1 Introduction

Vector fields are objects which you have encountered in previous labs, in the form of gradient vector fields. This lab is designed to let you work with more general fields and, in the last section, with line integrals, a topic that builds on the concept of a vector field.

2 Vector Fields in Two Dimensions

2.1 Definition of a Two Dimensional Vector Field

A vector field is defined by a function which associates a vector with each point in the domain of the function. In two dimensions a vector field $V$ can be represented by the expression

$$V(x, y) = (p(x, y), q(x, y))$$

The functions $p$ and $q$ may be arbitrarily defined, or they may depend on some other function, as in the case of a gradient vector field.

This demonstration shows a vector field, allowing you to define the field and its domain. It displays a vector field in two dimensions with the vectors represented by arrows. The arrow on each segment represents the head of the vector.

The functions $p(x, y)$ and $q(x, y)$ that define the vector field $V(x, y) = (p(x, y), q(x, y))$ are in the Controls window. The Vector Field window shows the vectors at each of the grid points. To make the vector field easier to see, the vectors are scaled down to be smaller than their true lengths; this prevents them from running over each other and creating a complete mess. There is also a window where you may observe the length of the vector field $|V(x, y)| = \sqrt{p^2 + q^2}$ as a function graph over the domain.
Exercise 1
Study the gradient vector field of a function $f(x, y)$. If you know the function whose gradient you want to look at and you don’t want to calculate the gradient by hand, you can enter it in the following way: suppose the function is $f(x, y) = x^2 + y^2$. Then you would enter $x^2 + y^2$ in the $f(x,y)$ field, and enter $\text{grad}(f)(x,y)$ in the $V(x,y)$ typein field.

Exercise 2
Study the vector field $V(x, y) = (x, y)$ and the rotational field $V(x, y) = (−y, x)$.

Exercise 3
Study the field $W(x, y) = \left(\frac{-y}{(x^2+y^2)}, \frac{x}{(x^2+y^2)}\right)$, defined to be $(0,0)$ at the origin. You can enter this field in the demo in the following way: if $(x,y) = (0,0)$ then $(0,0)$ else $(-y/(x^2+y^2), x/(x^2+y^2))$

Exercise 4
Study the vector field $V(x, y) = (-y^2, x^2)$. 
2.2 Divergence and Curl in Two Dimensions

We can define several quantities associated with a vector field that will help us characterize the field. Two of these are the divergence and the curl. Developing a sense of what these quantities mean takes some practice.

The divergence of a vector field $V(x, y) = (p(x, y), q(x, y))$ is defined to be

$$\text{div } V = \nabla \cdot V = p_x + q_y$$

The divergence is thus a scalar value.

The curl of a vector field is a quantity that can be obtained only from a three-dimensional vector field $V(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z))$. It is defined to be

$$\text{curl } V = \nabla \times V = (r_y - q_z, p_z - r_x, q_x - p_y)$$

The curl of a vector field is thus another vector field. When we are working in two dimensions, with $x$ and $y$ only, we consider the function $r$ to be defined as identically zero, and all partial derivatives with respect to $z$ to be zero. In other words, we write our two-dimensional vector field $V$ as $V = (p(x, y), q(x, y), 0)$. Thus the curl of a vector field in two dimensions is $(0, 0, q_x - p_y)$. Since the first two coordinates are always zero, it makes sense to graph the third coordinate of the curl, which is also the signed length of the curl, as a function graph over the domain.

![Demonstration 2. Divergence and curl in two dimensions](image)

This demonstration shows the divergence and the signed magnitude of the curl of a vector field in the plane. After entering the functions $p(x, y)$ and $q(x, y)$ that define the vector field $V(x, y) = (p(x, y), q(x, y))$, you can observe three different things. One window shows the vector field and domain. Another window shows the
divergence of the vector field $V(x, y)$, namely $\text{div } V = p_x + q_y$, as a function over the domain, graphed in red. The last window shows the signed length of the curl, equal to $-p_y + q_x$, as a function graph over the domain, graphed in beige.

**Exercise 5**
Study the divergence and curl of the vector field $V(x, y) = (x, y)$ and the rotational field $V(x, y) = (-y, x)$. Do you begin to get a sense of what these quantities measure (think of their names)? Start the vectors as very short and continuously increase their lengths. How does the field seem to be moving in each case?

**Exercise 6**
As above, study the field $W(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, defined to be $(0, 0)$ at the origin.

**Exercise 7**
What are the curl and divergence of the constant function $V(x, y) = (1, 1.5)$? Why?

**Exercise 8**
Examine the curl of a few gradient vector fields, such as those of Twin Peaks and Crater Lake. Make a generalization about the curl based on what you observe. (See exercise 1.1 for a convenient way to enter the gradient vector field of a particular function.)

**Exercise 9**
Examine the function $V(x, y) = (y \cos(x), x \sin(y))$. (This function due to Marsden and Tromba, Vector Calculus.)

### 3 Vector Fields in Three Dimensions

#### 3.1 Definition of a Three Dimensional Vector Field
In three dimensions a vector field $V$ can be represented by the expression

$$V(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z))$$
Demonstration 3. General vector fields in three dimensions

The functions $p$, $q$, and $r$ may be arbitrarily defined, or they may depend on some other function, as in the case of a gradient vector field.

This demonstration shows a vector field in three-space, allowing you to define the field and its domain. The vector field is defined over a rectangular domain and has the form $V(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z))$. You have the option of seeing the gradient vector field in the whole domain or of seeing the gradient vectors in one or more planes at a time, by using the checkbox and variable in the Controls window.

Exercise 10
Examine the default vector field, $V(x, y, z) = (-z^2, x^2, -y^2)$.

Exercise 11
Examine the vector field $V(x, y, z) = (-y, x, 0)$.

Exercise 12
Examine the gradient vector field of a function of three variables. You can enter this function the same way you did for the function of two variables (see Exercise 1).
3.2 Divergence and Curl in Three Dimensions

The same quantities that we used to characterize vector fields in two dimensions are useful in three dimensions.

The divergence of a vector field \( V(x, y, z) = (p(x, y, z), q(x, y, z), r(x, y, z)) \) is defined to be

\[
\text{div } V = \nabla \cdot V = p_x + q_y + r_z
\]

The divergence is thus a scalar value.

The curl of a vector field is defined to be

\[
\text{curl } V = \nabla \times V = (r_y - q_z, p_z - r_x, q_x - p_y)
\]

The curl of a vector field is thus another vector field.

**Demonstration 4. Divergence and curl in three dimensions**

This demonstration shows the divergence and the signed magnitude of the curl of a vector field in three-space along with the original vector field. It combines options that you have used for viewing general vector fields in three dimensions (previous demo) with options you have used for viewing slice surfaces of functions of three variables (in Lab 4).

**Exercise 13**
Examine the curl and divergence of the default vector field, \( V(x, y, z) = (-z^2 x, x^2 y, -y^2 z) \).
Exercise 14
Examine the divergence of a few curl vector fields. If you know the function whose curl you want to look at and you don’t want to calculate the curl by hand, you can enter it in the following way: suppose the function is \( W(x,y,z) = (x^2y, y^2z, -z^2x) \). Then you would erase everything in the \( V(x,y,z) \) function type-in in the input window, and enter: 
\[
curl(func(x,y,z){x^2, y^2, -z^2})(x,y,z)
\]

4 An Exposition of Nabla

Divergence and curl are both quantities associated with vector fields. We know how to compute them and we know that they must contain useful information about how the vector field behaves. We would now like to find out what this information is.

By now, you have seen the \( \nabla \) (the nabla symbol, often read “del”) operator associated with three quantities: the gradient, divergence, and curl. One way to look at \( \nabla \) is as a vector operator. We can say (in three dimensions) that
\[
\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})
\]

When we take the gradient of a function, we apply \( \nabla \) to it.
\[
\nabla f = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}).
\]

To find the divergence of a vector field, \( V = (p, q, r) \), we take the dot product of \( \nabla \) with \( V \) and apply each differential operator to the corresponding component of \( V \).
\[
\nabla \cdot V = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (p, q, r) = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}.
\]

The first summand, \( \frac{\partial p}{\partial x} \), measures the change in the \( x \)-component of \( V \), \( p(x, y, z) \), as \( x \) changes. In other words, it tells us how much the \( x \)-component of \( V \) stretches as we move in the \( x \) direction. The other two summands give similar information in the \( y \) and \( z \) directions. Thus the divergence at a point measures how much the vector field \( V \) “spreads out” (or “diverges” from itself) at the point. A positive divergence means the vector field has a net expansion from the point; a negative divergence means it has a net contraction into the point.

For this reason, the divergence of the vector field \( V = (x, y, z) \) is always positive (in fact it is identically equal to 3). At every point, the vector field “spreads out” away from the point. On the other hand, the divergence of the rotational field
\( V = (-y, x) \) is identically zero because the field never has a net expansion from (or contraction into) a point.

The curl of a vector field, \( V = (p, q, r) \) is defined as \( \nabla \times V \), the cross product of the \( \nabla \) operator with each vector in the field.

\[
\nabla \times V = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (p, q, r) = \left( \frac{\partial q}{\partial z} - \frac{\partial r}{\partial y}, -\frac{\partial p}{\partial z} + \frac{\partial r}{\partial x}, \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} \right).
\]

This is another vector field. In two dimensions, we simply let \( r(x, y, z) = 0 \), so \( \nabla \times V = (0, 0, \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x}) \).

The curl at a point measures the net rotation of the vector field around the point. The rotational vector field in two dimensions, \( V = (-y, x) \) has curl with magnitude \( |\nabla \times V| = 1 \) at all points; it has the same “net rotation” around all points in the plane. A way to visualize this is to imagine a very small paddle with four fins forming a cross at right-angles (i.e. like a + sign). Now imagine that the plane is filled with water, and that the vector field tells us the velocity of the water (in other words, a water particle at the point \( (x, y) \) has velocity \( V(x, y) \)). If we put the axis of the paddle (the center of the +) at a point, the curl at the point would tell us how fast the paddle rotates.

These three quantities, the gradient, divergence, and curl, are all interrelated in an interesting way. A gradient vector field is a special type of vector field. Not every field is a gradient. A gradient vector field, \( V = (p, q) \) (we’ll stick to two dimensions for now), has the property that there exists a family of functions, \( F(x, y) \), called the potential functions of \( V \), such that \( p = F_x \) and \( q = F_y \). Given this information, how do we discern whether a given field is a gradient?

A general vector field \( V \) is a gradient if the above conditions hold. One additional thing that we know about \( F \) is that, as long as it is well-behaved, \( F_{xy} = F_{yx} \). Consequently, we can say that \( p = F_x \), and \( q = F_y \) if and only if \( \frac{\partial p}{\partial y} = F_{xy} = F_{yx} = \frac{\partial q}{\partial x} \); that is, if \( 0 = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = \nabla \times V \) which is to say, the curl of \( V \) is identically zero. It follows that for all \( C^2 \) functions, \( \nabla \times \nabla F(x, y) = \nabla \times V(x, y) = 0 \), the curl of the gradient is always zero. These identities are also true in three dimensions.

We can then integrate \( V = (f_x, f_y) \) component-wise to find \( F \). We first integrate \( G(x, y) = \int f_x \, dx \) and we know that \( F(x, y) = G(x, y) + g(y) \) where \( g(y) \) is some function of the single variable \( y \) that disappears upon differentiation with respect to \( x \). Similarly, we integrate \( H(x, y) = \int f_y \, dy \) to find \( F(x, y) = H(x, y) + h(x) \). Then we solve \( F(x, y) = G(x, y) + g(y) = H(x, y) + h(x) \) to find what \( F(x, y) \) must be.
Example 1

Suppose we are given the vector field \( V = (2xy, x^2 + 1) \). We would like to find out if it is a gradient and if so what its potential function is.

We begin by differentiating. \( \frac{\partial p}{\partial y} = \frac{\partial(2xy)}{\partial y} = 2x \) and \( \frac{\partial q}{\partial x} = \frac{\partial(x^2+1)}{\partial x} = 2x \).

So the curl of \( V \) is given by

\[
\nabla \times V = \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = 2x - 2x = 0.
\]

implying that \( V \) is indeed the gradient of some potential function \( F(x, y) \).

To find \( F \) we set \( V = (2xy, x^2 + 1) = (F_x, F_y) = \nabla F \). Integrating, we have

\[
F(x, y) = \int F_x \, dx = \int 2xy \, dx = x^2y + g(y)
\]

and

\[
F(x, y) = \int F_y \, dy = \int (x^2 + 1) \, dy = x^2y + y + h(x).
\]

Setting these equal, we get \( g(y) = y + h(x) \). But since \( h(x) \) depends only on \( x \) and \( g(y) \) only on \( y \), we can conclude that \( h(x) \) must be a constant, \( h(x) = C \).

Thus, we get that \( F(x, y) = x^2y + y + C \).

A vector field that is a gradient (i.e. has a potential function) is called conservative.