Brown University Mathematics

Trigonometry Boot Camp

(Prepared by Dan Katz)

This module consists of a video, exercises, and solutions.

- It <u>IS</u> intended for students who have previously taken trigonometry and wish to review it in preparation for calculus.
- It <u>IS NOT</u> intended for students attempting to learn trigonometry for the first time.

Radians vs. Degrees

- There are two different ways to measure angles: radians + degrees

$$\text{One full rotation} = 360^\circ \text{ (degrees)} = 2\pi \text{ (radians)}$$
$$\text{Right angle} = 90^\circ \text{ (degrees)} = \frac{\pi}{2} \text{ (radians)}$$

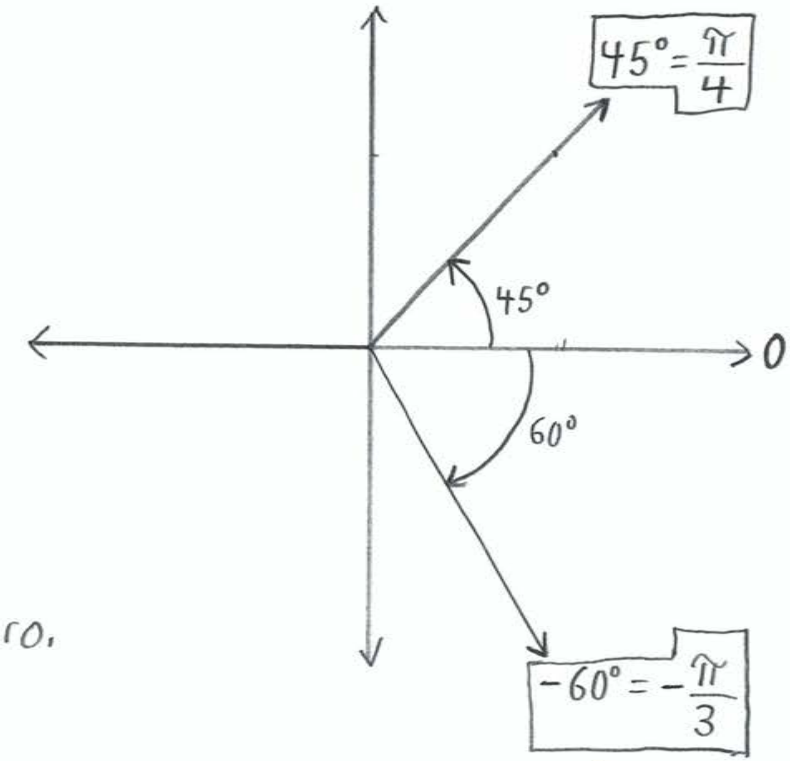
- To convert: $\boxed{210^\circ} = 210 \text{ deg} \times \dfrac{2\pi \text{ rad}}{360 \text{ deg}} = \dfrac{420\pi}{360} \text{ rad} = \boxed{\dfrac{7\pi}{6}}$

$$\boxed{\frac{3\pi}{4}} = \frac{3\pi}{4} \text{ rad} \times \frac{360 \text{ deg}}{2\pi \text{ rad}} = \frac{1080\pi}{8\pi} \text{ deg} = \boxed{135^\circ}$$

- When doing calculus, it's important to use radians!

(Otherwise, certain calculations give wrong answers.)

Angles as Directions



- We can identify any angle with a direction from the origin.
  - The angle zero points directly to the right.
  - Positive angles are rotated counterclockwise from zero.
  - Negative angles are rotated clockwise from zero.
  - Angles larger than 2π (or less than -2π) rotate through at least one full rotation before reaching a direction.

- Multiple angles can point in the same direction! For example, $-\frac{\pi}{2}$, $\frac{3\pi}{2}$, and $\frac{7\pi}{2}$ all point straight down.

Trigonometric Functions

- Trigonometric functions convert <u>angles</u> into <u>ratios</u>.

- There are six standard trig functions:

  - $\sin\theta$ (sine)
  - $\cos\theta$ (cosine)
  - $\tan\theta$ (tangent)
  - $\csc\theta$ (cosecant)
  - $\sec\theta$ (secant)
  - $\cot\theta$ (cotangent)

- In a way, $\sin\theta$ and $\cos\theta$ are the "real" trig functions, and the other four are just modifications of these two.

The unit circle, sine and cosine

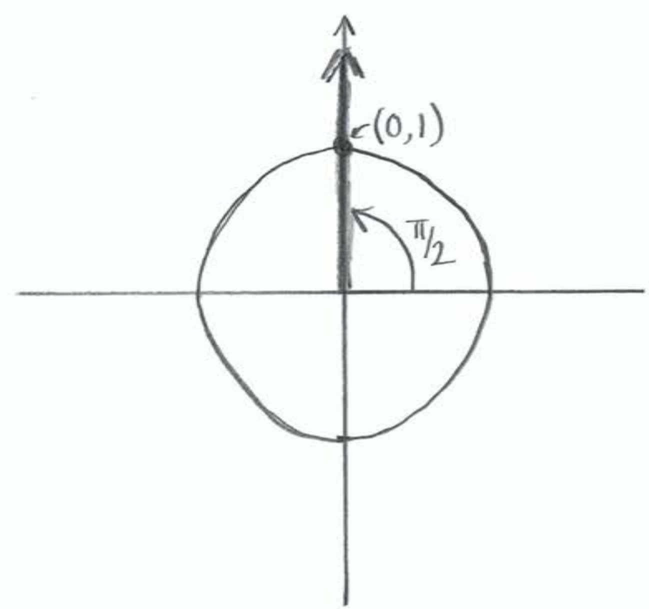
- The circle $x^2+y^2=1$ (the unit circle) is centered at the origin with radius 1.

- Every direction from the origin intersects the unit circle at exactly one point.

- Given an angle θ, if the direction θ intersects the circle at the point (x,y),

  then: $\cos\theta = x.$

  $\sin\theta = y.$

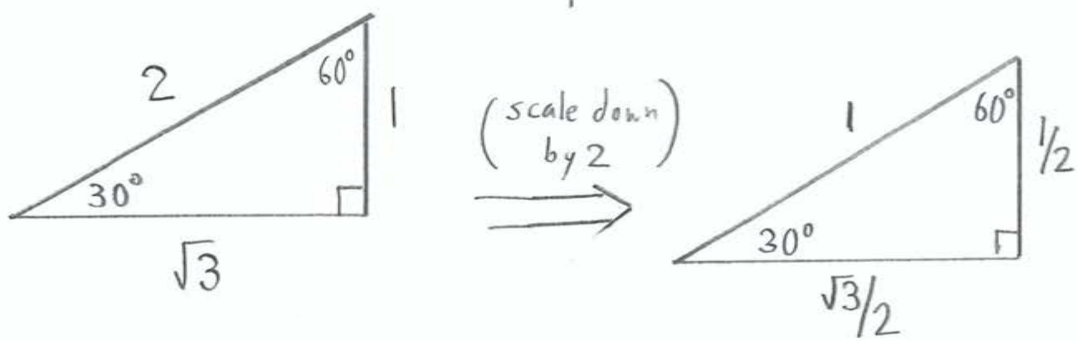


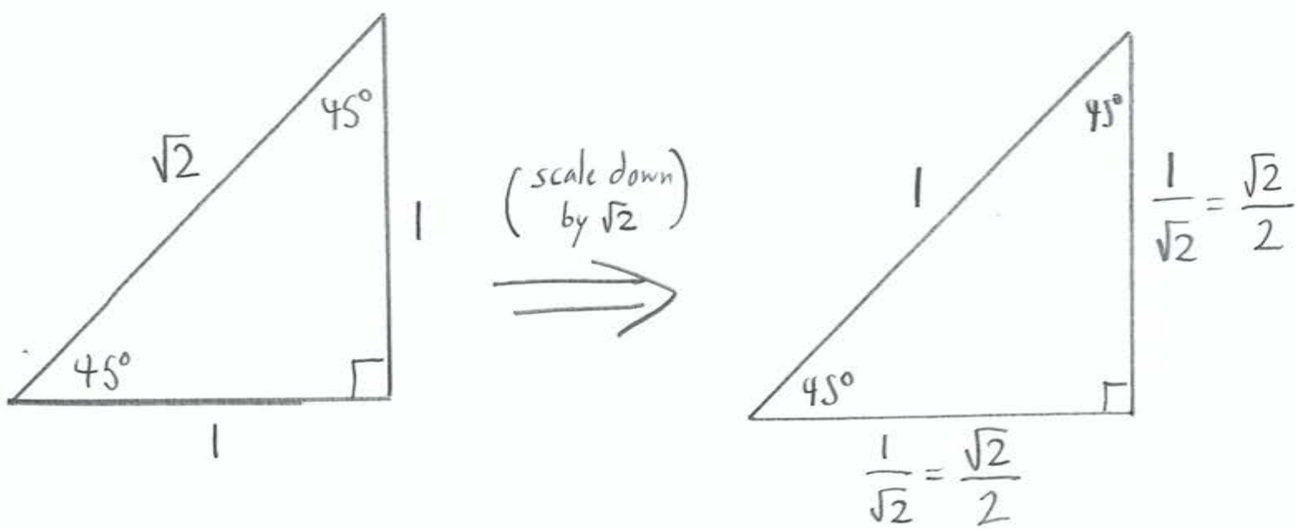
For example, the point in the direction $\frac{\pi}{2}$ is $(0,1)$, so $\cos\frac{\pi}{2}=0$ and $\sin\frac{\pi}{2}=1$.

Special angles between 0 and $\pi/2$

- For most angles, we need a calculator to find sine and cosine, because we can't determine the coordinates of the point in that direction by hand.

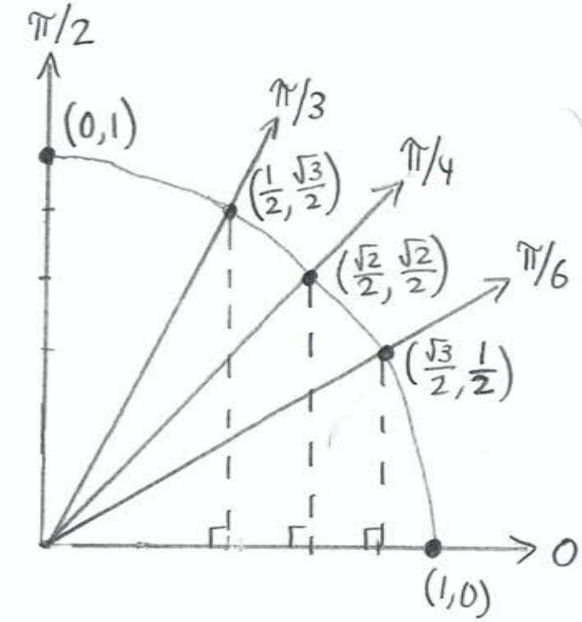
- But there are two special triangles we can use to determine the points in certain directions.





Special angles between 0 and $\pi/2$, continued

- Using those triangles, we can find points for the angles $0, \pi/6, \pi/4, \pi/3$, and $\pi/2$.

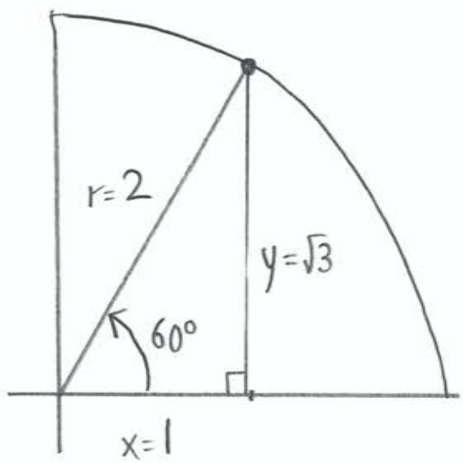


θ (in degrees)	θ (in radians)	$\cos\theta$	$\sin\theta$
$0°$	0	1	0
$30°$	$\pi/6$	$\sqrt{3}/2$	$1/2$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$60°$	$\pi/3$	$1/2$	$\sqrt{3}/2$
$90°$	$\pi/2$	0	1

Trigonometry using other circles (or triangles)

- We can actually define $\sin\theta$ and $\cos\theta$ using points on <u>any</u> circle. If the circle has radius r, and the point in the direction θ is (x,y), then: $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$.

For example:



so $\cos 60° = \dfrac{1}{2}$, $\sin 60° = \dfrac{\sqrt{3}}{2}$, as we know.

- If θ is one angle of a right triangle, we have x = length of adjacent side
  y = length of opposite side
  r = length of hypotenuse

Thus we have:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

Tangent, secant, cosecant, and cotangent

- The other four trig functions can be written in terms of $\cos\theta$, $\sin\theta$,
  or in terms of x, y, and r.

- $\cos\theta = \frac{x}{r}$
- $\sin\theta = \frac{y}{r}$
- $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$
- $\sec\theta = \frac{r}{x} = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{r}{y} = \frac{1}{\sin\theta}$
- $\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}$ $\left(\text{or } \frac{1}{\tan\theta}\right)$

- If a denominator is 0, the trig function is undefined.

  For example, $\sec\frac{\pi}{2}$ is undefined,

  since for all points in the direction $\pi/2$ (upward),

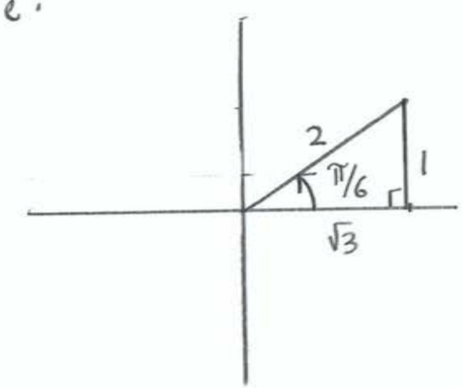
  the x-coordinate is 0.

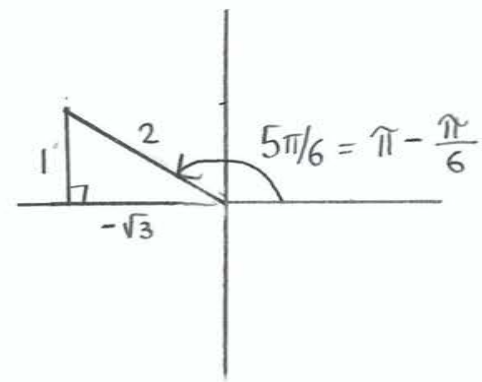
Special angles not between 0 and $\pi/2$

- Many angles not between 0 and $\pi/2$ are <u>reflections</u> of the special angles we've already seen.

  We can use this fact to find trig functions of these angles.

Example:





$$\sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

This angle has the same y (and r), but the opposite x!

$$\sin\frac{5\pi}{6} = \frac{1}{2}$$

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}}$$

Important identities (rules)

- NOTE: $\sin^2\theta$ is mathematical notation for $(\sin\theta)^2$

  (and $\sin^3\theta = (\sin\theta)^3$, etc.)

- Since $(\cos\theta, \sin\theta)$ is a point on the circle $x^2+y^2=1$, we know that

$$(*) \quad \boxed{\sin^2\theta + \cos^2\theta = 1}.$$

- If you divide both sides of $(*)$ by $\cos^2\theta$, you get:

$$\boxed{\tan^2\theta + 1 = \sec^2\theta}.$$

- Alternatively, if you divide both sides of $(*)$ by $\sin^2\theta$, you get:

$$\boxed{1 + \cot^2\theta = \csc^2\theta}$$

Other important identities

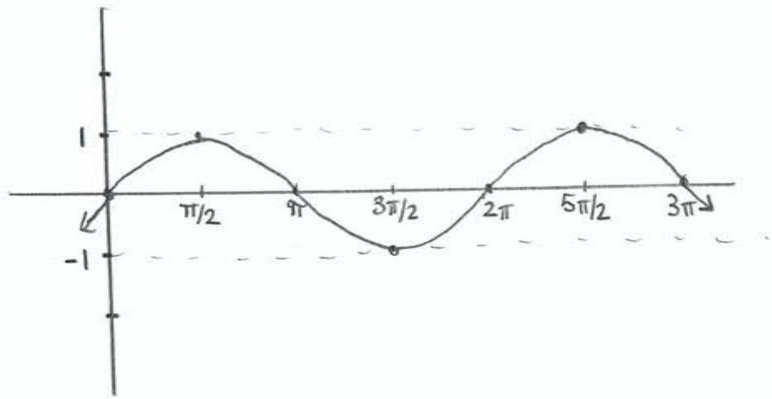
- There are formulas for $\sin(x \pm y)$ and $\cos(x \pm y)$ that we won't worry about here. They tell us that:
  - $\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}$ (Note that you can't just "factor out" the 2!)
  - $\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta} = 1 - 2\sin^2\theta$
    $= 2\cos^2\theta - 1$

- By solving for $\sin^2\theta$ and $\cos^2\theta$ in the $\cos(2\theta)$ identities, we also get:
  - $\boxed{\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))}$
  - $\boxed{\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))}$

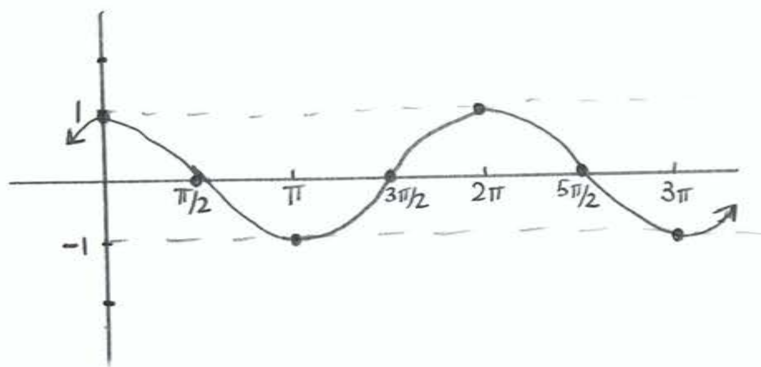
There are other trig identities, but these are the main ones used in Calculus I and II.

Graphs of trig functions

- $y = \sin x$

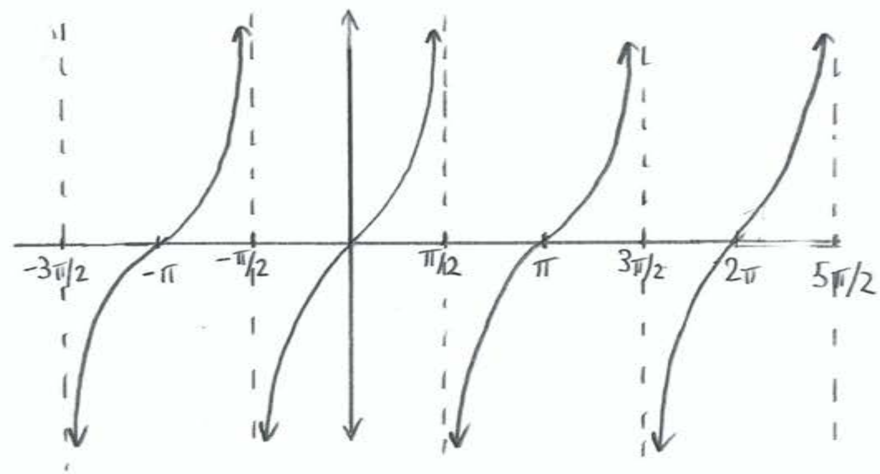


- $y = \cos x$



- Note that values of sine and cosine always fall between -1 and 1!

- $y = \tan x$



- Tangent can have any value,
  $\tan x$ is undefined for $x = \frac{\pi}{2} + 2\pi N$
  (for any integer N)
  and it becomes extremely high or low as you approach those x-values.