the concept of partial derivatives which is of unparalleled importance in multivariate calculus.