Midterm (I) Solutions

Problem 1. Let $A=(4,-1,1)$, $B=(0,3,2)$ and $C=(1,0,1)$.

(1) Find an equation of the plane containing A,B and C.

Solution: Let $\vec{u}=\overrightarrow{AB}=\langle -4,4,1\rangle$ and $\vec{v}=\overrightarrow{AC}=\langle -3,1,0\rangle$. The normal vector of the plane containing A,B and C is

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -4 & 4 & 1\\ -3 & 1 & 0\end{vmatrix}\\ &= \begin{vmatrix}4 & 1\\ 1 & 0\end{vmatrix}\vec{i}-\begin{vmatrix}-4 & 1\\ -3 & 0\end{vmatrix}\vec{j}+\begin{vmatrix}-4 & 4\\ -3 & 1\end{vmatrix}\vec{k}\\ &= -\vec{i}-3\vec{j}+8\vec{k}.\end{aligned}$$

Therefore an equation for the plane is

$$-x-3y+8z+d = 0.$$

To find d, we simply substitude one of the points into the equation. Lets choose B

$$-0-3\cdot 3+8\cdot 2+d = 0,$$

so $d=-7$. Therefore

$$-x-3y+8z-7 = 0$$

is an equation of the plane.

(2) Find a symmetric equation of the line passing through A and perpendicular to the plane containing A,B,C.

Solution: A line perpendicular to a plane is parallel to its normal vector. In this case the normal vector is $\vec{n}=\langle -1,-3,8\rangle$ as calculated in part (1). Therefore a symmetric equation of the line passing through $A=\langle 4,-1,1\rangle$ and parallel to $\vec{n}$ is

$$\frac{x-4}{-1} = \frac{y+1}{-3} = \frac{z-1}{8}.$$

(3) Find the area of the triangle ABC.

Solution: The area of the triangle ABC is half of the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and we know that this area is equal to

$$\begin{aligned}|\overrightarrow{AB}\times\overrightarrow{AC}| &= |\langle -1,-3,8\rangle|\\ &= \sqrt{1+9+64}\\ &= \sqrt{74}.\end{aligned}$$

Therefore the area of the triangle ABC is $\sqrt{74}/2$.

Problem 2. Find the distance from the origin to the line $x=1-2t$, $y=t-3$, $z=2t+1$.

Solution: The vector equation of this line is

$$\vec{r} = \vec{r_o}+t\vec{v}$$

where $\vec{r_o} = \langle 1, -3, 1\rangle$ and $\vec{v} = \langle -2, 1, 2\rangle$. Now let O denote the origin and let Q be the closest point on the line to the origin. So the vector $\overrightarrow{OQ}$ is perpendicular to the line and the distance from the origin to the line is simply $|\overrightarrow{OQ}|$. Let $P = (1, -3, 1)$ be the tip of $\vec{r_o}$ (which is on the line). We can see that the projection of $\overrightarrow{OP} = \vec{r_o}$ on to the line is $\overrightarrow{QP}$. In other words

$$\overrightarrow{QP} = \mathrm{proj}_{\vec{v}}\vec{r_o}$$

and so

$$\begin{aligned} |\overrightarrow{QP}| &= |\mathrm{comp}_{\vec{v}}\vec{r_o}| \\ &= \frac{|\vec{v}\cdot\vec{r_o}|}{|\vec{v}|} \\ &= \frac{|-2-3+2|}{\sqrt{4+1+4}} \\ &= 1. \end{aligned}$$

Now O,Q and P forms a right triangle with sides $\overrightarrow{OQ}$, $\overrightarrow{QP}$ and hypotenuse $\overrightarrow{OP} = \vec{r_o}$. Therefore by pythagorean theorem

$$\begin{aligned} |\overrightarrow{OQ}| &= \sqrt{|OP|^2 - |QP|^2} \\ &= \sqrt{1+9+1-1} \\ &= \sqrt{10}. \end{aligned}$$

Problem 3. Find the curvature of the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at the point (2,0).

Solution: First, we need to parametrize the ellipse. One parametrization is

$$\vec{r}(t) = \langle 2\sin(t), 5\cos(t)\rangle$$

where $t \in [0, 2\pi)$. Now we can use one of the curvature formulas:

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} \qquad \text{or} \qquad \kappa(t) = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$

Observe that $(2,0) = \vec{r}(\pi/2)$ so we need to calculate $\kappa(\pi/2)$. Lets calculate it in both ways.

`Using the 1st formula:` First lets write down what functions we need to calculate and then evaluate those functions at $\pi/2$. Now we know that

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

and so by product rule

$$\begin{aligned} \vec{T}'(t) &= \frac{1}{|\vec{r}'(t)|}\vec{r}''(t) + \left(\frac{1}{|\vec{r}'(t)|}\right)'\vec{r}'(t) \\ &= \frac{1}{|\vec{r}'(t)|}\vec{r}''(t) - \frac{|\vec{r}'(t)|'}{|\vec{r}'(t)|^2}\vec{r}'(t). \end{aligned} \tag{1}$$

Therefore we need $\vec{r}'(t), \vec{r}''(t), |\vec{r}'(t)|$ and $|\vec{r}'(t)|'$ and their values at $\pi/2$. Now $\vec{r}'(t) = \langle 2\cos(t), -5\sin(t)\rangle$ and $\vec{r}''(t) = \langle -2\sin(t), -5\cos(t)\rangle$. So

$$\vec{r}'(\pi/2) = (0, -5) \quad \& \quad \vec{r}''(\pi/2) = (-2, 0).$$

Using $\vec{r}'$ we get $|\vec{r}'(t)| = \sqrt{4\cos^2(t) + 25\sin^2(t)}$ and so

$$|\vec{r}'(t)|' = \frac{-8\cos(t)\sin(t) + 50\sin(t)\cos(t)}{2\sqrt{4\cos^2(t) + 25\sin^2(t)}}.$$

The good news is that $|\vec{r}\,'(\pi/2)| = 5$ and $|\vec{r}\,'(\pi/2)|' = 0$. Substituting these values into equation (1) we get

$$\begin{aligned}\vec{T}\,'(\pi/2) &= \frac{1}{5}(-2,0) - 0 \\ &= (-\frac{2}{5},0)\end{aligned}$$

and therefore

$$\begin{aligned}\kappa(\pi/2) &= \frac{|\vec{T}\,'(\pi/2)|}{|\vec{r}\,'(\pi/2)|} \\ &= \frac{2}{25}\end{aligned}$$

`Using the 2nd formula:` This formula involves $\vec{r}\,'(\pi/2)$ and $\vec{r}\,''(\pi/2)$ and $|\vec{r}\,'(\pi/2)|$ which we already calculated above as

$$\vec{r}\,'(\pi/2) = (0,-5) \quad , \quad \vec{r}\,''(\pi/2) = (-2,0) \quad \& \quad |\vec{r}\,'(\pi/2)| = 5\,.$$

Observe that $\vec{r}\,'(\pi/2)$ and $\vec{r}\,''(\pi/2)$ are perpendicular vectors. Therefore the length of their cross product is equal to the product of their lengths. Therefore

$$\begin{aligned}\kappa(\pi/2) &= \frac{|\vec{r}\,'(\pi/2) \times \vec{r}\,''(\pi/2)|}{|\vec{r}\,'(\pi/2)|^3} \\ &= \frac{|\vec{r}\,'(\pi/2)| \cdot |\vec{r}\,''(\pi/2)|}{|\vec{r}\,'(\pi/2)|^3} \\ &= \frac{5 \cdot 2}{5^3} \\ &= \frac{2}{25}.\end{aligned}$$

Problem 4. A particle starts at the origin with the initial velocity $3\vec{i} - \vec{j} + \vec{k}$ and its acceleration is $\vec{a}\,(t) = -6t\vec{k} + 12t^2\vec{j} + 6t\vec{i}$. Find its position vector at $t = 2$.

Solution: First we calculate the velocity function $\vec{v}\,(t)$ by integrating $\vec{a}\,(t)$. So

$$\begin{aligned}\vec{v}\,(t) &= \int (-6t\vec{k} + 12t^2\vec{j} + 6t\vec{i}\,)dt \\ &= -3t^2\vec{k} + 4t^3\vec{j} + 3t^2\vec{i} + \vec{C}.\end{aligned}$$

Substituting $t = 0$ on both sides of the equation and using the fact that $\vec{v}\,(0) = 3\vec{i} - \vec{j} + \vec{k}$, we get $\vec{C} = 3\vec{i} - \vec{j} + \vec{k}$ and so

$$\vec{v}\,(t) = (-3t^2 + 1)\vec{k} + (4t^3 - 1)\vec{j} + (3t^2 + 3)\vec{i}\,.$$

Now we integrate $\vec{v}\,(t)$ to get $\vec{r}\,(t)$.

$$\begin{aligned}\vec{r}\,(t) &= \int [(-3t^2 + 1)\vec{k} + (4t^3 - 1)\vec{j} + (3t^2 + 3)\vec{i}\,]dt \\ &= (-t^3 + t)\vec{k} + (t^4 - t)\vec{j} + (t^3 + 3t)\vec{i} + \vec{D}.\end{aligned}$$

Similarly we set $t = 0$ and since $\vec{r}\,(0) = \langle 0,0,0\rangle$ we get $\vec{D} = \langle 0,0,0\rangle$. Therefore

$$\begin{aligned}\vec{r}\,(2) &= (-2^3 + 2)\vec{k} + (2^4 - 2)\vec{j} + (2^3 + 3\cdot 2)\vec{i} \\ &= -6\vec{k} + 14\vec{j} + 14\vec{i}\,.\end{aligned}$$

Problem 5. Find the arc length of the curve $\vec{r}\,(t) = \langle 2\cos 2t, 2\sin 2t, 2t^{3/2}\rangle$, $0 \leqslant t \leqslant 1$.

Solution: We integrate the speed function between 0 and 1. Now

$$\vec{r}\,'(t) = \langle -4\sin 2t, 4\cos 2t, 3\sqrt{t}\rangle \quad \text{so} \quad |\vec{r}\,'(t)| = \sqrt{16 + 9t}.$$

So we need to calculate $\int_0^1 \sqrt{16+9t}dt$. Substituting $u=16+9t$, we get $du=9dt$ and we get

$$\begin{aligned}
\int_0^1 \sqrt{16+9t}dt &= \int_{16}^{25} \frac{\sqrt{u}}{9}du \\
&= \frac{2u^{3/2}}{27}\Big]_{16}^{25} \\
&= \frac{2}{27}(125-64) \\
&= \frac{122}{27}.
\end{aligned}$$

Problem 6. Show that for given vectors $\vec{u}$ and $\vec{v}$, if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular to each other, then $\vec{u}$ and $\vec{v}$ have the same length.

Solution: Since $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular to each other, their dot product is zero. Thus

$$\begin{aligned}
0 &= (\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) \\
&= |\vec{u}|^2 - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} - |\vec{v}|^2 \\
&= |\vec{u}|^2 - |\vec{v}|^2.
\end{aligned}$$

Therefore $|\vec{u}|^2 = |\vec{v}|^2$ and so $|\vec{u}| = |\vec{v}|$.

Problem 7. Find a parametrization of the space curve obtained by intersecting the plane given by $6z-8y=0$ and the surface given by $\frac{x^2}{4}+\frac{y^2}{3}-\frac{z^2}{8}=1$.

Solution: An equation of a surface means a condition for a point to lie on the surface. Therefore a point lies on the intersecting curve if it satisfies both conditions simultaniously. If we rewrite the first condition as

$$z = \frac{4}{3}y$$

and combine this with the second equation, we get

$$\begin{aligned}
1 &= \frac{x^2}{4}+\frac{y^2}{3}-\frac{(\frac{4}{3}y)^2}{8} \\
&= \frac{x^2}{4}+\frac{y^2}{3}-\frac{2y^2}{9} \\
&= \frac{x^2}{4}+\frac{y^2}{9}.
\end{aligned}$$

So we get an equation involving only x and y which represents an ellipse. Note that this ellipse is the projection of the intersection curve onto the xy-plane. We parametrize this ellipse as

$$x(t) = 2\cos t \quad \text{and} \quad y(t) = 3\sin t\ .$$

To find $z(t)$ we plug $y(t)$ into the equation of the plane and solve $z(t)$

$$6z(t) - 8(3\sin t) = 0.$$

Thus

$$z(t) = 4\sin t.$$