Integration of Functions of Three Variables

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1 Introduction

In the study of functions of two variables $f(x, y)$ defined over a region in the plane, the double integral has two interpretations: the signed volume between the graph of $z = f(x, y)$ and the $xy$-plane or the total mass of a lamina with density determined by the function $f(x, y)$. In the case of a function of three variables $f(x, y, z)$, the graph $w = f(x, y, z)$ is a subset of four-dimensional space, and the analogue of the interpretation of the double integral as signed volume is the interpretation of the triple integral as signed hypervolume under the graph. This is not as easy to visualize as its analogue is in three-space. However, there are other uses of the triple integral. For example, we can still consider a region in three-dimensional space with a density $f(x, y, z)$ and look at the triple integral of the density function over the region as the total mass of the region.

2 Triple Integrals and Volume

The triple integral is calculated as an iterated integral just as double integrals were. That is, the integral of $f(x, y, z)$ over a three-dimensional region, $R$, in space given by $a \leq x \leq b$, $c \leq y \leq d$, and $g \leq z \leq h$ is $\int \int \int_{R} f(x, y, z) dz dy dx$ and is calculated iteratively, one integral at a time, starting from the inner one ($dz$). This integral gives the hypervolume under the hypersurface $f(x, y, z)$ over the region.

Computing hypervolumes is not all that useful but, fortunately, triple integrals have other uses. One of these is to compute three-dimensional volumes. In one-dimension, the integral $\int_{I} 1 ds$ gives the length of the interval $I$. In two-dimensions, the integral $\int \int_{R} 1 dA$ gives the area of the region $R$. Likewise, in three-dimensions, the integral $\int \int \int_{R} 1 dV$ gives the volume of the three-dimensional region $R$.

Although this volume can be computed with a double integral, in some cases it is easier to use a triple integral.
Example 1

The volume of the rectangular solid given by \(a \leq x \leq b, \ c \leq y \leq d, \) and \(f \leq z \leq g\) is the value of the triple iterated integral \(\int_a^b \int_c^d \int_f^g \, dzdydx\)

This method is most useful, however, when dealing with more complicated objects which we represent with more complicated domains. As in the case of functions of two variables defined over a non-rectangular domain, we can integrate over non-rectangular regions in space. We can define the limits of a family of such regions by: \(z\) goes from \(p(x, y)\) to \(q(x, y)\), \(y\) goes from \(c(x)\) to \(d(x)\), and \(x\) goes from \(a\) to \(b\). As we were previously able to define some regions in the plane with two functions of \(x\), \(c(x)\) and \(d(x)\), and other regions with two functions of \(y\), \(a(y)\) and \(b(y)\), so here we can define other regions by giving \(y\) or \(z\) constant limits and defining the limits of the other two variables in terms of functions. However, these domains still have the same basic form as the family we will be investigating here.

As with double integrals, we often want to change coordinate systems to facilitate integration. For this, we need a three-dimensional equivalent to the Jacobian Determinant. You might expect that this will be a three-dimensional determinant similar to the Jacobian. In fact it is.

For a change of variables given by \(x = g(u, v, w)\), \(y = h(u, v, w)\), and \(z = j(u, v, w)\), we have

\[
\begin{vmatrix}
g_u & g_v & g_w \\
h_u & h_v & h_w \\
j_u & j_v & j_w \\
\end{vmatrix}
\]

\[
= g_u \cdot (h_v j_w - h_w j_v) - g_v \cdot (h_u j_w - h_w j_u) + g_w \cdot (h_u j_v - h_v j_u)
\]
Optional Material
The derivation of this formula is very similar to what we did in one less dimension, but it is available for the curious.

We extend our development of the Jacobian to three dimensions as follows.

Our change of variables is given by the mapping $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = j(u, v, w)$. The rectangular frame with lower-front-lefthand corner $(x, y, z)$, with edges of length $dx$, $dy$, and $dz$, in the positive $x$, $y$, and $z$ directions, respectively, and volume $dxdydz$, is mapped to a parallelopiped with the three vectors

$$(g(u + du, v, w) - g(u, v, w), h(u + du, v, w) - h(u, v, w), j(u + du, v, w) - j(u, v, w)),$$

$$(g(u, v + dv, w) - g(u, v, w), h(u, v + dv, w) - h(u, v, w), j(u, v + dv, w) - j(u, v, w)),$$

and

$$(g(u, v, w + dw) - g(u, v, w), h(u, v, w + dw) - h(u, v, w), j(u, v, w + dw) - j(u, v, w))$$

as bases.

We divide these by the differentials $dx$, $dy$, and $dz$, respectively, and take the limit as they approach zero. Our new vectors are $(g_u, h_u, j_u)$, $(g_v, h_v, j_v)$, and $(g_w, h_w, j_w)$. Linear Algebra tells us that the volume of a parallelopiped with bases $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ has volume equal to $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

The volume of our parallelopiped is then

$$\det(g_u, h_u, j_u; g_v, h_v, j_v; g_w, h_w, j_w)$$

2.1 Spherical Coordinates

To demonstrate the use of this method, we introduce a new coordinate system, spherical coordinates. This, like cylindrical coordinates, is an extension of polar coordinates to three dimensions, but instead of being based on a cylinder, it is based on the unit sphere. We will show how to translate a point $(x, y, z)$ into spherical coordinates.

The first coordinate, $\rho$ (rho), is the distance, in space, of the point from the origin. (It differs from $r$ in cylindrical coordinates in that the $r$ is the distance in the $xy$-plane.) The distance formula gives $\rho = \sqrt{x^2 + y^2 + z^2}$.

The second coordinate, $\theta$, is the same as it is in cylindrical coordinates, the counter-clockwise angle between the line connecting the point to the origin and the positive $x$-axis, measured in radians.
The third coordinate, \( \phi \) (phi), like \( \theta \), is an angle. It is the angle between the line connecting the point to the origin and the \( z \)-axis, again measured in radians. \( \phi \) is limited to the domain \(-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\), to avoid multiple definitions of a point (other than the origin) and by convention \( \phi = -\frac{\pi}{2} \) corresponds to the negative \( z \)-axis and \( \phi = \frac{\pi}{2} \) to the positive \( z \)-axis. (Some people use the alternate convention that \( 0 \leq \phi \leq \pi \), where \( \phi = 0 \) corresponds to the positive \( z \)-axis and \( \phi = \pi \) to the negative \( z \)-axis, so be careful).

The formulae for the last two variables are \( \theta = \tan^{-1} \frac{y}{x} \) and \( \phi = \tan^{-1} \frac{z}{\rho} \), where \( \rho \) is defined as above. The changes of coordinates in both directions are then:

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \tan^{-1} \frac{y}{x} \\
\phi &= \tan^{-1} \frac{z}{\rho} \\
x &= \rho \cos \theta \cos \phi \\
y &= \rho \sin \theta \cos \phi \\
z &= \rho \sin \phi
\end{align*}
\]

A straightforward calculation gives \( dV \) for this change of coordinates to be \( dV = (\rho^2 \sin \phi) d\phi d\theta d\rho \).
Example 2

We want to calculate the volume of a sphere of radius \( r \). This would be messy in rectangular or cylindrical coordinates so we do it in spherical coordinates.

Our domain is as follows: \( 0 \leq \rho \leq r \), \( 0 \leq \theta \leq 2\pi \), \( 0 \leq \phi \leq \pi \).

The volume integral is then
\[
V = \int_0^r \int_0^{2\pi} \int_0^\pi (\rho^2 \sin \phi) \, d\phi \, d\theta \, d\rho.
\]
Integrating gives
\[
V = \frac{4}{3}\pi r^3,
\]
which is the well-known formula for the volume of a sphere.

3 Total (Signed) Mass of Three-Dimensional Domains

As in the case of a rectangular region in the plane with variable density given by a function defined on the region, we can approximate the total mass of a rectangular box with variable density by a triple sum, where we divide the \( x \)-domain by \( m \) equally spaced points into \( m - 1 \) subintervals, and similarly divide the \( y \)-domain into \( n - 1 \) subintervals and the \( z \)-domain into \( p - 1 \) subintervals. This divides the entire domain into \((m-1)(n-1)(p-1)\) small rectangular boxes, on each of which the variable density can be approximated by its value at the lower left bottom vertex. Thus, for a box whose sides lie between \( x_{i-1} \) and \( x_i \), \( y_{j-1} \) and \( y_j \), and \( z_{k-1} \) and \( z_k \), we approximate the density by \( f(x_{i-1}, y_{j-1}, z_{k-1}) \). The total mass is then approximated by a triple sum

\[
\sum \sum \sum f(x_{i-1}, y_{j-1}, z_{k-1})(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})
\]
which converges to the triple integral

\[
\iiint f(x, y, z) \, dz \, dy \, dx.
\]

This technique, of course, is not limited to rectangular domains. We can equally find the mass of non-rectangular Cartesian domains, and of domains in curvilinear coordinates.

It is important to note that if the function can assume both positive and negative values, the regions where \( f(x, y, z) \) is negative will be counted as having negative mass. This is why we refer to the result of the integral as the “signed” mass. As before, you should realize that negative densities are purely theoretical and models of real systems should have always-positive density functions.

4 Demonstrations of the Total (Signed) Mass of Various Three-Dimensional Domains

There are three demonstrations (accessed by clicking on the corresponding image) of calculating the mass of a domain with variable density. They get progressively
more complicated. It is not necessary for you to do all three, but it is advised to do one of the first two before the third, since they will clarify what it is doing.

Do not forget that if you want to calculate the volume of the domain, instead of the mass, enter \( f = 1 \) for the density function. Then the integral will be equal to the volume as well as the mass.

In all three demos, the density is shown by color; the densest points are red, and the least dense are blue. In between, the color gradually shifts from blue to green to yellow to red, according to the density. To see how this works, consider a piece of metal heated over a direct flame. As the metal heats up, the hottest points become white, the points with medium temperature become orange, and the points that are least hot remain silver or black. Here, instead of temperature, we are measuring density, and the densest points are not white, but red. The least dense points are not black or silver, but blue.

This method of coloring a three-dimensional domain by the value of a function is an important way of picturing functions of three variables. This technique allows us to use color to picture the fourth dimension. In this case we are calling the function density, but it could represent anything we want - for instance, electrical charge, shaded from blue in the less charged regions to red in the more charged regions, or temperature, shaded from blue in the cooler regions to red in the warmer regions.

### 4.1 Rectangular Domains

This demo shows the total signed mass of a rectangular domain in three-space with variable density. The single viewing window shows the domain, colored by the value of the density function \( f(x, y, z) \) at each point from blue to green to yellow to red. That is, points in the domain where the density is lowest are colored blue. As the density function gets larger, the color of the point gradually goes from blue to green, then to yellow, and finally to red, the color of points with the largest density. In order to see points in the inside of the domain, you can drag up and down the white point on the top face of the box. As you do so, the appropriate slice of the domain is shown on this face.

**Exercise 1**

Plug in various density functions and compare the corresponding total masses of the domain. For example try \( f(x, y, z) = ax^2 + by^2 + cz^2 \) for various values of \( a, b, \) and \( c \), both positive and negative. For which values of the constants over what domains is the density function always positive and thus a realistic density function?
4.2 General Cartesian Domains

This demo shows the total signed mass of a non-rectangular Cartesian domain in three-space with variable density. The single viewing window shows the domain, colored by the value of the density function $f(x, y, z)$. The domain is colored by the density function $f$ just like in demonstration 1.

The domain is defined by $a \leq x \leq b$, $c(x) \leq y \leq d(x)$, $p(x, y) \leq z \leq q(x, y)$

It is worthwhile to spend some time trying various functions to get used to the way these limits define the shape of a region.
4.3 Curvilinear Domains

Domains In Curvilinear Coordinates

START : caption
The domain of the function \( f(r, \theta, z) = r^2 \theta \) graphed in cylindrical coordinates with variable density (color) representing the fourth dimension. END : caption

This demonstration is more complicated than its siblings, because of the more complicated definition of the domain. The density function is input as usual in a Type-In. Now, however, it is a function of \( u, v, \) and \( w \), instead of \( x, y, \) and \( z \). The mapping from \((u, v, w)\)-space to \((x, y, z)\)-space is defined by three functions, \( x(u, v, w), y(u, v, w), \) and \( z(u, v, w) \).
The domain is defined, similar to the General Cartesian demo, with $u$ traversing an interval $[a, b]$, $v$ bounded by two functions of $u$, called $c(u)$ and $d(u)$, and $w$ bounded by functions of $u$ and $v$, called $p(u, v)$ and $q(u, v)$.

Because of the change of coordinates, the Mass computation (displayed in the Total (Signed) Mass of Domain Printer) may be inaccurate. To boost the accuracy, the resolutions of all three variables should be sufficiently high.

There are two view windows. One, The Domain (uvw-space), shows the original domain, colored according to density (red being denser than yellow). The second, The Domain (xyz-space) shows the domain mapped to rectangular coordinates, also colored by density. It is the latter domain whose total mass we are calculating.

This demo shows the total signed mass of a domain in user-defined coordinates in three-space with variable density. One viewing window shows the domain, another the domain mapped into Cartesian coordinates, each colored by the value of the density function $f(x, y, z)$ at each point from yellow to red. That is, points in the domain where the density is lower are colored yellow, gradually shading to parts where the density is greater, colored red.

It is worthwhile to spend some time trying various functions to get used to the way they define the shape of a region.