Orbits of lattices on a symmetric space and the Furstenberg boundary

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(Joint work with Alex Gorodnik)

Hyperbolic case

- $\mathbb{D} = \{z \in \mathbf{C} : |z| < 1\}$: the hyperbolic unit disc.

- $\mathbb{D}(\infty) = \mathbb{S}^1$: the geometric boundary

- $\Gamma < \mathrm{Isom}(\mathbb{D})^\circ = \mathsf{PSL}_2(\mathbb{R})$: torsion-free lattice (i.e., discrete subgroup of finite co-volume)

Fix $y \in \mathbb{D}$ and $b \in \mathbb{S}^1$. Then

- $y\Gamma$ is discrete in $\mathbb{D}$;

- $b\Gamma$ is dense in $\mathbb{S}^1$.

Goal: Understand the distribution of $y\Gamma$ and $b\Gamma$ and their relation.

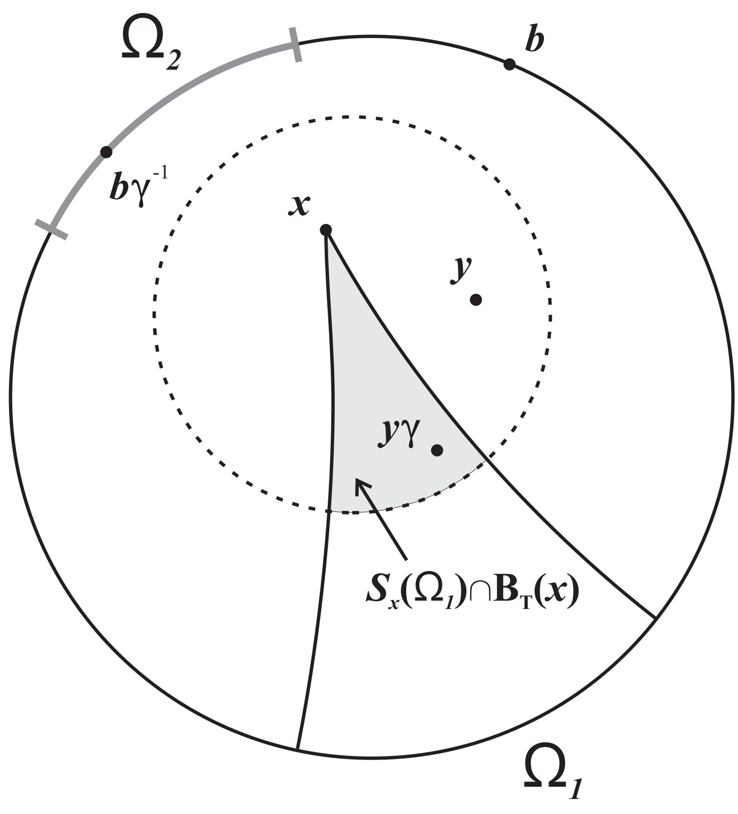
For $x \in \mathbb{D}$,

$$B_T(x) = \{z \in \mathbb{D} : d(x, z) < T\}.$$

For $\Omega \subset \mathbb{S}^1$,

$$S_x(\Omega)$$

: the sector at x consisting of $z \in \mathbb{D}$ such that the geodesic ray from x to z has the end pt in Ω



Let m_x denote the probability measure on $\mathbb{S}^1$ invariant by rotations about x.

Theorem 1. Let $x, y \in \mathbb{D}$, and $\Omega \subset \mathbb{S}^1$.

(A) [Nicholls 83]

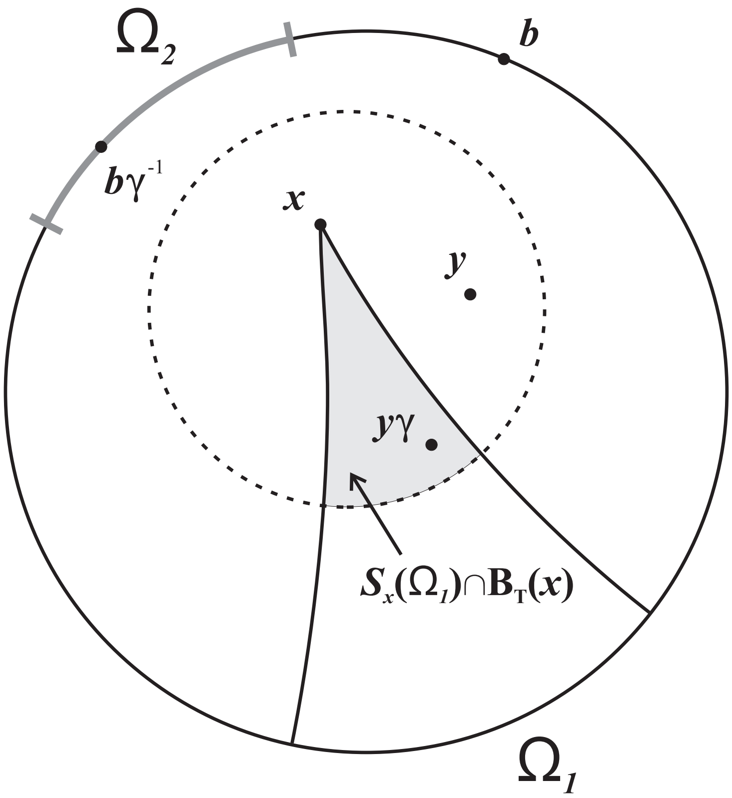
$$\#\{\gamma \in \Gamma : y\gamma \in B_T(x) \cap S_x(\Omega)\} \sim c \cdot m_x(\Omega)e^T$$

(B) [Good 83] $b \in \mathbb{S}^1$, $\Omega \subset \mathbb{S}^1$.

$$\#\{\gamma \in \Gamma : y\gamma \in B_T(x), b\gamma \in \Omega\} \sim c \cdot m_x(\Omega)e^T$$

(C) $\Omega_1, \Omega_2 \subset \mathbb{S}^1$.

$$\#\{\gamma \in \Gamma : y\gamma \in B_T(x) \cap S_x(\Omega_1), b\gamma^{-1} \in \Omega_2\} \sim c \cdot m_x(\Omega_1) m_y(\Omega_2) e^T$$



If Γ is co-compact, then (A) and (B) are also proven by Margulis in his thesis (69).

General Riemannian symmetric space case

- X: Riemannian symmetric space of non-compact type

- $X(\infty)$: the geometric boundary of X

- $G = \mathrm{Isom(X)}^{\circ}$

- $\Gamma \subset G$: lattice

For $y \in X$, and $b \in X(\infty)$,

- $y\Gamma$: discrete in X
- $b\Gamma$: dense in $bG \subset X(\infty)$.

  Note $bG = X(\infty)$ iff rank(X) = 1 and

  $$bG \simeq \text{Parabolic}\backslash G.$$

Goal: Understand the distribution of $y\Gamma$ and $b\Gamma$ and their relation.

Sector $S_x(\Omega)$: Let $x \in X$.

- W_x: a closed Weyl chamber at x
- $K_x = Stab_G(x)$
- $X = W_x K_x$ [Cartan decomposition].

For $\Omega \subset K_x$,

$$S_x(\Omega) := W_x \Omega$$

the sector about x associated to Ω.

Note if $M_x = Stab_{K_x}(W_x) = Cent_{K_x}(A^+)$,

$$S_x(\Omega) = S_x(M_x \Omega)$$

Proposition For any $x \in X$,

$$\mathrm{Vol}(B_T(x)) \sim T^{(\mathrm{rk(X)}-1)/2} e^{\delta T}$$

where $\delta = \max\{2\rho(t) : t \in \mathfrak{a}_1\}$.

[Knieper 97.]

In the following, we set

$$\mathrm{Vol}(B_T) = T^{(\mathrm{rk(X)}-1)/2} e^{\delta T}.$$

Theorem 2

Let $x, y \in X$ and $b \in X(\infty)$.

For $\Omega_1 \subset K_x$ and $\Omega_2 \subset bG \subset X(\infty)$,

$$\#\{\gamma \in \Gamma : y\gamma \in B_T(x) \cap S_x(\Omega_1),\, b\gamma^{-1} \in \Omega_2\}$$

$$\sim m_x(M_x\Omega_1) \cdot m_{y,b}(\Omega_2) \cdot \mathrm{Vol}(B_T)$$

where m_x: the prob. Haar measure on K_x, and

$m_{y,b}$: the unique K_y-inv. prob. measure on bG.

Corollaries (+thm)

A. Equidistribution in sectors

Let $x, y \in X$ and $\Omega \subset K_x$. Then

$$\#(y\Gamma \cap B_T(x) \cap S_x(\Omega)) \\ \sim \#(\Gamma \cap K_y) \cdot m_x(M_x\Omega) \cdot \mathrm{Vol}(B_T)$$

- Nicholls (83) for $X = \mathbb{H}^n$
- Eskin-McMullen (93) for $\Omega = K_x$ and $x = y$.

B. Equidistribution in Cpt. homoge. sp.

- G: connected s.s Lie group with no compact factors

- $Q < G$ s.t $Q \supset AN$ for $G = KAN$ Iwasawa decomp.

  e.g. Q parabolic subgp or algebraic co-cpt subgp of G

Hence the compact sp. of the form G/Q includes the projective homogeneous varieties of s.s real alg. gp.

Examples of G/Q for $G = SL_n(\mathbb{R}^n)$: $P(\mathbb{R}^n)$, Grassmanian variety $\mathrm{Gr}_k(\mathbb{R}^n)$, Flag variety $\mathcal{F}(\mathbb{R}^n)$.

Proposition For a lattice $\Gamma < G$, Γ-action on G/Q is minimal, i.e., for any $b \in G/Q$, Γb is dense in G/Q

[Mostow, 73]

Let $\Gamma_T = \{\gamma \in \Gamma : d(K, K\gamma) < T\}$, and m_K denote the unique K-inv. prob. measure on G/Q.

Theorem

Let $b \in G/Q$ and $\Omega \subset G/Q$. Then

$$\#\{\gamma \in \Gamma_T : \gamma b \in \Omega\} \sim m_K(\Omega) \cdot \mathrm{Vol}(G_T).$$

- For Q parabolic, different pf given by Gorodnik-Maucourant.

C. Statements in terms of Lie groups

- $G = KA^+K$: Cartan decomposition.
- $G = KQ$: Iwasawa decomposition (for any $Q \supset AN$).
- Let $M_1 = Cent_K(A^+)$ and $M_2 = K \cap Q$.

Theorem Let $\Omega_1, \Omega_2 \subset K$. Then

(1)

$$\#(\Gamma_T \cap KA^+\Omega_1 \cap \Omega_2 Q) \sim m_K(M_1\Omega_1) m_K(\Omega_2 M_2) \, \mathrm{Vol}(G_T)$$

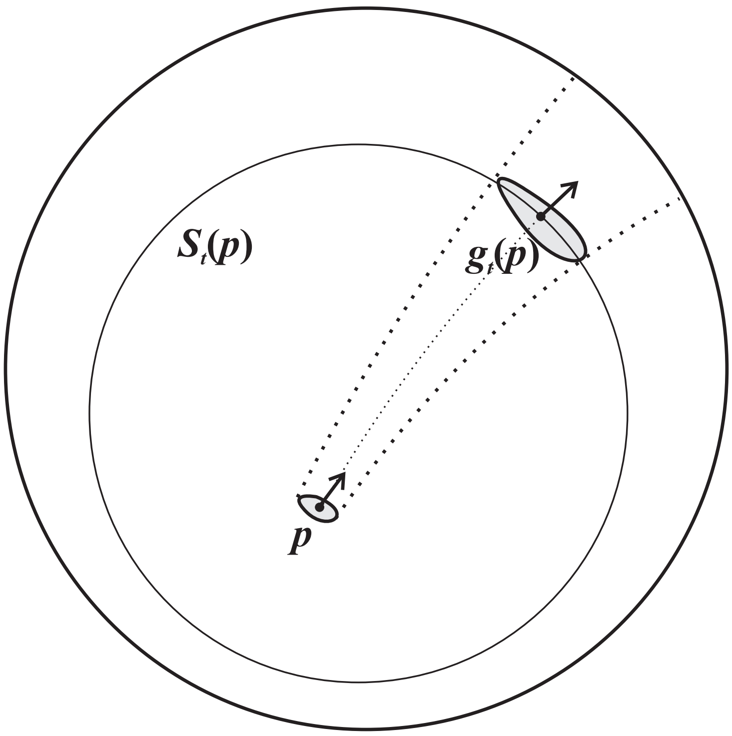
(2)

$$\#(\Gamma_T \cap \Omega_1 A^+ \Omega_2) \sim m_K(M_1\Omega_1) m_K(M_1\Omega_2) \, \mathrm{Vol}(G_T)$$

Main ingredients of proof

Strong Wavefront lemma: hyperbolic case

Let $T^1(\mathbb{D})$ be the unit tangent bundle and $\pi : T^1(\mathbb{D}) \to \mathbb{D}$ the natural projection. Let g_t be the geodesic flow on $T^1(\mathbb{D})$. Let $p \in \mathbb{D}$, and let $K = \pi^{-1}(p)$. Then $g_t(K)$ consists of unit normal vectors to the sphere $S_t(p)$.


$S_t(p)$
$g_t(p)$
p

- Wavefront lemma [Eskin-McMullen, 93]: $\exists$ a nbd $\mathcal{O}$ of K in $T^1(\mathbb{D})$ s.t. $g_t(\mathcal{O}) \sim g_t(K)$ $\forall t$.

- Strong wavefront lemma: $\forall\ v \in \pi^{-1}(p)$, $\exists$ a nbd $\mathcal{O}$ of v in $T^1(\mathbb{D})$ s.t. $g_t(\mathcal{O}) \sim g_t(v)$ $\forall t$.

Recall $G = KA^+K$ (Cartan decomposition).

<u>**Strong wavefront lemma**</u>

Let $\mathcal{C} \subset A^+$ be such that $d(\mathcal{C}, \text{Walls of} A^+) > 0$.

Then $\forall$ nbd U_1, U_2 of e in K and $\forall$ nbd V of e in A, $\exists$ a nbd $\mathcal{O}$ of e in G s.t.

$$g\mathcal{O} \cup \mathcal{O}g \subset (k_1U_1)(aV)(k_2U_2)$$

$\forall\ g = k_1ak_2 \in K\mathcal{C}K$.

Using the Strongwave front lemma, we may study Q-action of $\Gamma \backslash G$ instead of Γ action on $K\backslash G \times G/Q$.

Equidistribution of solvable flows

Consider $Q < G < L$ where $Q < G$ as before, L Lie gp, $\Lambda < L$ lattice

Theorem 3 Spse $x_0 G$ is dense in $\Lambda\backslash L$. Then for any $f \in C_c(\Lambda\backslash L)$,

$$\lim_{T\to\infty} \frac{1}{\rho(Q_T)} \int_{Q_T} f(x_0 q^{-1})\, d\rho(q) = \int_{\Lambda\backslash L} f(h)\, dh$$

where ρ denotes right inv. Haar measure on Q and

$$Q_T = \{q \in Q : d(K, Kq) < T\}.$$

Remarks

- If $Q \neq G$, Q is NOT generated by unipotent flows.

- General case follows from the case when $Q = AN$. Though $Q = AN$ is solvable, $\{Q_T\}$ are NOT Fölner sets.

- The same holds for $Q_T(\Omega) = Q_T \cap KA^+\Omega$ for any $\Omega \subset K$. (We need this for Thm 2)

Proof of Theorem 3 is based on:

- a theorem of Shah [95]: For any open $\Omega \subset K$, as $a \to \infty$,

$$\int_{\Omega} f(x_0 k a)\, dk \to m_K(\Omega) \int_{\Lambda \backslash L} f(h) dh.$$

  (using Ratner's classification of measures inv. under unipotent flows and the work of Dani-Margulis.)

- Proximal property of Furstenberg boundary G/B: every regular element in a positive Weyl chamber acts on an open subset of full measure on G/B as a contraction