1 Introduction

In this laboratory we will treat functions of three variables. Each function \( f \) will have a domain, the collection of points \((x, y, z)\) in three-dimensional space for which it is defined. The graph will be the collection of points in four-dimensional space \((x, y, z, w)\) where \( w = f(x, y, z) \). We will extend as best we can the techniques we used with functions of two variables to functions of three variables. The problem of visualizing four-dimensional space limits our ability to thoroughly examine these functions, but with the aid of the tools developed in the previous labs, we can learn much about them.

2 Graphs of Functions

Viewing the graph of a function of 3 variables is not easy, since humans live in a 3-dimensional world and cannot see in 4-space. We will be examining functions mainly by looking at lower-dimensional objects such as slice curves, slice surfaces, and gradient vectors, which, as we have seen, can give us information about the functions.

We will also look at objects in 4-space by projecting them down into 3-space, which is then shown in the 2-dimensional space of the computer screen. If you think about how you view three-dimensional objects on a computer screen or a blackboard, you will see that you have no problem imagining something to be three-dimensional even though you are actually looking at it in two dimensions. What you have done is projected the object from 3-space into 2-space. There are mathematical formulae which describe methods of projection that allow for easy viewing, and the computer uses such formulae. We can perform the analogous operation on a four-dimensional object, using a mathematical formula to project it into three-space; the computer then projects it into two-space, to make the final product which appears on your screen.
For a function of three variables defined over a rectangular domain, the domain is a rectangular parallelepiped and it is of the form

\[ a_1 \leq x \leq b_1, \]
\[ a_2 \leq y \leq b_2, \]
\[ a_3 \leq z \leq b_3. \]

We will be looking at surfaces defined over this type of domain.

3 Slice Curves and Surfaces

As in the calculus of two variables, in the calculus of three variables a special role is played by the slice curves. An

\( xy \)-slice curve, for example, is obtained by selecting a particular value

\( x_0 \) of

\( x \) and

\( y_0 \) of

\( y \)

and then displaying the curve

\((x_0, y_0, z, f(x_0, y_0, z))\)

on the graph. Displaying these curves in four-dimensional space is difficult, but we can easily show the graph of

\((z, w)\), where

\( w = w(z) = f(x_0, y_0, z) \), in two-dimensional space. We can learn a great deal by studying these slice curves and their derivatives. The derivatives of the three slice curves are the first partial derivatives

\( f_x \),

\( f_y \), and

\( f_z \), just as the the derivatives of the two slice curves for a function of 2 variables are the partial derivatives of that function.

We see an

\( xz \)-slice curve similarly to the way we saw the

\( xy \)-slice curve, by displaying the curve

\((x_0, y, z_0, f(x_0, y, z_0))\); we see a

\( yz \)-slice curve by displaying the curve

\((x, y_0, z_0, f(x, y_0, z_0))\). We display the two-dimensional graphs

\((x, w)\)

and

\((y, w)\) where \( w \) is defined analogously for \( x \) and \( y \) to the way it is defined for \( z \).

Slice Curves

Left: The domain with a box around a selected point. Middle: The slice curves of

\( x + y^2 + z^3 \)

and two slicing planes. Right: The \( xz \)-slice curve.
blue. This will help you match what you see in 2-space to what you see in 4-space.

In addition, the horizontal axis in the Slice Curve in Slicing Plane window is always labeled with the name of the variable which is being allowed to run.

There are three things you should keep in mind about the window which contains a four-dimensional graph: first, pressing ‘z’ will not cause you to look down the z-axis but will return you to the default position for that window, with the w-axis vertical. Second, when you rotate the graph the objects in the window may appear to change in ways you didn’t expect. This is a function of the fact that you are looking at a piece of four-dimensional space. Third, remember that you are not seeing the entire hypersurface but only one particular projection of it. You can rotate the display with the mouse in the projective space, but you cannot see the entire function graph. The view is there solely to delineate how the three slice curves meet and the shape of the graph near their intersection.

There are many CheckBoxes in the Control Panel window, but do not be intimidated. Their functionality is relatively simple.

The \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \): TypeIn contains the definition of the function of the three variables, \( x \), \( y \), and \( z \). Its domain is defined in the three TypeIns entitled X Domain, Y Domain, and Z Domain.

If the Point from Mouse CheckBox is on, you can select a point in the The Domain window with the middle mouse button. If it is off, you can select the point with the three Sliders called X position slider, Y position slider, and Z position slider. The coordinates of the point are displayed in a printer under the domain window.

The point selected in the domain determines which slice curves are shown. They are the curves in the slicing planes that go through the point.

This point in the domain and the domain itself can be displayed in several different ways. These are controlled by the six CheckBoxes called Bounding Box, Floors, Shadows, Domain Points, Point as Box, and Spindles. You should experiment with different combinations of these features to see which gives the clearest representation for you.

The central Slider 1=xy-curve... can be used to select which slice curve and corresponding slicing plane is displayed in the Slice Curve in Slicing Plane... window. The three values the Slider can attain represent the \( xy-, xz-, \) and \( yz-\) curves.

The three CheckBoxes called Do XY-Slice, etc. may be used to select which slice curve(s) are displayed in the Projected Graph of the Slice Curves window. These are independent of which curve is shown in the other window. The three CheckBoxes Do ZW-Plane, etc.
toggles the display of the slicing planes in the Projected Graph window.

This demo shows the graphs of the slice curves of a function of three variables. These slice curves are shown in two ways: as curves in two-space, that is, in their slicing planes; and as curves in four-space, as they would appear on the graph of the function (a hypersurface), projected into three-space.

**Exercise 1**
Describe the slice curves of the function \( f(x, y, z) = e^{xyz} \).

**Exercise 2**
Describe the slice curves of the function \( f(x, y, z) = (\sin x)(\sin y)(\sin z) \).

**Exercise 3**
Examine the slice curves of the function \( f(x, y, z) = (x^2 + y^2)(1 - x^2 - y^2 - z^2) \).

Notice that we had to choose constant values for two variables and let one variable run in order to obtain a curve in four-space. What happens if we hold only one variable constant and let the other two run? We get a slice surface rather than a slice curve. In the calculus of three variables a special role is played by the slice surfaces. A

- \( z \)-slice surface is obtained by selecting a particular value \( z_0 \) of \( z \) and then displaying the surface \( (x, y, w) \) for \( w = f(x, y, z_0) \) as a graph in three-dimensional space. Similarly we get an \( x \)-slice surface by displaying the surface \( (y, z, f(x_0, y, z)) \) and a \( y \)-slice surface by displaying the surface \( (x, z, f(x, y_0, z)) \).

One reason that slice surfaces are interesting is that the union of any set of consecutive parallel slice surfaces (by parallel, we mean generated by the same pair of variables) is the entire hypersurface (or a piece of it). This is a similar idea to reconstructing a function of two variables by looking at a group of \( x \)- or \( y \)-slice curves.

Slice Surfaces **START : caption**

Left: The domain with a box around a selected point. Middle: The \( x,y \), and \( z \)-slicing surfaces for
The x-surface is always shown in red, the y-surface in yellow, and the z-surface in blue. This will help you match what you see in 3-space to what you see in 4-space. In addition, the axes in the A Slice Surface Window is always labeled with the names of the variables which are being allowed to run.

There are two things you should keep in mind about the window which contains a four-dimensional graph: first, pressing ‘z’ will not cause you to look down the z-axis but will return you to the default position for that window, with the w-axis vertical. Second, when you rotate the graph the objects in the window may appear to change in ways you didn’t expect. This is a function of the fact that you are looking at a piece of four-dimensional space.

This demo is exactly analogous to the previous one, displaying slice surfaces instead of slice curves.

Exercise 4
Examine the slice surfaces of the functions in exercises 2.1-2.3.

Exercise 5
Describe the slice surfaces (especially the z-slices) of the following functions.
\[ f(x, y, z) = -x^4 + 2x^2 - y^2 + zx \]
\[ f(x, y, z) = -(x^2 + y^2)^2 + 2(x^2 + y^2) + zx. \]

4 Contours and Level Sets

In addition to slices where
\[ x , \]
\[ y , \text{ or} \]
\[ z \text{ are held constant, it is extremely important to study the slices by horizontal hyperplanes , where} \]
\[ w \text{ is held constant. The points of the graph lying in the hyperplane} \]
\[ w = k , \text{ where} \]
\[ k \text{ is a constant, make up the k-level set , and the set of points} \]
\[ (x, y, z) \text{ in the domain for which} \]
\[ f(x, y, z) = k \text{ is called the k-contour . For most values of} \]
\[ k , \text{ we expect that the} \]
\[ k \text{-level set will be a surface, or a collection of surfaces, but it may be more complicated.} \]
We are especially interested in locating the heights of the level sets that have singularities, manifested, for example, by the surface crossing itself or degenerating to a curve or a point, because at these heights the hypersurface has critical points.

Contours and Level Sets

Left: The $f(x,y,z) = k = 1$ contour surface for $x + y^2 + z^3$ with the x, y, and z-slice curves. Right: The function graph projected into fours space for $f(x,y,z) = k = 1$. END: caption

This demo shows a level set (also called a level surface since it is a surface) and a contour of a function which is input by the reader. The value of $k$ is input with a slider. The Level Set Under 4-Space Projection window displays the level set in 4-space projected down into 3-space and the The Contour window shows the contour in 3-space (xyz-space). Note that this means that the pictures in the two graphing windows are not of the same object! The two objects have the same $(x, y, z)$ -coordinates, but the contour surface depends on the particular value of $k = w$ selected whereas the 4-Space Projection is only displaying a subset of the entire object and it is really all of these contour surfaces put together. The contour is the part of the domain which, when the function is applied to it, yields the level set.

Level sets of functions of three variables are hard to calculate, so you will need to be patient and give fnord a few seconds to display the surfaces and to display any changes you make in the control panel.

You can also show slice curves on the level surface as it is cut by planes parallel to the $XY$-, $XZ$-, and $YZ$-coordinate planes. These curves are defined by setting one of $x$, $y$, or $z$, as well as $w$, equal to a constant. The Do X-Slice, Do Y-Slice, and Do Z-Slice CheckBoxes toggle the display of these curves in both windows. The height of each slice curve on the level surface is determined by the Slice Scale Sliders. This number is relative to each particular domain.

This demo shows a level set (also called a level surface since it is a surface) and a contour of a function which is input by the reader.

**Exercise 6**

Examine the level sets of the function $f(x, y, z) = -x^4 + 2x^2 - y^4 + 2y^2 - z^4 + 2z^2$.

**Exercise 7**

Examine the level sets of the function $xy(1 - x^2 - y^2 - z^2)$.
5 The Tangent Hyperplane to a Graph

If the slice curves are differentiable, it is possible to display their tangent lines. The tangent lines to the

- $xy$ -slice curve, the
- $yz$ -slice curve, and the
- $xz$ -slice curve at a point determine a hyperplane in four-space, the tangent hyperplane to the graph of the surface at the point, analogous to the tangent plane to the graph of a function of two variables. Analogous to that case, a critical point occurs when the tangent hyperplane is perpendicular to the $w$-axis.

The equation for the tangent hyperplane at a point $(x_0, y_0, z_0, f(x_0, y_0, z_0))$ will be given by

$$w = f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) + f(x_0, y_0, z_0)$$

The Tangent Hyperplane

The $x$, $y$, and $z$ -slicing curves with their tangent lines at the origin accompanied with the tangent planes formed by these same tangent lines and the tangent hyperplane.

You can use the checkboxes to turn on the slice curves and their tangents and to display the

- $xw$ -
- $yw$ -
- $zw$ - slicing planes, just like in the Slice Curves demo except that now the tangent lines are always displayed along with the slice curve.

You can turn the tangent planes on and off with checkboxes in the lower half of the control panel. The

- $XZ/YZ$
tangent plane, determined by the yellow tangent to the
- $xz$ -slice and the blue
- $yz$ -slice, is shown in green; the
- $XY/YZ$
tangent plane, determined by the red tangent to the
- $xy$ -slice and the blue
- $xz$ -slice, is shown in purple; The
- $XY/XZ$
tangent plane, determined by the red tangent to the
- $xy$ -slice and the yellow
xz -slice, is shown in orange. There is also a checkbox in the control panel to toggle the display of the hyperplane. The control panel also includes a scaling value for the length of the tangent vectors and therefore the size of the tangent planes and the tangent hyperplane. When you hit the space bar in the graph window the tangent planes will get solid, but the tangent hyperplane will remain in wire frame mode in order to allow you to see the tangent planes inside it.

This demo shows the tangent hyperplane to a graph. Each set of two out of the three tangent lines to slice curves determines a tangent plane at the point \( f(x_0, y_0, z_0) \), since two vectors always determine a plane. The three tangent planes determine a tangent hyperplane, shown as a parallelepiped.

### 6 The Chain Rule, Gradient Vector, and Directional Derivative

As in the case of functions of two variables, if we have a curve
\[
(x(t), y(t), z(t))
\]
in the domain of a function, we may form the new function
\[ w(t) = f(x(t), y(t), z(t)) \]
and differentiate it with respect to \( t \). The tangent line to the curve
\[
(x(t), y(t), z(t), w(t))
\]
at the point
\[
(x_0, y_0, z_0, f(x_0, y_0, z_0)) = (x(t_0), y(t_0), z(t_0), w(t_0))
\]
will lie in the tangent hyperplane at that point, so we have
\[
w'(t_0) = f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \]
This is the three-dimensional form of the chain rule.

Analogously to the two-variable case, we define a gradient vector at each point \((x, y, z)\) of the domain of a function by
\[
\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z))
\]
The chain rule then has the form
\[
w'(t) = \nabla f(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))
\]
Critical points occur when the gradient vector is the zero vector.

The Gradient Vector

The XY, XZ, and YZ -Gradient Vector Planes for
\[ f(x, y, z) = x + y^2 + z^3 \]
where the x, y, and z position sliders are set to 0. When the demo initially comes up you will see only the gradient vector at the current point in the domain. You will be aided in finding critical points by the

---

8
option of viewing the gradient vectors in planes, chosen with the
Do XY-Plane of Gradients
, for example. To see the entire vector field, turn on the
Do Domain of Gradients
checkbox.

The gradient vectors are shaded from blue at their tails to red at their heads,
in the direction they are pointing.

Under the domain window is a Printer that displays the value of the gradient
vector at the selected point. You may move around the point (with the mouse or
the Sliders) and try to locate the points where this value is zero. These are the
critical points of the function.

This demo shows the gradient vector field in the domain. You should examine
the patterns made by gradient vectors near critical points of a function. Recall that
you can identify the critical points by the fact that the gradient vector is zero there.

In the calculus of three variables, we generalize the directional derivative by
choosing a unit vector
\((u_1, u_2, u_3)\) and computing the derivative of
\(w(t) = f(x(t), y(t), z(t))\) , where
\(x(t) = x_0 + tu_1\ ,\)
\(y(t) = y_0 + tu_2\ and\)
\(z(t) = z_0 + tu_3\ ,\) with respect to
\(t\) . It follows from the general chain rule that

\[ D_u(f(x, y, z)) = \nabla f(x, y, z) \cdot (u_1, u_2, u_3) \]

The Directional Derivative START : caption

The directional derivative for
\(f(x, y, z) = x + y^2 + z^3\) as a function of
\(\theta\) and
\(\phi\) . END : caption

Just as in the directional derivatives demo for the functions of two variables ,
the slice line in the direction of the unit vector is shown in yellow in the domain
window. The graph of the height function \(w(t)\), which is calculated along the line in
the domain, is shown in another window, and the graph of the directional derivative
as a function of two variables is also shown. A vertical orange line in the
Graph of the Directional Derivative
window marks the current values of
\(\theta\) and
\(\phi\) chosen with sliders in the input window (If you can’t see the orange line it is
probably because the angles are chosen so that the directional derivative is small and the orange line is small as a consequence.

The Do Gradient at Point CheckBox toggles the display of the gradient vector of the function at the point (scaled by 0.5) in the domain window. The value of this vector is printed in the

The Gradient at the Point Printer beneath this window. The value of the directional derivative at the point in the direction of the angles you have chosen is displayed in a Printer under the The Graph of the Directional Derivative window.

As you change the angles determining the directional derivative its value will change. You will find that when the line the angles determine is parallel to the gradient vector (which you can observe in the domain window), the directional derivative is either at its maximum or its minimum.

For functions of three variables we need two angles to specify the unit vector which is the directional vector along which the directional derivative is computed. The directional derivative is thus a function of two variables, \( \theta \) and \( \phi \). The angle \( \theta \) determines how far around the vector swings in the xy-plane, and the angle \( \phi \) determines how far it swings out of the xy-plane. For \( \phi \), the value \(-\pi/2\) causes the vector to point parallel to the negative z-axis and the value \(\pi/2\) causes it to point parallel to the positive z-axis.

Also note that for a curve \((x(t), y(t), z(t))\) in a level set of a function, we have

\[
0 = w'(t) = \nabla f(x(t), y(t), z(t)) \cdot (x'(t), y'(t), z'(t))
\]

. This is true for all tangent vectors to the surface at any point \((x(t), y(t), z(t))\) in the surface, so it follows that the gradient vector \(\nabla f(x(t), y(t), z(t))\) is perpendicular to the tangent plane at each point. Thus for any point \((x_0, y_0, z_0)\), we have

\[
\nabla f(x(t), y(t), z(t)) \cdot (x - x_0, y - y_0, z - z_0) = 0
\]

As a particular case of this result, consider a function of three variables given by

\[
F(x, y, z) = z - f(x, y)
\]

, where \(f\) is some function of two variables. The level set \(F(x, y, z) = 0\) of this function is the graph of the original function of two variables, \(f(x, y)\). We note that
\[ \vec{\nabla} F(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)) = (-f_x(x, y), -f_y(x, y), 1) \]

It follows that the tangent plane to the level set at a point \((x, y, z)\) can be written
\[ 0 = (-f_x(x, y), -f_y(x, y), 1) \cdot (x-x_0, y-y_0, z-z_0) = -f_x(x, y)(x-x_0) - f_y(x, y)(y-y_0) + (z-z_0) \]
which agrees with our earlier formula
\[ z = f_x(x, y)(x-x_0) - f_y(x, y)(y-y_0) + z_0 \], where
\[ z_0 = f(x_0, y_0) \].

### 7 LaGrange Multipliers for Functions of Three Variables

In Lab 3, we discussed LaGrange Multipliers for Functions of Two Variables. They were used as a technique for locating the critical points of a function \(f(x, y)\), subject to a constraint curve given by \(g(x, y) = 0\). The method easily generalizes to functions of three or more variables.

Given a contour of a function \(g(x, y, z)\), we can find the maximum value of the function \(f(x, y, z)\) on this contour by looking for points at which a contour to \(f\) is tangent to the contour of \(g\). At such a point the gradient of \(f\) is parallel to the gradient of \(g\). This leads to three equations \(f_x = \lambda g_x, f_y = \lambda g_y, \) and \(f_z = \lambda g_z\). We may eliminate \(\lambda\) to get three conditions \(f_x g_y = f_y g_x, f_z g_x = f_x g_z, f_z g_y = f_y g_z\), one of which is a consequence of the other two. This gives two surfaces in space that intersect in a collection of curves. Where these curves cut the original contour surface of \(g\) will be the critical points of \(f\) restricted to that contour (assuming that none of the critical points of \(f\) itself lie on the contour).

A variation on LaGrange multipliers asks for the critical points of function \(f\) of three variables restricted to a curve in space. If it is a parametrized curve, then the same analysis as given above will show that the tangent line at the curve must be perpendicular to the gradient vector to the contour of \(f\) at any critical point.

If the curve is given by the intersection of two surfaces \(g(x, y, z) = c\) and \(h(x, y, z) = d\), then at a critical point of \(f\) on the intersection of these contours, we must have \(\nabla f\) perpendicular to the cross product of \(\nabla g\) and \(\nabla h\). This will happen when \(\nabla f\) is a linear combination of \(\nabla g\) and \(\nabla h\), leading to three equations, \(f_x = \lambda g_x + \mu h_x, f_y = \lambda g_y + \mu h_y, \) and \(f_z = \lambda g_z + \mu h_z\). Eliminating both \(\lambda\) and \(\mu\) leads to a single equation that must be satisfied, and this locus will intersect the original curve of intersection of the two contours at a finite number of points (in general), thus identifying the critical points of \(f\) on the intersection of the contours.
LaGrange Multipliers for Functions of Three Variables

In this demonstration you are to find the correct values of the variables for which the function is maximized (or minimized) with the constraint.

The function may be entered in the The Function $f$ Type-In. It should be a function of three variables, but we are only looking at one of its level sets at a time. The domain of this function may be entered in the x domain for $f$, the y domain for $f$, and the z domain for $f$ Type-Ins.

A level set of this function, chosen with the $k$ Slider as a scale value in the $k$ domain, is displayed in blue in the The Function and Constraint window. This is the only view window in the demonstration.

A point on the level set may be chosen with the middle mouse button in this window. The point on the set closest to where you click will be chosen. Beware that this is a very delicate method of choosing points. Sometimes more than one point will be selected and sometimes none will. If either of these cases occurs, click again nearby.

The constraint function is less general here than in the two dimensional case. This marks the major theoretical difference between the two demonstrations. The constraint is a function $z = g(x, y)$ of two variables. It may be entered in the The Constraint Type-In, and its domain may be entered in the x domain for $g$ and y domain for $g$ Type-Ins. The constraint surface is displayed in red in the view window.

You may choose a point on the constraint with the X-Coordinate (scaled) on Constraint and the Y-Coordinate (scaled) on Constraint Sliders. When the Sliders are at 0, the point is at one corner of the surface and when the Sliders are at 1, it is at the opposite corner.

At the points chosen on the surfaces, you may display three things, crosses to mark the points, the gradient vectors of the functions at the points, or sections of the tangent planes to the surfaces at the points.

The coordinates of each point as well as the gradient vector there are displayed in the Information window.

Your goal is to adjust the height of the level set of $f$ with the $k$ Slider so the blue surface is tangent to the red surface. When you think it is, move the points on both curves to the point of tangency.

If it is indeed tangent, the gradients will be parallel, as will the tangent planes. The Solved? Printer in the Information window will then read "True" and the value of $\lambda$ will be displayed in the Lambda Printer. This value is the ratio of lengths of the two gradient vectors.
The Stringency Slider controls how difficult the problem is. This is how close to parallel the gradients must be and how close the points must be. You might want to start with a high stringency and then lower it once the Solved? Printer reads ”True”. In this way, you can get closer and closer to correct.

When the problem is ”solved”, it means that you have found the value of k, and values of x, y, and z such that the function f is maximized (or minimized) subject to the constraint function g = 0.

One thing to note is that three Printers Function point , Gradient of Function at Point , and Lambda may contain more than one value. These values should all be equal so just look at the first one. The reason they have more than one has to do with how we calculate level sets in Fnord.

This is a demonstration of LaGrange multipliers for functions of three variables in which you locate the values of the variables for which the function is maximized (or minimized) with the constraint geometrically. The demonstration will inform you when you are correct.

This demonstration is an extension of the demonstration on LaGrange multipliers for functions of two variables. The constraint curve here, however, is limited to a function of two variables.

Exercise 8

Using LaGrange Multipliers, find the closest point on the graph of $g(x, y, z) = xyz - 4$ to

1) the origin
2) the point (1,0,2)

8 The Hessian Determinant

There is an analogue of the Hessian determinant for functions of three variables, but it is much more difficult to use than in the case of two variables. In order to find the Hessian determinant, it would be necessary to form the matrix of second partials, where the first row is $(f_{xx}, f_{xy}, f_{xz})$, the second is $(f_{yx}, f_{yy}, f_{yz})$, and the third is $(f_{zx}, f_{zy}, f_{zz})$, and take its determinant (if you are not familiar with this terminology, don’t worry; you don’t have to know it). There are, however, some special cases where the Hessian determinant is easy to compute, for example $f(x, y, z) = Ax^2 + By^2 + Cz^2$. For this function, we have $\nabla f(x, y, z) = (2Ax, 2By, 2Cz)$ and $f_{xx} = 2A, f_{yy} = 2B$, and $f_{zz} = 2C$ for all x, y, and z. The Hessian determinant will be positive if $ABC > 0$ and negative if $ABC < 0$. If all coefficients are positive, then the function has a local minimum at the origin and the Hessian is positive. If all coefficients are negative, then the origin is a local maximum but the Hessian is negative. If one of the coefficients
is negative and the others are positive, then the level sets near the critical point are hyperboloids of one sheet, and if exactly two are negative, the level sets are hyperboloids of two sheets.