Brown University Vector Boot Camp

Part 3: The Cross Product

In some cases, it's useful to have a product of vectors that produces a direction as well as a magnitude. There isn't a nice way to do this in two-dimensional space, but in three-dimensional space, there is a vector product called the **cross product**.

We write the cross product of two vectors $\vec{v}$ and $\vec{w}$ (which is a third vector) as $\vec{v} \times \vec{w}$.

To define the cross product, we have to determine how large the resulting vector should be, and also in which direction it should point.

Recall that for two vectors $\vec{v}$ and $\vec{w}$, if the angle between the vectors is θ, the dot product of the vectors is

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta.$$

We define the magnitude of the cross product of $\vec{v}$ and $\vec{w}$ as

$$|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin\theta.$$

For example, the vectors $\langle 2, 0, 0\rangle$ and $\langle 0, 3, 0\rangle$ are orthogonal, so the angle between them is $\pi/2$. This means that the magnitude of $\langle 2, 0, 0\rangle \times \langle 0, 3, 0\rangle$ should be $2 \cdot 3 \cdot \sin(\pi/2) = 6$.

Given two three-dimensional vectors, unless the vectors are parallel (meaning that they point in the same or in opposite directions), there is exactly one two-dimensional plane that contains both of them.

We choose the direction of the cross product of the two vectors so that it is orthogonal to both vectors (and to every vector in the plane containing them).

There are two directions that would work, and we choose one of them using the "right-hand rule": if you hold your right hand such that your fingers curl from $\vec{v}$ toward $\vec{w}$ (through an angle less than π), then your thumb points in the direction of $\vec{v} \times \vec{w}$.

Using this rule, $\langle 2, 0, 0\rangle \times \langle 0, 3, 0\rangle$ should point in the positive z-direction, so

$$\langle 2, 0, 0\rangle \times \langle 0, 3, 0\rangle = \langle 0, 0, 6\rangle \,.$$

Note that if the vectors are parallel, we cannot choose a direction this way. But in that case, the angle between the vectors is either 0 or π, and since $\sin 0 = \sin \pi = 0$, which means that the cross product is the zero vector, which has no direction.

In terms of unit component vectors ($\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$), we can define the cross product of each pair of unit component vectors. The cross product is distributive over vector addition, so this determines the product of any pair of vectors.

Note that because of the right-hand rule, switching the order of the multiplied vectors reverses the direction of the result!

$$\vec{\imath} \times \vec{\jmath} = -\left(\vec{\jmath} \times \vec{\imath}\right) = \vec{k}$$

$$\vec{\jmath} \times \vec{k} = -\left(\vec{k} \times \vec{\jmath}\right) = \vec{\imath}$$

$$\vec{k} \times \vec{\imath} = -\left(\vec{\imath} \times \vec{k}\right) = \vec{\jmath}$$

$$\vec{\imath} \times \vec{\imath} = \vec{\jmath} \times \vec{\jmath} = \vec{k} \times \vec{k} = \vec{0}$$

If we break a cross product up using these $\vec{\imath}, \vec{\jmath}, \vec{k}$ rules, we can derive the following formula for the cross product.

$$\langle v_1, v_2, v_3 \rangle \times \langle w_1, w_2, w_3 \rangle = \langle v_2 w_3 - w_2 v_3,\ v_3 w_1 - w_3 v_1,\ v_1 w_2 - w_1 v_2 \rangle$$

One way to remember this formula is as the *determinant* of the following 3×3 matrix.

$$\left(v_1 \vec{\imath} + v_2 \vec{\jmath} + v_3 \vec{k} \right) \times \left(w_1 \vec{\imath} + w_2 \vec{\jmath} + w_3 \vec{k} \right) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

We can use this formula to confirm (as we showed earlier) that

$$\langle 2, 0, 0 \rangle \times \langle 0, 3, 0 \rangle = \langle 0 - 0, 0 - 0, 6 - 0 \rangle = \langle 0, 0, 6 \rangle$$

The following properties apply to the cross product:

- The cross product is **anti-commutative**: $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$

- The cross product is compatible with scalar multiplication: $c_1\vec{v} \times c_2\vec{w} = c_1c_2(\vec{v} \times \vec{w})$

- The dot product is distributive over vector addition: $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

- The cross product is not associative: $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

- The cross product of any vector and the zero vector is the zero vector: $\vec{v} \times \vec{0} = \vec{0}$

There are other cross product properties listed in some textbooks, but most of them result from the ones above.

One application of the cross product is to find the area of a parallelogram. We say a parallelogram is determined by two vectors if the two vectors form adjacent sides of the parallelogram, as shown below.

The area of a parallelogram is its base length times its height. In this case, that product is $|\vec{v}||\vec{w}|\sin\theta$, which is the magnitude of $\vec{v} \times \vec{w}$.

For example,

$$\langle 1, 2, 3\rangle \times \langle 4, 5, 6\rangle = \langle -3, 6, -3\rangle$$
$$|\langle -3, 6, -3\rangle| = \sqrt{9 + 36 + 9} = 3\sqrt{6}$$

This means that the parallelogram determined by $\langle 1, 2, 3\rangle$ and $\langle 4, 5, 6\rangle$ has area $3\sqrt{6}$.

Two vectors determine a parallelogram. Three vectors determine a parallelepiped, as shown.

The volume of a parallelepiped is the area of its parallelogram base times its height. The base area in the picture above is $|\vec{u} \times \vec{v}|$, and the height is $|\vec{w}| \cos\theta$. The product of these is

$$|\vec{u} \times \vec{v}||\vec{w}| \cos\theta = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

The quantity above is known as either the triple scalar product or the box product.

If the order of the three vectors are "cycled", the value stays the same, though if their order is reversed, the value is negated (because the cross product is anti-commutative).

$$A = (\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v}$$

$$-A = (\vec{v} \times \vec{u}) \cdot \vec{w} = (\vec{w} \times \vec{v}) \cdot \vec{u} = (\vec{u} \times \vec{w}) \cdot \vec{v}$$

So depending on the chosen order of the three vectors, the box product may be negative, but the volume of the corresponding parallelepiped will always be the absolute value of the box product.

For example, the parallelepiped determined by $\langle 1, 2, 3\rangle$, $\langle 4, 5, 6\rangle$, and $\langle 7, 0, 0\rangle$ has volume

$$|(\langle 1, 2, 3\rangle \times \langle 4, 5, 6\rangle) \cdot \langle 7, 0, 0\rangle| = |\langle -3, 6, -3\rangle \cdot \langle 7, 0, 0\rangle| = |-21| = 21.$$