Brown University Vector Boot Camp

Part 2: The Dot Product

There are a couple of ways to define the product of two vectors. In this part of the boot camp, we'll discuss the scalar product, or **dot product**.

The dot product can be applied to two 2-dimensional vectors, two 3-dimensional vectors, or in general, two n-dimensional vectors.

We write the dot product of two vectors $\vec{v}$ and $\vec{w}$ as $\vec{v} \cdot \vec{w}$.

The result of the dot product is a scalar, which is why it's also called the scalar product. Later we'll see a way to "multiply" two three-dimensional vectors and get a vector.

We calculate the dot product by multiplying the two vectors' components in each dimension, and adding the products.

$$\langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2$$

$$\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

So, for example:

$$\langle -1, 2, 3 \rangle \cdot \langle 4, -5, 6 \rangle = (-1)(4) + (2)(-5) + (3)(6) = 18 - 4 - 10 = 4.$$

In terms of unit component vectors ($\vec{i}$, $\vec{j}$, and $\vec{k}$), we can define the dot product of each pair of unit component vectors. The dot product is distributive over vector addition, so this determines the product of any pair of vectors.

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$$

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{i} = \vec{i} \cdot \vec{k} = \vec{k} \cdot \vec{i} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{j} = 0$$

Here's a more visual interpretation of the dot product. Let θ be the angle between two vectors $\vec{v}$ and $\vec{w}$. Then:

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$$

This means that if we want to find the angle between two vectors, we can use the formula

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$$

and then apply the inverse cosine function to calculate the angle. The angle between two vectors is always between 0 and π, so any possible cosine value we find this way will correspond to exactly one possible angle.

Why does multiplying the vectors' components and adding the products give the same result as multiplying the vectors' magnitudes and the cosine of the angle between them?

We can see why using the Law of Cosines. We'll show that this works for 3-dimensional vectors, but the same argument can be used in any dimension.

Assume $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$.

$$
\begin{aligned}
|\vec{v} - \vec{w}|^2 &= |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}| \cos\theta \\
2|\vec{v}||\vec{w}| \cos\theta &= |\vec{v}|^2 + |\vec{w}|^2 - |\vec{v} - \vec{w}|^2 \\
&= (v_1^2 + v_2^2 + v_3^2) + (w_1^2 + w_2^2 + w_3^2) - [(v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_2 - w_2)^2] \\
&= [v_1^2 + w_1^2 - (v_1^2 + w_1^2 - 2v_1 w_1)] + [v_2^2 + w_2^2 - (v_2^2 + w_2^2 - 2v_2 w_2)] \\
&\quad + [v_3^2 + w_3^2 - (v_3^2 + w_3^2 - 2v_3 w_3)] \\
&= 2v_1 w_1 + 2v_2 w_2 + 2v_3 w_3 \\
|\vec{v}||\vec{w}| \cos\theta &= v_1 w_1 + v_2 w_2 + v_3 w_3
\end{aligned}
$$

So we see that the two definitions of the dot product match.

The following properties apply to the dot product:

- The dot product is commutative: $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

- The dot product is distributive over vector addition: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

- The dot product of a vector with itself is its magnitude squared: $\vec{v} \cdot \vec{v} = |\vec{v}|^2$

- The dot product is compatible with scalar multiplication: $c\vec{v} \cdot \vec{w} = c(\vec{v} \cdot \vec{w})$

- The dot product of any vector and the zero vector is zero: $\vec{v} \cdot \vec{0} = 0$

There are other dot product properties listed in some textbooks, but most of them result from the ones above.

If the angle between two vectors is a right angle, we say that the two vectors are **orthogonal** (or perpendicular).

Since $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$, and the cosine of a right angle is 0, we can detect whether two vectors are orthogonal by checking whether their dot product is zero.

For example, the vectors $\langle 1, 2, 3 \rangle$ and $\langle 7, -2, -1 \rangle$ are orthogonal, since

$$\langle 1, 2, 3 \rangle \cdot \langle 7, -2, -1 \rangle = 7 - 4 - 3 = 0.$$

In two-dimensional space, two lines are orthogonal if the product of their slopes is -1. We can prove this using the dot product. Since slope is "rise over run," two lines of slopes m_1 and m_2 can be represented by the vectors $\langle 1, m_1 \rangle$ and $\langle 1, m_2 \rangle$. The dot product of these two vectors is

$$\langle 1, m_1 \rangle \cdot \langle 1, m_2 \rangle = 1 + m_1 m_2$$

The vectors are orthogonal when $1 + m_1 m_2 = 0$, which happens when $m_1 m_2 = -1$.

The dot product can be used to project one vector onto another vector. Given two vectors, we might want to know how far one vector points in the direction of another vector, as shown below.

The **vector projection** of $\vec{v}$ onto $\vec{w}$ is the portion of $\vec{v}$ that points in the direction of $\vec{w}$. It is given by the formula below.

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$$

The **scalar projection** of $\vec{v}$ onto $\vec{w}$ is the magnitude of the vector projection, or the "amount" of $\vec{v}$ that points in the direction of $\vec{w}$. It is given by the formula below.

$$\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$$

For example, suppose we want to project the vector $\vec{v} = \langle 10, 7, 6 \rangle$ onto the vector $\vec{w} = \langle 2, 4, 6 \rangle$.

$$\vec{v} \cdot \vec{w} = 20 + 28 + 36 = 84$$
$$\vec{w} \cdot \vec{w} = |\vec{w}|^2 = 4 + 16 + 36 = 56$$

So the vector projection of $\vec{v}$ onto $\vec{w}$ is $(84/56)\vec{v} = \langle 3, 6, 9 \rangle$.

The scalar projection of $\vec{v}$ onto $\vec{w}$ is $84/\sqrt{56} = 3\sqrt{14}$, the magnitude of $\langle 3, 6, 9 \rangle$.

Note that $\vec{v}$ can be written as $\langle 3, 6, 9 \rangle + \langle 7, 1, -3 \rangle$, where the first vector points in the direction of $\vec{w}$ and the second is orthogonal to $\vec{w}$.

Reminder: The vector projection of $\vec{v}$ onto $\vec{w}$ is $\dfrac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}$.

Reminder: The scalar projection of $\vec{v}$ onto $\vec{w}$ is $\dfrac{\vec{v} \cdot \vec{w}}{|\vec{w}|}$.

Where do these formulas come from? If θ is the angle between the vectors, the scalar projection should be equal to $|\vec{v}| \cos \theta$.

Once we know what length the vector projection should have, we can "rescale" $\vec{w}$ appropriately.