1 Double Integral

1. Definition:

Suppose D is a region in R^2, $f(x,y)$ is a scalor function of two variable, we will define the double integral of f over D in the following way

$$\int\int_D f(x,y)dxdy \tag{1}$$

Double Integral is always used to calculate the volume of some solid in R^3.

The way to evaluate this double integral TOTALLY depends on the region D, basically we have the following 3 big cases

1. $D=[a,b]\times[c,d]$ is a rectangle, then $\int\int_D f(x,y)dxdy = \int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy$ by Fubini's theorem, usually different order of integration will result in different difficulties, you can check Example 3 on page 962 as one example. Remark: Match the right bound with the right variable, and the order to do iterated integral is from inside to outside.

2. D belongs to type I: $D=\{a\le x\le b, f_1(x)\le y\le f_2(x)\}$ or type II $D=\{g_1(y)\le x\le g_2(y), c\le y\le d\}$, the way to do this double integral is again iterated integral, for example, if you have type I, the way to do the double integral is $\int\int_D f(x,y)dxdy = \int_a^b\int_{f_1(x)}^{f_2(x)} f(x,y)dydx$.

Remark: By reversing the order of integration, the double integral maybe simplified, when you reverse the order, remember to change the bounds accordingly.

3 D is easier to describe in polar coordinate system,(You really need to be familiar with the polar coordinate system, r,θ, especially the geometry meaning of these two variables) for example when you have polar rectangles, like circles, annulus, a slice of the circle or even part of the slice, we should use polar integral, here the idea of change of variable is involved. (When you do change of variable, you need to change EVERYTHING, including the integral bounds, integrand, and the variable.), the formula is the following when your D is a polar rectangle $0\le a\le r\le b, c\le\theta\le d$

$$\int\int_D f(x,y)dxdy = \int_a^b\int_c^d f(rcos\theta, rsin\theta)rd\theta dr \tag{2}$$

Remark: When you have the "pizza" shape thing, PLEASE use polar coordinate.

2 Triple Integral

The story of the triple integral is almost the same with the double integral We start with setting up the triple integral

1. Definition: Suppose you have region $E \in R^3$, a scalar function $f(x,y,z)$ of three variable, we define the triple integral of f over E as following

$$\int\int\int_E f(x,y,z)dxdydz \tag{3}$$

Again the way to evaluate the triple integral TOTALLY depend on the region E, we have the following three cases

1. $E = [a,b] \times [c,d] \times [r,s]$ is a rectangular box, as before, we need to do iterated integral, here we have 6 different ways to do it by Fubini's theorem.

2 E belongs to type I, II or III, the way to deal with such region is to REDUCE the triple integral into double integral, in another world, you need to identify which type does your region belongs to, generally E belongs to all three types, but you need to pick one which may simplify your integal, the idea is to do PROJECTION, project your region E onto $x-y$ plane, $y-z$ plane and $x-z$ plane respectively, and see in which way you will have the simplest projection, for example, if you have the paraboloid, $z = x^2 + y^2$, if you project the paraboloid onto $x-y$ plane, you will get a circle, onto $x-z$ plane, you will have a parabola, same if you project onto $y-z$ plane, since we have learned the polar coordinate, the circle is easier to deal with, so we would like to project everything onto $x-y$ plane, int his way you reduce your triple integral into a double integral in the following way

$$\int\int\int_E f(x,y,z)dxdydz = \int\int_D \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)dzdxdy \tag{4}$$

after you do the inner integral with respect to z, you will be left with a double integral of functionx,y on a circle, if you have a particular problem, you need to figure out the center and radius of the circle. You can check Example 3 on page 994.

3 Change of variable, when you do change of variable, you need to change EVERYTHING,(the bounds, integrand and the variable) here is the formula. Suppose your new variable is u and v,

$$\int\int_R f(x,y)dxdy = \int\int_S f(x(u,v),y(u,v)) \mid \frac{\partial(x,y)}{\partial(u,v)} \mid dudv \tag{5}$$

where $\mid \frac{\partial(x,y)}{\partial(u,v)} \mid$ if the determinate of the Jacobian matrix, R is the region for x, y, and s if the region for u, v, the polar coordinate is a special case of this formula, we also have the three dimention change of variable, it is almost the same with the previous formula

$$\int\int\int_R f(x,y,z)dxdydz = \int\int_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \mid \frac{\partial(x,y,z)}{\partial(u,v,w)} \mid dudvdw \tag{6}$$

where $\mid \frac{\partial(x,y,z)}{\partial(u,v,w)} \mid$ if the determinate of the Jacobian matrix, R is the region for x, y, z, and s if the region for u, v, w.

We also have two important change of variables:

Spherical coordinate (ρ, θ, ϕ), you really need to know the geometry meaning and the domain of these three varibles, otherwise you will have difficulty to figure out the bounds of integral, especially when you have part of the sphere. The Jacobian matrix for this change of variable is $\rho^2 sin\phi$ check Example 4 on page 1009.(The sphere is shifted upward in this example, which make things a little bit harder)

cylindrical coordinate(r, θ, z), you also need to know the geometry meaning of these three variables, and the Jacobian matrix for this change of variable is r.

3 Line integral

We have the line integral of scalar functions and vector fields,, but no matter which, your first step is ALWAYS parameterize your line (You need to be familiar with some simple parameterization, like line segment and circles, ellipse), after you get the parameterization, here is the formula for line integral of

1 Scalar functions $f(x, y)$,

$\int_L f(x,y)ds = \int_{t_0}^{t_1} f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}dt$
$\int_L f(x,y)dx = \int_{t_0}^{t_1} f(x(t), y(t))x'(t)dt$
$\int_L f(x,y)dy = \int_{t_0}^{t_1} f(x(t), y(t))y'(t)dt$

so when you do line integral of scalar functions, you need to be careful which formular you need to use.

2 Vector fields

Suppose you already know the parameterization $r(t) =< x(t), y(t) >$, and the vector field $F =< P, Q >$, then we define the line integral in the following way

$$\int_L F \cdot dr = \int_{t_0}^{t_1} < P, Q > \cdot < x'(t), y'(t) > dt = \int Pdx + \int Qdy \qquad (7)$$

This formula applies to ALL vector field, as long as your already have the parameterization of the line.

But we have a more convenient formula for special vector fields, namely, the fundamental theorem for conservative vector fields as following. (If your vector field F is conservative, namely, there is a scalar function f such that $F = grad(f)$, and the scalar function f is called the potential function for this vector field.)

Suppose c is the line you want to integrate over, and the parameterization is $r(t) =< x(t), y(t) >, a \leq t \leq b$, then

$$\int_c \bigtriangledown f \cdot dr = f(r(b)) - f(r(a)) \qquad (8)$$

From the formula, we immediately know that the line integral of conservative vector field is independent of paths, specially, if we have a CLOSED curve, the line integral of the conservative vector field over a closed curve is always 0.

Since this Fundamental theorem only applies to conservative vector field, so here is a natural question 1, Is every vector field conservative? 2 If it is conservative, how to find the potential function?

The answer to 1: NO, not ever vector field is conservative, actually, we have some special technique to detect if the given vector field conservative or not.

If the vector field is from $R^2 \to R^2$, has component $< P, Q >$, then it is conservative if $\partial R/\partial y = \partial R/\partial x+$some restrictions of the domain, you can check the detail in your book.

If the vector field$F$ is from $R^3 \to R^3$, then it is conservative if and only if $curl F = 0$.

If you want to know the answer to question2, here are two examples you may look at. Example 4 on P1050, and example 5 on page 1051.

After you answer these two questions, you are free to use the fundamental theorem for conservative vector field.

4 Surface integral

If you want to do surface integral, the FIRST STEP is again parameterize the surface, you need to know how to parameterize the following surfaces.

1 Plane given a point and two vectors on the plane

2 sphere

3 cylinder

4 graph, namely $z = f(x, y)$ $y = f(x, z)$ or $x = f(y, z)$.

5 revolutions

Suppose you have the parameterization, $r(u, v) =< x(u, v), y(u, v), z(u, v) >$, we should also consider the orientation of the surface, how would we define the "positive" direction of a surface, for surface, we will consider the normal vector of tangent surface at each point, since the normal vector has to change continuously, basically we have two ways to orient a surface, generally, when you are given a surface, they will specify the orientation, but when you have a closed surface, it is convention to make the normal vectors pointing outward, the other case is if you have a gragh $z = f(x, y)$, then $n = \frac{r_x \times r_y}{|r_x \times r_y|}$

Another point here is to know the surface area, we use ds to denote the change of area on the surface, we care about the relation between ds and $dudv$ because we need to do integral with respect to u, v. later. Here is the relation $ds =| r_u \times r_v |dudv$

After these preparation, we are ready to define surface integral

1 Scalar function, we first set up the integral $\int\int_s f(x, y, z)ds = \int\int_D f(r(u, v))| r_u \times r_v |dudv$, where D is the region for u, v, so we end up with a double integral.

There is a nice example you may look at on your book, Example 3 on page 1084.

2 Vector field, I really recommend you to read the intuition for Surface integral of vector field on page 1087. Here is the formula

$$\int\int_S F \cdot nds = \int\int_D F \cdot (r_u \times r_v)dudv \tag{9}$$

You will use the righr hand side of the formular to calculate the surface integral in most of the cases, since normal vectors at different points are different usually, but when your normal vector is easy to tell and have easy expression, like $(0, 0, 1)$, you can consider use the left hand side of the formular, but do NOT forget to change $ds =| r_u \times r_v |dudv$.

We have learnt

1 Double integral

2 Triple Integral

3 Line integral

4 Surface integral

The following theorem will link them together

1 Green's theorem. You need to know the story of Green's theorem, basically, you transform a line integral into a double integral, but here you can NOT use it for any line integral since the line here has to be CLOSED.

In most of the cases, if we have a line integral of vector field over a closed curve, we would like to use Green's theorem, because we "prefer" double integral.

Also it is helpful when you calculate the area of some region, you can check the red box on page 1058, and there is some similar homework problem (the polygon one)

2 Stroke's Theorem

This theorem will link the line integral and surface integral, it is helpful when

1 The line integral has messy terms, by taking the curl operation, you have much nicer form, namely, you will transfrom the line integral into surface integral

2 The surface has nice boundary, and you integrand is already in the form of $curlF$. You will transform the surface integral into line integral

Remark: when you use strokes's theorem, be careful about the orientation of the surface and the line.

3 Divergent Theorem

It will link the surface integral and triple integral, but you can only use this theorem when the surface is CLOSED, so if you have a vector field integrate over a closed surface PLEASE use divergent theorem. We love it.