Minimally Clued Latin Square Puzzles

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My Puzzle Biography in Four Stages

- Magazines: GAMES and Dell/Penny Press
- MIT Mystery Hunt
- National Puzzlers' League
- International logic puzzle community

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FAIR WARNING: You never know where a puzzle might be hidden...
Especially if you're at a t[A]lk given by a puzzle enthusiast.

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- Other hunt-style puzzles: BAPHL, Gen Con, WIRED
- Abstract logic: MAA FOCUS, Wordplay, Grandmaster Puzzles
- Traditional logic: *The Puzzle Files of Larry Logic*

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- Currently tester, occasional constructor for Grandmaster Puzzles

Latin Square Completions

A **Latin square** is an *N*-by-*N* arrangement of the n[U]mbers 1 through *N* such that no row or column contains any repeats.

A **critical set** in a Latin square is a collection of numbers in a grid that can be extended to exactly one Latin square, and which has no proper subsets with that same property.

A critical set can be viewed as a puzzle (with no unnecessary clues) where the goal is to complete the Latin square.

Example: Sudoku

Most Latin square completion puzzle [T]ypes have additional information given.

Goal of traditional Sudoku: Complete a 9-by-9 Latin square, where each 3-by-3 “box” must also contain a full set of numbers.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Defining a “Puzzle”

Let’s define a **clue** to be any constraint that applies to certain N-by-N Latin squares.

Any set of clues that describes a unique Latin square (and thus could be solved) is an ***N*-puzzle**. Unlike a critical set, an N-puzzle could include unnecessary clues.

For example, sudoku has:

- Given number clues that appear in specific sudoku
- 3-by-3 box clues that appear in all sudoku

Minimal Clue Sets

Others have asked: How small can a critical set for a Latin square be?

I want to ask: How small can a clue set for an N-puzzle be? (And what does “small” mean?)

For the rest of this talk, we’ll discuss different types of Latin square puzzles, and look for examples of these puzzles that can be constructed with minimal clue sets.

Futoshiki

Futoshiki is a Latin square puzz[L]e type that originated in Japan and appears in some British newspapers. Clues include both given numbers and “greater than” comparisons between certain squares.

1			
		3	
4		>	

1	2	4	3
3	4	1	2
2	1	3	4
4	3	2 >	1

Types of Clues in Futoshiki

Let's begin to form a taxon[O]my of puzzle clues:

Definition

An **entry clue** *requires a particular number to be placed in a particular grid square.*

Definition

An **ordering clue** *specifies an ordered pair of grid squares and requires their contents to be related by a total or partial ordering (determined by the puzzle type).*

Types of Clues in Futoshiki

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A **futoshiki** ***N*****-puzzle** *is an N-puzzle consisting of entry clues and orderin[G] clues relating adjacent numbers (using the usual ordering for the numbers 1 to N).*

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This is not the only possible type of ordering clue. Other puzzle types could give:

- Information on whether numbers are divisible by adjacent numbers
- Ordering relationships between non-adjacent numbers

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- Information on whether numbers are divisible by adjacent numbers
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We'll refer to the ordering clues in futoshiki as **futoshiki ordering clues.**

Minimally Clued Futoshiki Puzzles

We could look for futoshiki puzzles with the l[O]west total number of clues, but since entry and ordering clues function differently, there is a two-dimensional search space.

Definition

An (h, k)**-futoshiki** *N***-puzzle** *is an N-puzzle with h entry clues and k futoshiki ordering clues.*

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Definition

*An (h, k)-**futoshiki** N**-puzzle** is an N-puzzle with h entry clues and k futoshiki ordering clues.*

Question

For each $k \geq 0$, what is the minimum h needed for a valid futoshiki N-puzzle?

In other words, how many entry clues are needed to make a solvable N-by-N futoshiki with only k ordering clues?

A Preliminary Fact

Proposition

If an (h, k)-futoshiki N-puzzle exists, the[N]

$$h \geq (N-1) - \frac{3}{2}k.$$

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Proof: The clues must provide information about at least $(N-1)$ rows and at least $(N-1)$ columns, or else we could permute two rows or two columns in one solution to obtain another.

Each entry clue adds information about at most one row and one column, and each ordering clue adds information about at most two rows and a column (or vice versa). This implies that $2h + 3k \geq 2(N-1)$, which is equivalent to the statement above.

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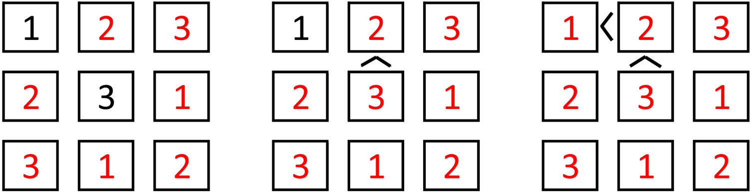
This limits the search space enough to start experimenting with low values of N.

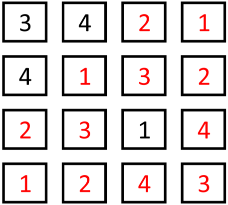
Futoshiki 3-puzzles



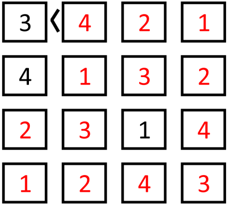
(2,0) (1,1) (0,2)

Note that any pairs with $h + k < 2$ would violate the inequality from earlier.

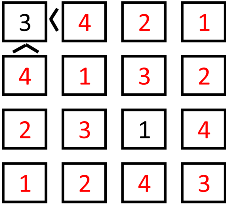
Futoshiki 4-puzzles



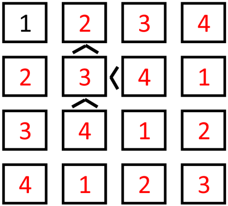
(4,0)



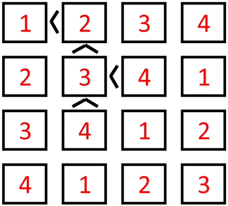
(3,1)



(2,2)

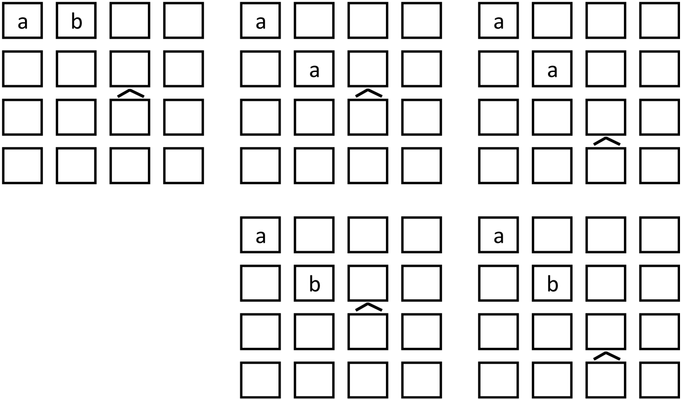


(1,3)

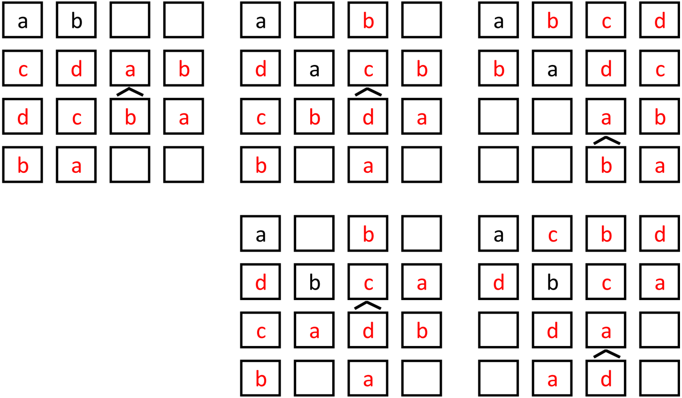


(0,4)

Showing There is No (2,1)-futoshiki 4-puzzle

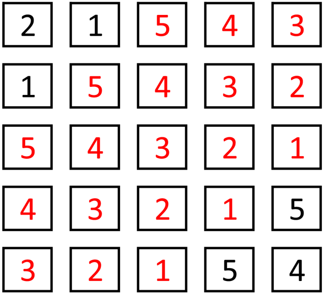


Showing there is no (2,1)-futoshiki 4-puzzle



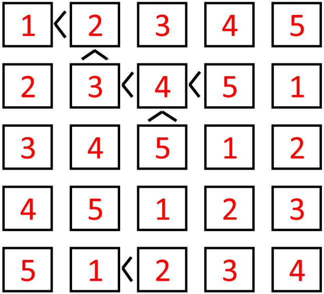
We can similarly show that $(3,0)$, $(1,2)$, and $(0,3)$ cannot be achieved.

Futoshiki 5-puzzles



(6,0)

[Can be modified to (5,1), (4,2), (3,3), (2,4)]



(0,6)

[Can be modified to (1,5), (2,4), (3,3), (4,2)]

Potential Patterns

There exists an $(h, 4-h)$-futoshiki 4-puzzle for every $h = 0, 1, \ldots, 4$, achievable using one of two structures.

There exists an $(h, 6-h)$-futoshiki 5-puzzle for every $h = 0, 1, \ldots, 6$, achievable using one of two structures.

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There exists an $(h, 6-h)$-futoshiki 5-puzzle for every $h = 0, 1, \ldots, 6$, achievable using one of two structures.

Question

Let H be the minimal value for which there exists an $(H, 0)$-futoshiki N-puzzle. Does there always exist an $(h, H-h)$-futoshiki N-puzzle for $h = 0, 1, \ldots, H$?

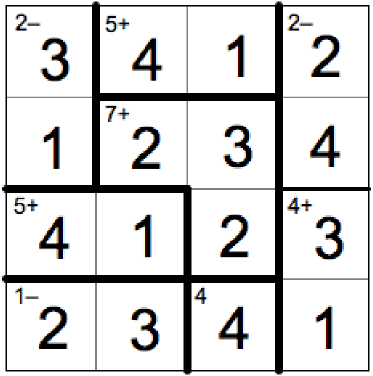
Question

For a given N, how many structures are needed to construct all futoshiki N-puzzles that have minimal h with respect to k?

KenKen

KenKen is another Latin square puzzle ty[P]e, created by Tetsuya Miyamoto. This type usually has no given numbers, but has “cages” marked with arithmetic operations and numbers.

When the operation is applied to the numbers in the cage, the given number should be the result.

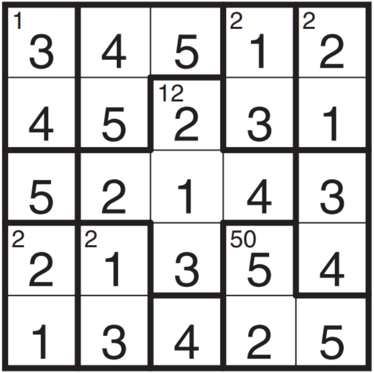


The terms KenKen and Kendoku are trademarked, so KenKen puzzles in international puzzle competitions were often called Calcudoku. Until...

TomTom

Thomas Snyde[R] adapted KenKen and renamed the result **TomTom**, with three key changes:

- Cages may specify a result without an operation
- Not all squares are necessarily part of a cage
- Cages with more than two squares can use subtraction or division as an operation



Example by Bryce Herdt

Types of Clues in TomTom and KenKen

Adding to our tax[O]nomy of clues:

Definition

An **unambiguous content clue** *requires a particular set of grid squares to contain (in an unspecified order) a specified multiset of numbers.*

Definition

An **ambiguous content clue** *requires a particular set of grid squares to contain (in an unspecified order) one of a collection of specified multisets of numbers.*

Types of Clues in TomTom and KenKen

Definition

*A k-**TomTom** N-**puzzle** is an N-puzzle consisting of k content clues where for each clue, the set of grid squares forms a contiguous region, and the specified multisets are determined by a given arithmetic result (with or without a given operation).*

These particular content clues may or may not be am[B]iguous.

In a TomTom 5-puzzle:

- A 2-square "20"clue must contain $\{4, 5\}$
- A 2-square "7" clue could contain either $\{2, 5\}$ or $\{3, 4\}$.

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From here on, we will narrow our focus to unambiguous content clues.

Minimally Clued TomTom Puzzles

How many cont[E]nt clues (of any size) can successfully clue a puzzle?

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Question

Let $f(N)$ be the minimum number k for which there exists a k-TomTom N-puzzle. How does $f(N)$ grow as N does?

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How many cont[E]nt clues (of any size) can successfully clue a puzzle?

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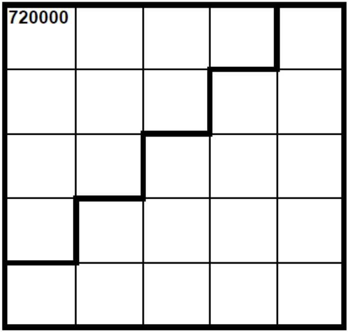
Let $f(N)$ be the minimum number k for which there exists a k-TomTom N-puzzle. How does $f(N)$ grow as N does?

Answer: It doesn't.

A TomTom with One Clue!



2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4
1	2	3	4	5

$$720000 = 5^4 \cdot 4^3 \cdot 3^2 \cdot 2^1$$

Using this same structure, we can construct a 1-TomTom N-puzzle for any N, with the clue region containing $(N^2 - N)/2$ squares.

Minimizing TomTom Clue Area

We can always clue a TomTom with one clue, but how large does that clue have to be?

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Question

Given N, what is the minimum area in the single clue in a 1-TomTom N-puzzle?

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Question

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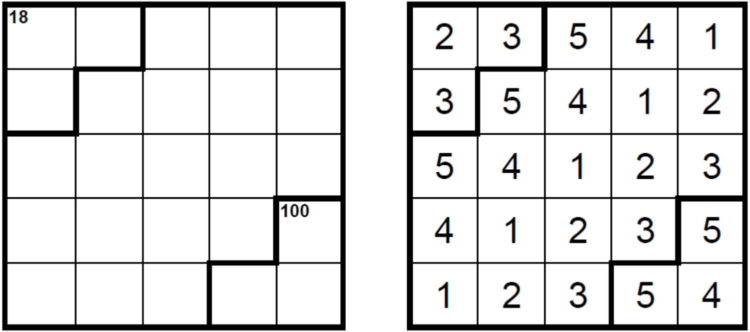
Question

Given N and k, what is the minimum total area of the clues in a k-TomTom N-puzzle? Average area?

A TomTom with Two Clues but Less Clue Area



Using this same structure, we can construct a 2-TomTom N-puzzle with the clue areas totaling $\lfloor N^2/4 \rfloor$ squares.

This structure covers about one-fourth of the grid, while the previous structure covered almost one-half.

Two Clues Might Already Be Optimal For Area

Definition

A **strongly completable critical set** *of a Latin square is one where the remaining entries are independently "forced" in some order by the given entries.*

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Theorem

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So even with more than two clue regions, we can't cover less than $\lfloor N^2/4 \rfloor$ squares... unless the resulting structure is *weakly* completable. (Those are hard to construct, and even harder to generalize.)

Back to Sudoku

How can we characterize [S]udoku with our new cluing terminology?

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
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Definition

A **traditional sudoku 9-puzzle** *is a 9-puzzle consisting of nine unambiguous content clues (where the regions are disjoint 3-by-3 boxes and the specified set for each is* $\{1,2,3,4,5,6,7,8,9\}$*) and some number of entry clues.*

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5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
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(Note: An entry clue is a type of content clue. But it's common enough to deserve its own name.)

Minimally Clued Sudoku Puzzles

Theorem

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The McGuire/Tugemann/Civario paper did not show why 16-to-17 is a "special" border... Instead, it strategically narrowed the search space of possible 16-entry-clue sudoku until the remaining 5.4 billion options could be checked using 7.1 million hours(!) of computing time.

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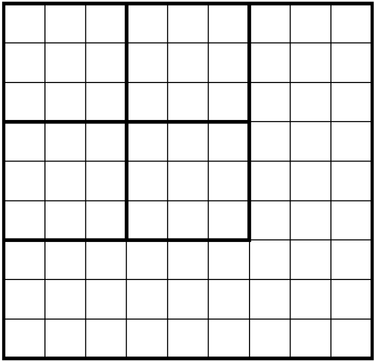
Can we minimize the number of content clues? That’s silly, since traditional sudoku inherently have nine of these. **Or do they?**

The Dark Secret of Sudoku

Five of the content clues in a traditiona[L] sudoku are unnecessary!



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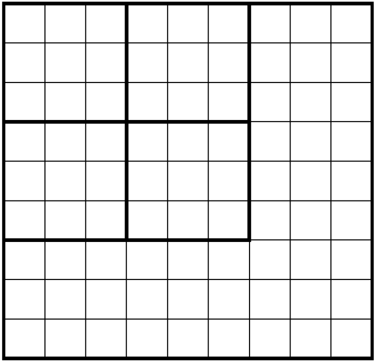

A short combinatorics exercise for the audience: How many of the 126 different sets of four content clues force all nine content clues?

Irregular Sudoku

In irregular sudoku, the “boxes” are replaced by irregul[A]r regions.

		9		8	2		5	
		7	4			1		3
		8	5	2				
1								
				3				2
7								
						4		
							3	

3	6	9	1	8	2	7	5	4
5	2	7	4	6	9	1	8	3
9	1	8	5	2	4	3	7	6
1	9	2	3	5	8	6	4	7
8	5	4	7	3	6	9	1	2
4	7	6	2	1	3	8	9	5
7	8	3	6	4	1	5	2	9
2	3	5	8	9	7	4	6	1
6	4	1	9	7	5	2	3	8

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5	2	7	4	6	9	1	8	3
9	1	8	5	2	4	3	7	6
1	9	2	3	5	8	6	4	7
8	5	4	7	3	6	9	1	2
4	7	6	2	1	3	8	9	5
7	8	3	6	4	1	5	2	9
2	3	5	8	9	7	4	6	1
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Definition

An **irregular sudoku N-puzzle** *is an N-puzzle consisting of N unambiguous content clues (where for each clue the region is contiguous and the specified set is $\{1, 2, 3, \ldots, N\}$) and some number of entry clues.*

Minimally Clued Irregular Sudoku Puzzles

Claim: An irregular sudoku N-puzzle must have at least $(N - 1)$ en[T]ry clues.

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Proof: If there are at most $(N - 2)$ entry clues, there are at least two numbers that do not appear as entry clues. In any valid solution, these two numbers can be permuted to form another solution, so the solution cannot be unique.

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Much More Audacious Claim: For every N, there exists an irregular sudoku N-puzzle with $(N - 1)$ entry clues!

An Irregular Sudoku N-puzzle with $(N-1)$ givens

					1
					2
					3
					4
					5

2	3	4	5	6	1
3	4	5	6	1	2
4	5	6	1	2	3
5	6	1	2	3	4
6	1	2	3	4	5
1	2	3	4	5	6

Questions about Generalized Sudoku

Definition

An (h, k)**-generalized sudoku** N**-puzzle** *is an* N*-puzzle consisting of* k *unambiguous content clues (where for each clue the region is contiguous and the specified set is* $\{1, 2, 3, \ldots, N\}$*) and* h *entry clues.*

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- We have seen that $(4, 17)$-generalized sudoku 9-puzzles exist (through traditional sudoku).
- We have seen that $(N-1, N-1)$-generalized sudoku N-puzzles exist (through irregular sudoku).

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- We have seen that $(N - 1, N - 1)$-generalized sudoku N-puzzles exist (through irregular sudoku).

Question

For a given N, what (h, k)-generalized sudoku N-puzzles exist? How low can h be for a given k, and how low can k be for a given h?

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- A pair of adjacent squares containing a 1 and a 2 may have either color dot.
- Any pair of adjacent squares not connected by a dot **must not have either property**.

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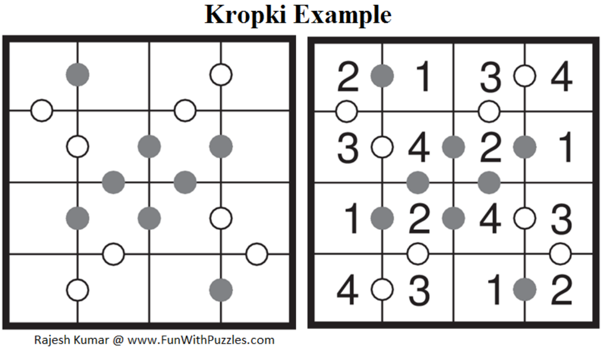
- Any pair of adjacent squares not connected by a dot **must not have either property**.

A **Kropki Pairs** puzzle omits the last rule. (In this variant, a lack of dot does not give any information.)

Example of a Kropki



Kropki Sudoku (sudoku with kropki-style dot clu[E]s) is also a popular variant.

Types of Clues in Kropki

Definition

A **graph adjacency clue** *specifies an pair of grid squares and requires their conte[N]ts to be adjacent on a specified graph.*

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A graph adjacency clue is a specific type of ambiguous content clue.

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An (h, k)**-kropki** N**-puzzle** *is an* N*-puzzle consis[T]ing of h graph adjacency clues specifying adjacent grid squares and the "consecutive graph," k graph adjacency clues specifying adjacent grid squares and the "doubling graph," and* $2N(N-1) - h - k$ *graph adjacency clues specifying adjacent grid squares and the "neither graph."*

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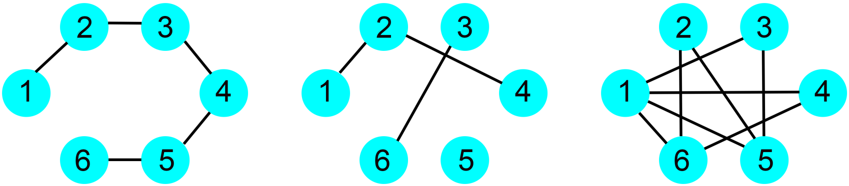

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Question

Given N*, what is the minimum possible value of* $h + k$ *for a valid* (h, k)*-kropki* N*-puzzle?*

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One Last Puzzle

	2		1	
	5			
			3	
		4		

The grid above can be filled with information from earlier in this presentation to yield a one-word answer associated with Brown University.

One Last Puzzle

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	5			
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Thank you, and enjoy the rest of SUMS!