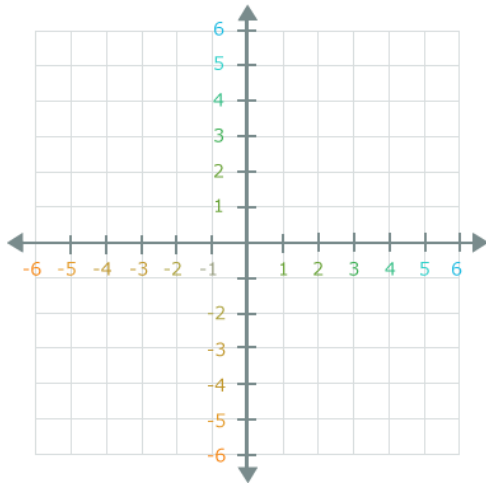


Brown University Vector Boot Camp

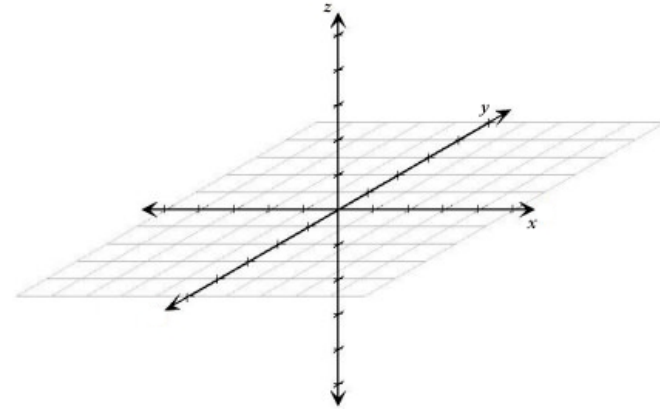
Part 1: Vectors and Scalars

Vector calculations are frequently done in two-dimensional or three-dimensional space.

Two-dimensional space (\mathbb{R}^2):



Three-dimensional space (\mathbb{R}^3):

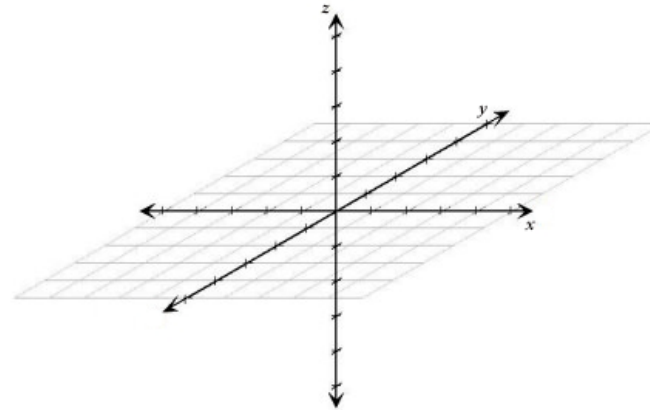
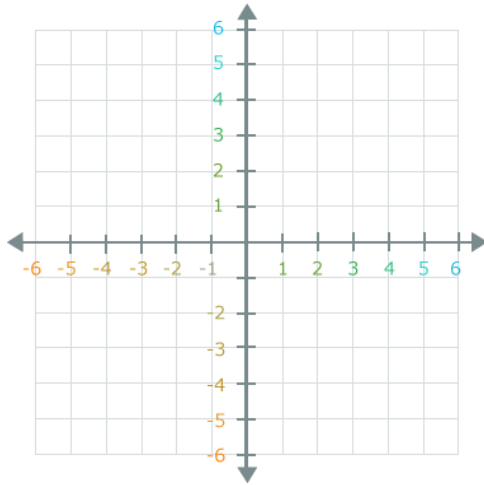


In \mathbb{R}^3 , the direction of the z -axis relative to the x -axis and y -axis matters.

If you curl the fingers on your right hand from the positive x -direction toward the positive y -direction, your thumb points in the positive z -direction.

We can solve math problems in higher-dimensional spaces (\mathbb{R}^4 , \mathbb{R}^5 , and so on). We just can't easily visualize those spaces.

A **vector** is a mathematical object with magnitude (size) and direction. Think of it as an arrow for which the length and direction both matter.



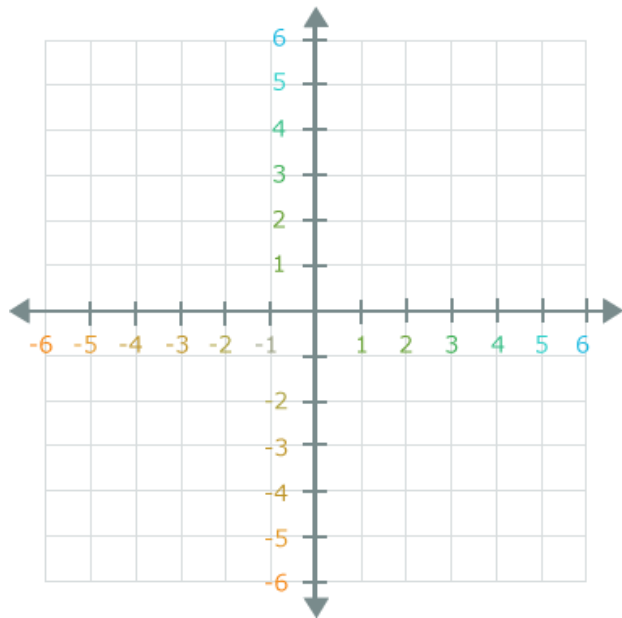
A vector can exist in \mathbb{R}^2 , in \mathbb{R}^3 , or in higher-dimensional spaces.

The one vector that doesn't have a direction is the "zero vector," often written as $\vec{0}$. A vector with magnitude zero points in no particular direction.

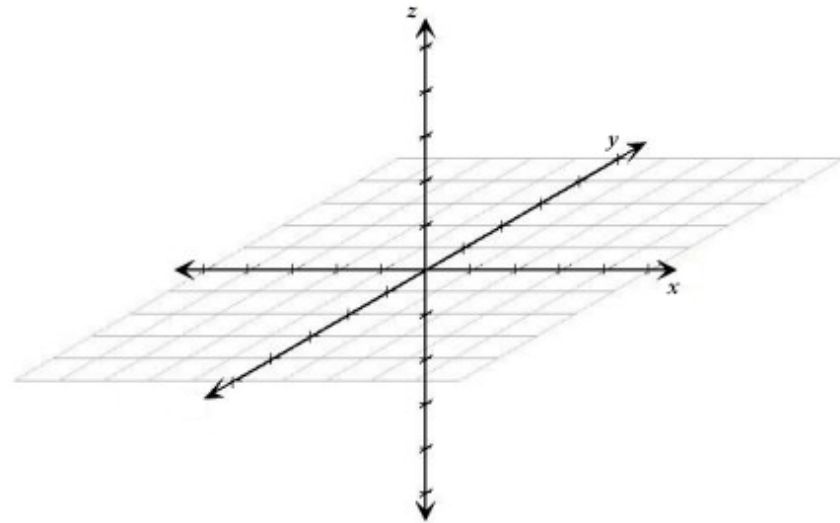
A **scalar** is just a number which has size but no direction. In a vector context, we call numbers scalars because they can be used to scale the size of a vector, as we'll see later.

We often write vectors in terms of their component values (essentially how far they extend along each axis).

For example:

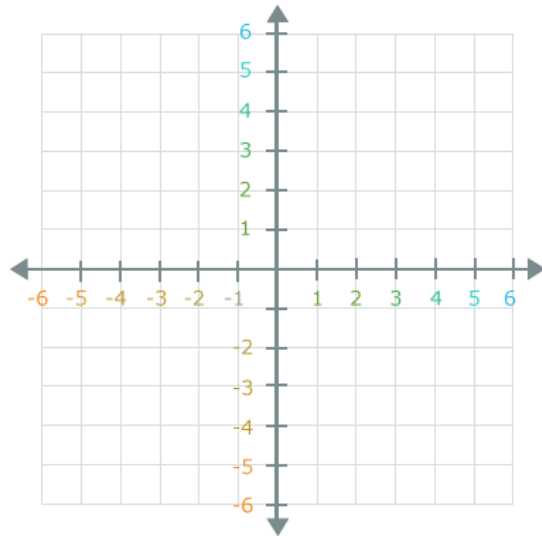


$$\vec{v} = \langle -3, 2 \rangle$$



$$\vec{w} = \langle 1, 2, 3 \rangle$$

We can **add** two vectors by placing them head-to-tail, with the first vector starting at the origin. (Think of the sum of two vectors as the result of following one arrow, and then following the second from where you ended up.)

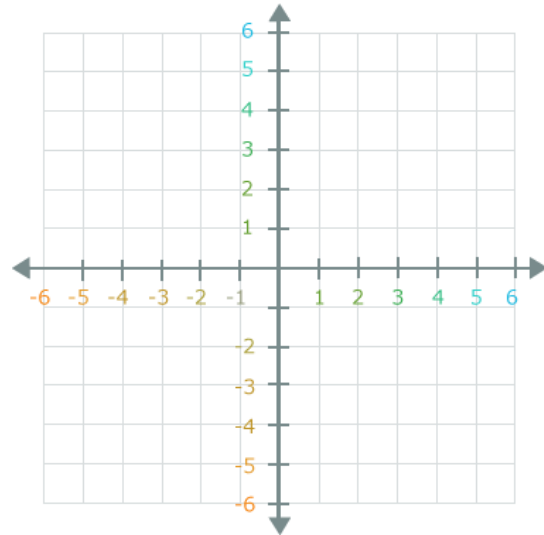


You can calculate the sum of two vectors by adding their component values.

$$\langle -3, 2 \rangle + \langle 2, 1 \rangle = \langle (-3 + 2), (2 + 1) \rangle = \langle -1, 3 \rangle$$

$$\langle 1, 2, 3 \rangle + \langle 4, 5, 6 \rangle = \langle (1 + 4), (2 + 5), (3 + 6) \rangle = \langle 5, 7, 9 \rangle$$

To **subtract** a vector from another vector, place them head-to-head and follow the subtracted vector to its tail. (Think of this as “un-adding” the second vector by following it backward.)



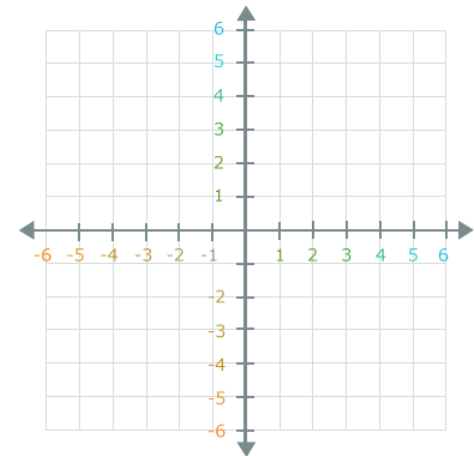
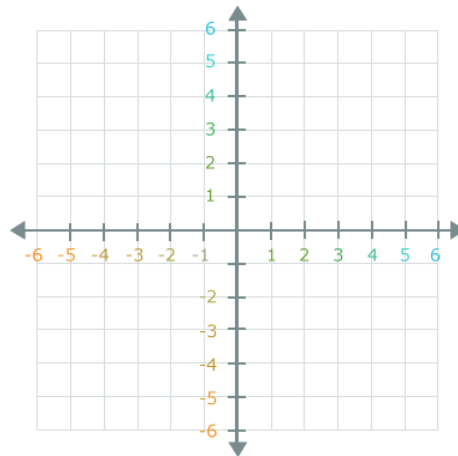
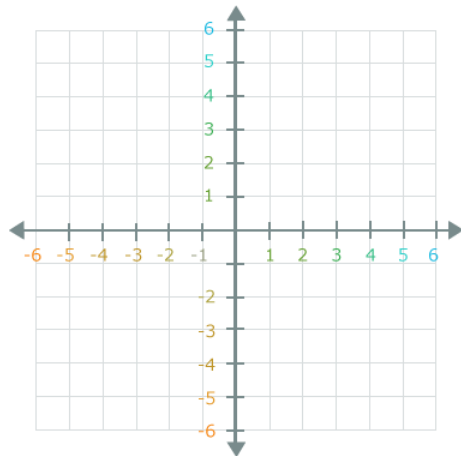
You can calculate the difference of two vectors by subtracting their component values.

$$\langle -3, 2 \rangle - \langle 2, 1 \rangle = \langle (-3 - 2), (2 - 1) \rangle = \langle -5, 1 \rangle$$

$$\langle 1, 2, 3 \rangle - \langle 4, 5, 6 \rangle = \langle (1 - 4), (2 - 5), (3 - 6) \rangle = \langle -3, -3, -3 \rangle$$

A vector can be multiplied by another vector (we'll get to that later), but it can also be **multiplied by a scalar**.

If the scalar is positive, this scales the vector's magnitude and leaves its direction unchanged.
If the scalar is negative, it reverses the vector's direction.

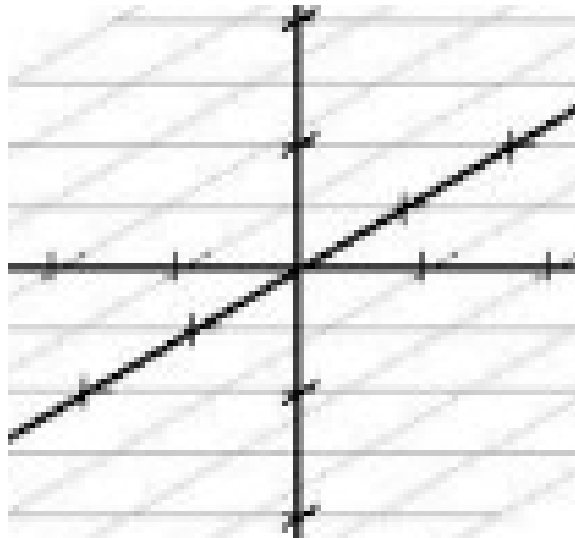


You can calculate the result of a scalar multiplication by multiplying all of the vector's component values by the scalar.

$$3 \langle 2, 1 \rangle = \langle 6, 3 \rangle$$

$$(-5) \langle 1, 2, 3 \rangle = \langle -5, -10, -15 \rangle$$

Sometimes vectors are written in terms of the unit component vectors \vec{i} , \vec{j} , and (in three dimensions) \vec{k} . These vectors each have magnitude 1, and they point in the positive directions on the x -axis, y -axis, and z -axis respectively.



The component values of a vector tell you how to write the vector in terms of \vec{i} , \vec{j} , and \vec{k} .

$$\langle -3, 2 \rangle = -3\vec{i} + 2\vec{j}$$

$$\langle 1, 2, 3 \rangle = \vec{i} + 2\vec{j} + 3\vec{k}$$

In general, a vector with magnitude 1 (in any direction) is called a **unit vector**.

The notation for the magnitude of a vector \vec{v} is $|\vec{v}|$.

You can calculate the magnitude of any vector from its components using the Pythagorean Theorem. The magnitude is the square root of the sum of the squares of the component values.

$$|\langle -3, 2 \rangle| = \sqrt{(-3)^2 + (2)^2} = \sqrt{13} \qquad \left| \vec{i} + 2\vec{j} + 3\vec{k} \right| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{13}$$

Given any vector, you can find a unit vector pointing in the same direction by dividing that vector by its own magnitude.

For example, $\frac{1}{\sqrt{13}} \langle -3, 2 \rangle = \left\langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$ points in the same direction as $\langle -3, 2 \rangle$, and its magnitude is:

$$\sqrt{\left(-3/\sqrt{13}\right)^2 + \left(2/\sqrt{13}\right)^2} = \sqrt{9/13 + 4/13} = \sqrt{13/13} = 1$$

The following properties apply to vector algebra:

- Vector addition is commutative: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Vector addition is associative: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
- Repeatedly scaling a vector scales the vector by the scalars' product: $a(b\vec{v}) = (ab)\vec{v}$
- Scalar multiplication is distributive over vector addition: $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$
- Scalar multiplication is distributive over scalar addition: $(a + b)\vec{v} = a\vec{v} + b\vec{v}$

There are other vector algebra properties listed in some textbooks, but most of them result from the ones above.

For a two-dimensional vector, if a vector \vec{v} has magnitude r and direction θ (the angle of rotation counterclockwise from the positive x -axis), you can calculate its component values using trigonometry.

$$\vec{v} = \langle r \cos \theta, r \sin \theta \rangle$$

One application of this is to prove the Law of Cosines, which you may have seen in a geometry or trigonometry course (and which will come in handy in the next video).

The Law of Cosines says that for the triangle shown, $a^2 + b^2 - 2ab \cos \theta = c^2$.

$$\begin{aligned}c^2 &= |(b \cos \theta - a, b \sin \theta)|^2 \\&= (b \cos \theta - a)^2 + (b \sin \theta)^2 \\&= b^2 \cos^2 \theta - 2ab \cos \theta + a^2 + b^2 \sin^2 \theta \\&= a^2 + b^2 - 2ab \cos \theta\end{aligned}$$