DISCRETE SUBGROUPS GENERATED BY LATTICES IN OPPOSITE HOROSPHERICAL SUBGROUPS

Hee Oh

Contents

0. INTRODUCTION

Let G be a center-free connected semisimple real algebraic group with no compact factors. The unipotent radical of a proper parabolic subgroup of G is called *horospherical*. Two horospherical subgroups are called *opposite* if they are the unipotent radicals of two opposite parabolic subgroups.

We recall the following theorem:

Typeset by $\mathcal{AMS}$-TeX

Theorem 0.1. *(*[12, Theorem 7.1.1]*, also see* [18, Theorem 4.2]*) Suppose that the real rank of G is at least 2. Then for any irreducible non-uniform lattice Γ in $G(\mathbb{R})$, there exists a pair of opposite horospherical subgroups U_1 and U_2 defined over $\mathbb{R}$ such that $\Gamma \cap U_i(\mathbb{R})$ is a lattice in $U_i(\mathbb{R})$ for $i = 1, 2$.*

This theorem was one of the main steps in proving the arithmeticity of a non-uniform lattice in such groups, without the use of the superrigidity theorem [13] which had settled the arithmeticity of both uniform and non-uniform lattices at once.

In this paper we study the converse problem, which may be stated as follows: suppose that one is given opposite horospherical real subgroups U_1 and U_2 and lattices F_1 and F_2 inside $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively. Then under what conditions is the group generated by F_1 and F_2 discrete? What discrete subgroups of G can arise in this way?

Our main result is that if G is absolutely simple, then (under some additional assumptions on U_1 and U_2) any discrete group generated by F_1 and F_2 is a non-uniform lattice in $G(\mathbb{R})$. In particular we prove the following:

Theorem 0.2. *Let G be an adjoint absolutely simple $\mathbb{R}$-algebraic group with real rank at least 2, U_1, U_2 a pair of opposite horospherical $\mathbb{R}$-subgroups of G. Suppose that G is split over $\mathbb{R}$ and that U_1 is not the unipotent radical of a Borel subgroup in a group of type A_2.*

Let F_1 and F_2 be lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively . If F_1 and F_2 generate a discrete subgroup, then there exists a $\mathbb{Q}$-form of G with respect to which U_1 and U_2 are defined over $\mathbb{Q}$ and F_i is commensurable to $U_i(\mathbb{Z})$ for each $i = 1, 2$. Furthermore the discrete subgroup Γ_{F_1,F_2} generated by F_1 and F_2 is commensurable to $G(\mathbb{Z})$.

Let us remark that Theorem 0.2 is not true in the group of real rank one; in fact there exist discrete subgroups of the form Γ_{F_1,F_2} which are not lattices. For example, the subgroup Γ_n of $SL_2(\mathbb{R})$ generated by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for some nonzero $n \in \mathbb{Z}$ is not a lattice if $n > 2$. To see this, note that the subgroup Γ_n is contained in the subgroup generated by the elements $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the fundamental domain in the upper half plane for the latter subgroup is the set $\{z \in \mathbb{H}^+ \mid |z| > 1, |Re(z)| < n/2\}$ which has infinite volume when $n > 2$.

We can extend Theorem 0.2 in some cases by dropping the assumption that G is split over $\mathbb{R}$, giving the following more technical result.

Theorem 0.3. *Let Z_i denote the center of U_i for each $i = 1, 2$. In Theorem 0.2 the assumption that G is split over $\mathbb{R}$ may be replaced by one of the following:*

(1) *U_1 is either commutative (in that case also assume that the $\mathbb{R}$-form of* **G** *is not*

of type ${}^1E_{6,2}^{28}$) or Heisenberg, and the commutator subgroup of $N(U_1) \cap N(U_2)$ is non-trivial and has no $\mathbb{R}$-anisotropic factors.

(2) $[U_1, U_1] = Z_1$, *Z_1 is not the root group of a highest real root, the commutator subgroup of $N(U_1) \cap N(U_2) \cap G_0$ has no $\mathbb{R}$-anisotropic factors and the $\mathbb{R}$-form of G_0 is not of type ${}^1E_{6,2}^{28}$, where G_0 is the subgroup generated by Z_1 and Z_2.*

(3) $[U_1, U_1] \neq Z_1$, *Z_1 is the root group of a highest real root, the commutator subgroup of $N(U_1') \cap N(U_2') \cap G_0'$ has no $\mathbb{R}$-anisotropic factors and the $\mathbb{R}$-form of G_0' is not of type ${}^1E_{6,2}^{28}$, where U_i' is the centralizer of $\tilde{U}_i = \{g \in U_i \mid gug^{-1}u^{-1} \in Z_i \text{ for all } u \in U_i\}$ in U_i and G_0' is the subgroup generated by the center of U_1' and of U_2'.*

If the commutator subgroup of $N(U_1) \cap N(U_2)$ has no $\mathbb{R}$-anisotropic factors, then the assumptions on $\mathbb{R}$-anisotropic factors in (2) and (3) are to be automatically satisfied (Remark 4.3.5). Weakening this assumption to those in (2) and (3) makes our result cover a greater deal of important cases. Since Theorem 0.2 follows from Theorem 0.3, we will refer to Theorem 0.3 as the main theorem hereafter.

As a corollary, we obtain a complete classification of discrete subgroups generated by lattices in opposite horospherical subgroups U_1 and U_2 considered in the main theorem. In particular, we note the following:

Corollary 0.4. *Let G, U_1 and U_2 be as in the main theorem. Then any discrete subgroup generated by two lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ is an arithmetic subgroup.*

Therefore for the $\mathbb{R}$-split groups, combining Theorem 0.2 with Theorem 0.1, we obtain the following criterion for a discrete subgroup to be a non-uniform lattice:

Corollary 0.5. *Let G be an adjoint absolutely simple $\mathbb{R}$-split group with rank at least 2 and Γ a discrete subgroup. In addition, assume that G is not of type A_2. Then Γ is a non-uniform lattice if and only if there exists a pair of opposite horospherical subgroups U_1 and U_2 of G such that $\Gamma \cap U_i$ is Zariski dense in U_i for each $i = 1, 2$.*

(Note that any arithmetic subgroup of $G(\mathbb{R})$ which has a non-trivial unipotent element is a non-uniform lattice in $G(\mathbb{R})$ (e.g. [16, Theorem 10.18]).)

Remark. We note that the main theorem presents a strong necessary condition for discreteness of a subgroup generated by lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$. On the other hand it is clear that this condition is not sufficient for discreteness; see e.g. example 2.2.4. We refer the readers to [9] for discreteness criteria of this kind in $SL_2(\mathbb{R})$.

We call a horospherical subgroup U $\mathbb{R}$-Heisenberg if $[U, U]$ is equal to the center $Z(U)$ of U, i.e., 2-step nilpotent and $Z(U)$ is the root group of a highest real root of G. If U is

$\mathbb{R}$-Heisenberg and dim$Z(U) = 1$, then U is Heisenberg. It should be noted that the main theorem would not cover the cases when U_1 is either $\mathbb{R}$-Heisenberg with $\dim(Z(U)) > 1$ or Heisenberg in an $\mathbb{R}$-split group of type A_2, even if we were to drop the assumption on $\mathbb{R}$-anisotropic factors. On the other hand we can see that the main theorem (Theorem 0.3) implies Theorem 0.2 as follows: if G is split over $\mathbb{R}$, none of the subgroups G_0, G_0' and H has $\mathbb{R}$-anisotropic factors. Therefore the case when U_1 is either commutative or Heisenberg follows from (1). When U_1 is neither of those, U_1 satisfies assumption (2) or (3) according to whether U_1 is 2-step nilpotent or not, respectively.

The proof of the main theorem is given in three parts according as the horospherical subgroups involved are commutative (Theorem 4.1.1), Heisenberg (Theorem 4.2.11) or non-$\mathbb{R}$-Heisenberg (Theorem 4.3.4). One of the main ideas for the first two cases is to use Raghunathan's conjecture proved by Ratner (Theorem 3.3.1) for the action of the group of real points of the commutator subgroup of $N(U_1) \cap N(U_2)$ on the space of lattices in $U_i(\mathbb{R})$ for each $i = 1, 2$. A theorem of Margulis on the construction of a representation (Theorem 2.4.2) enables us to reduce the non-$\mathbb{R}$-Heisenberg horospherical subgroup cases to the commutative cases. In fact the assumptions on $\mathbb{R}$-anisotropic factors in (1)-(3) arise because of the dependence of our proof on Ratner's theorem.

The main theorem of this paper was announced in [14] together with a detailed sketch of the proof, in the case when $G(\mathbb{R}) = SL_n(\mathbb{R})$, $n \geq 3$ and the horospherical subgroups involved are commutative.

Acknowledgment. I would like to express my deep gratitude to my advisor, G. Margulis, for suggesting this problem and for his constant encouragement and guidance. I would very much like to thank G. Prasad for many valuable comments and for providing the proof of Proposition 1.4.2. I am also very thankful to T. N. Venkataramana for useful discussions and especially for pointing out that Corollary 0.5 follows from the main theorem.

1. PRELIMINARIES

1.1. Notation and terminology

1.1.1. As usual, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the complex numbers, the reals, the rationals, the integers and the positive integers.

1.1.2. For a group H, $Z(H)$ denotes the center of H. For any subset $F \subset H$, $N(F)$ and $C(F)$ denote the normalizer and the centralizer of F in H respectively.

For a Lie algebra $\mathfrak{h}$, the commutator of two elements X, Y of $\mathfrak{h}$ is denoted by $[X, Y]$ and for any two subsets A and B of $\mathfrak{h}$, $[A, B]$ denotes the linear subspace generated by all the commutators $[X, Y]$, $X \in A, Y \in B$. $Z(\mathfrak{h})$ denotes the center of $\mathfrak{h}$, that is $\{X \in \mathfrak{h} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{h}\}$.

1.1.3. Two subgroups H_1 and H_2 are called *commensurable* if $H_1 \cap H_2$ has a finite index in both of H_1 and H_2. $H_1 \times H_2$ denotes the direct product of H_1 and H_2 and $L \ltimes N$ the semi-direct product of L and a normal subgroup N.

1.1.4. For an algebraic group H, $R(H)$ and $R_u(H)$ denote the radical and the unipotent radical of H respectively. For a Lie group H, we denote by H^0 the connected component of the identity of H.

1.1.5. For a subfield k of $\mathbb{R}$, a linear algebraic $\mathbb{R}$-group $G \subset GL_n(\mathbb{C})$ is defined over k if it consists of all matrices whose entries annihilate some set of polynomials on $M_n(\mathbb{C})$ with coefficients in k. In this case, we denote by $G(J)$ the subgroup $\{g = (g_{ij}) \in G \mid g, g^{-1} \in GL_n(J), g_{ij} = \delta_{ij} \bmod J\}$ for any subring J of k and $G(\mathbb{R}) = G \cap GL_n(\mathbb{R})$.

1.1.6. A k-form of an algebraic $\mathbb{R}$-group G is a pair $(\tilde{G}, f)$ where $\tilde{G}$ is an algebraic group defined over k and f an isomorphism $G \to \tilde{G}$ defined over $\mathbb{R}$. For a given k-form $(\tilde{G}, f)$ of G, we denote $f^{-1}(\tilde{G}(J))$ by $G(J)$ for a subring J of k. If an algebraic $\mathbb{R}$-subgroup H of G is such that $f(H)$ is a k-subgroup of $\tilde{G}$, we say that H is defined over k and denote $H \cap G(J)$ by $H(J)$. A subgroup commensurable to $G(\mathbb{Z})$ is called an arithmetic subgroup of $G(\mathbb{R})$.

1.1.7. We extend the definition of an arithmetic group to a semisimple Lie group with finite center as follows. Let G be a connected semisimple Lie group with finite center. A discrete subgroup Γ of G is called an arithmetic subgroup if there exist a connected adjoint semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ and an isogeny $p : G \to \mathbf{G}(\mathbb{R})^0$ such that $p(\Gamma)$ is commensurable to $\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0$.

1.1.8. A connected algebraic k-group G is called absolutely almost simple if it has no connected normal subgroup, and almost k-simple if it has no connected normal k-subgroup. A connected semisimple algebraic group G is called simply connected (resp. adjoint) if every central isogeny $\phi : G' \to G$ (resp. $\phi : G \to G'$), for G' connected, is an algebraic group isomorphism. In characteristic 0 case, a connected semisimple algebraic group is adjoint if and only if its center is trivial.

1.1.9. For a locally compact group G, a discrete subgroup Γ of G is called a lattice if the quotient G/Γ has a finite invariant measure. A lattice Γ in G is called *uniform* if G/Γ is compact, and *non-uniform*, otherwise. A lattice Γ in a connected semisimple Lie group G with finite center is called irreducible if for every normal subgroup of positive dimension N, Γ is dense when projected onto G/N.

1.1.10. A discrete subgroup F in a real vector space V is called a quasi-lattice and in particular, if F spans V, then we call it a lattice. The determinant of a lattice F in V is the volume of the quotient V/F.

1.1.11. A connected semisimple k-group G is *isotropic over* k if it contains a non-trivial k-split torus and is *anisotropic over* k otherwise.

For $k = \mathbb{R}$, G is anisotropic over $\mathbb{R}$ if and only if $G(\mathbb{R})$ is compact.

1.1.12. Let G be a connected semisimple algebraic k-group. We call a subgroup U a horospherical k-subgroup if it is the unipotent radical of a parabolic k-subgroup P such that $P \cap G'$ is a proper parabolic subgroup of G' for each semisimple normal k-subgroup of G. It is known that the normalizer of a horospherical k-subgroup is a parabolic k-subgroup.

We note that, in our definition, the existence of a horospherical $\mathbb{R}$-subgroup of a semisimple $\mathbb{R}$-group G implies that G does not have any $\mathbb{R}$-anisotropic factors.

1.1.13. In a connected simple algebraic group G, a horospherical subgroup U is called *Heisenberg* if $[U, U] = Z(U)$ and $Z(U)$ has dimension one. We call a horospherical subgroup U $\mathbb{R}$-Heisenberg if $[U, U] = Z(U)$ and $Z(U)$ is the root group of a highest real root of G (also see section 1.3.1).

A parabolic (resp. horospherical) subgroup of a semisimple algebraic group is called *reflexive* if its conjugacy class contains an opposite parabolic (resp. horospherical) subgroup to it.

1.1.14. Let G be a connected semisimple algebraic k-group and P a parabolic k-subgroup. A Levi decomposition $P = LR_u(P)$ of P is called k-Levi decomposition if L is defined over k. We denote by $Ad : L \to GL(\mathcal{U})$ the representation of L in $\mathcal{U} = \mathrm{Lie}(R_u(P))$ which is the restriction of the adjoint representation of P.

1.1.15. Let G be a connected semisimple algebraic $\mathbb{R}$-group and U_1 and U_2 a pair of opposite horospherical $\mathbb{R}$-subgroups. For lattices F_1 and F_2 in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively, we denote by Γ_{F_1,F_2} the subgroup of $G(\mathbb{R})$ generated by F_1 and F_2.

1.1.16. Let (G, U_1, U_2) be a triple where G is a connected semisimple adjoint algebraic $\mathbb{R}$-group with no $\mathbb{R}$-anisotropic factors and U_1 and U_2 a pair of opposite horospherical $\mathbb{R}$-subgroups. In order to avoid the lengthy repetition, we say the triple (G, U_1, U_2) has property (A) if every discrete subgroup of the form Γ_{F_1,F_2} is an arithmetic subgroup of $G(\mathbb{R})^0$ where F_1 and F_2 are lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$, respectively . We use this terminology only in chapter 4 where we prove the main theorem.

1.1.17. All algebraic groups are assumed to be connected and all fields, usually denoted by k or K, have characteristic 0. We freely use terminology from the theory of algebraic groups in ([1], [5]) and of Lie algebras in [8].

1.2. Some known algebraic lemmas

1.2.1. (cf. [5, Ch 3-4]) Let G be a connected semisimple algebraic k-group, S a maximal k-split torus of G and T a maximal k-torus containing S. Denote by ${}_k\Phi = \Phi(S, G)$ (resp. $\Phi = \Phi(T, G)$) the set of roots of G with respect to S (resp. T). An element in ${}_k\Phi$ is called a k-root. We choose compatible orderings on Φ and ${}_k\Phi$, and let ${}_k\Delta$ and Δ be the simple roots for these orderings. Let j be the map $\Phi \to {}_k\Phi \cup \{0\}$ induced by the restriction onto S.

For each $b \in \Phi(T, G)$, we denote by U_b the unique one parameter root subgroup associated with b. We observe that $\mathrm{Lie}(U_b) = \mathcal{U}_b := \{v \in \mathrm{Lie}(G) \mid \mathrm{Ad}t(v) = b(t)v, t \in T\}$, $\dim(\mathcal{U}_b) = 1$ and $\mathrm{Lie}(G) = \bigoplus_{b\in\Phi(T,G)} \mathcal{U}_b \oplus \mathrm{Lie}(T)$.

A subset $\Psi \subset {}_k\Phi$ is called closed if $a, b \in \Psi$ and $a + b \in {}_k\Phi$ imply $a + b \in \Psi$. If Ψ is closed, then we denote by G_Ψ the subgroup generated by T and the subgroups U_a, $a \in j^{-1}(\Psi \cup \{0\})$, and by G^*_Ψ the subgroup generated by all the subgroups U_a, $a \in j^{-1}(\Psi)$. The subgroups G_Ψ and G^*_Ψ are algebraic and do not depend on the choice of a maximal torus T containing S. The groups G_Ψ and G^*_Ψ are defined over k if and only if Ψ is invariant under $\mathrm{Gal}(K/k)$, K the separable closure of k [5, Proposition 3.14]. If G^*_Ψ is unipotent, then it will be also denoted by U_Ψ and the set Ψ in this case will be called unipotent. For a closed unipotent subset $\Psi \subset {}_k\Phi$, the unipotent subgroup U_Ψ is defined and split over k.

1.2.2. For $\Theta \subset {}_k\Delta$, $[\Theta]$ denotes the $\mathbb{Z}$-linear combinations of Θ which are k-roots.

We define the following closed subsets of ${}_k\Phi$:

$$\pi_\theta = [\Theta] \cup {}_k\Phi^+, {\pi_\theta}^- = [\Theta] \cup {}_k\Phi^-, \beta_\theta = {}_k\Phi^+ - [\Theta], {\beta_\theta}^- = {}_k\Phi^- - [\Theta].$$

For the sake of simplicity, we shall denote by ${}_kP_\Theta, {}_kP_\Theta{}^-, {}_kV_\Theta, {}_kV_\Theta{}^-$, the subgroups $G_{\pi_\Theta}, G_{\pi_\Theta{}^-}, U_{\beta_\Theta}, U_{\beta_\Theta{}^-}$. The subgroups ${}_kP_\Theta, {}_kP_\Theta{}^-, {}_kV_\Theta$ and ${}_kV_\Theta{}^-$ are connected and defined over k.

The subgroups ${}_kP_\Theta$ (resp. ${}_kV_\Theta$), $\Theta \subset {}_k\Delta$ are called standard parabolic (resp. horospherical) k-subgroups of G associated with S and ${}_k\Phi^+$. Every parabolic (resp. horospherical) k-subgroup of G is conjugate by an element of $G(k)$ to a unique standard parabolic (resp. horospherical) subgroup.

Lemma.

(1) $R_u({}_kP_\Theta) = {}_kV_\Theta$, $R_u({}_kP_\Theta{}^-) = {}_kV_\Theta{}^-$ *for some* $\Theta \subset {}_k\Delta$.

(2) *Any pair of opposite parabolic (resp. horospherical) k-subgroups is conjugate by an element of $G(k)$ to the pair ${}_kP_\Theta, {}_kP_\Theta{}^-$ (resp. ${}_kV_\Theta, {}_kV_\Theta{}^-$) for some $\Theta \subset {}_k\Delta$.*

(3) *Any two parabolic k-subgroups opposite to P are conjugate by a unique element of $R_u(P)(k)$.*

1.2.3. The root system $\Phi(S,G)$ (resp. $\Phi(T,G)$) is irreducible if and only if G is almost k-simple (resp. absolutely almost simple). The type of an irreducible root system Φ is by definition the type of its Dynkin diagram. For a k-simple k-group G, we refer the type of $\Phi(T,G)$ by the absolute type, or simply the type of G, and the type of $\Phi(S,G)$ by the k-type of G.

1.2.4. Proposition. [21, 3.1.2]

(1) *A connected simply connected (resp. adjoint) semisimple k-group decomposes uniquely into a direct product of simply connected (resp. adjoint) almost k-simple normal k-groups.*

(2) *A connected semisimple k-group decomposes into an almost direct product of almost k-simple normal k-groups.*

1.2.5. Proposition. *For a connected semisimple k-group G, there exists a sequence $\tilde{G} \to_{\tilde{p}} G \to_{\bar{p}} \bar{G}$, where $\tilde{G}$ is a simply connected k-group, $\bar{G}$ is an adjoint k-group and $\tilde{p}$ and $\bar{p}$ are central k-isogenies. The groups $\tilde{G}$ and $\bar{G}$ and the isogenies $\tilde{p}$ and $\bar{p}$ are determined uniquely up to k-isomorphism.*

1.2.6. Let k' be a finite separable field extension of k. We denote by $R_{k'/k}$ the restriction of scalar functor from k' to k. (For definition, see [5, 6.17-6.21]).

Proposition.

(1) *If H is k'-group, then the functor $R_{k'/k}$ defines a bijection of the set of parabolic (resp. horospherical) k'-subgroups of G onto the set of parabolic (resp. horospherical) k-subgroups of the k-group $R_{k'/k}(H)$ and if H is reductive, then $rank_{k'}(H) = rank_k(R_{k'/k}(H))$.*

(2) *If G is a simply connected (resp. adjoint) almost k-simple k-group, then there exists a finite separable field extension k' of k and a connected simply connected (resp. adjoint) absolutely almost simple k'-group G' such that $G = R_{k'/k}G'$.*

1.2.7. The next proposition is useful when we want to determine whether some some algebraic groups and algebraic maps are defined over a sub-field.

Proposition. *(see* [27, 3.1.8, 3.1.10]*)*

(1) *Suppose that $G \subset GL_n(\mathbb{C})$ is an algebraic group and that $G \cap GL_n(k)$ is Zariski dense in G for some subfield k of $\mathbb{C}$. Then G is defined over k.*

(2) *Suppose that V, W are k-varieties and that $f : V \to W$ is a regular map. Suppose also that there is a set $A \subset V_k$ which is Zariski dense in V such that $f(A) \subset W_k$. Then f is defined over k.*

1.3. Adjoint representation and Maximal subgroups

1.3.1. Let G be a connected almost k-simple k-group, S a maximal k-split torus, and $\Phi(S,G)$ a corresponding root system. We fix a basis ${}_k\Delta$ of $\Phi(S,G)$. Since G is k-simple, $\Phi(S,G)$ is an irreducible (but not necessarily reduced) root system. We note that a horospherical k-subgroup U k-Heisenberg if and only if U is conjugate to ${}_kV_\Theta$ such that $[{}_kV_\Theta, {}_kV_\Theta] = Z({}_kV_\Theta)$ and $Z({}_kV_\Theta) = U_{\alpha_h}$ where α_h is the highest root in $\Phi(S,G)$. Note that the center of a k-Heisenberg subgroup is the root group of a highest k-root.

We note that if U_{α_h} has dimension one, a k-Heisenberg subgroup is in fact a Heisenberg subgroup. If ${}_kV_\Theta$ is k-Heisenberg (resp. commutative), we call ${}_k\Delta - \Theta$ the set of k-Heisenberg (resp. commutative) roots. In an irreducible root system $\Phi(S,G)$, there exist a unique highest k-root α_h and a unique set of simple roots ${}_k\Delta_H \subset {}_k\Delta$ such that ${}_kV_{{}_k\Delta - {}_k\Delta_H}$ is the unique k-Heisenberg standard horospherical k-subgroup of G.

It is not difficult to prove the following lemma which characterizes the commutative and k-Heisenberg roots in each irreducible root system.

Lemma.

(1) *A simple root α is a commutative root if and only if its coefficient in α_h is 1.*

(2) *The set ${}_k\Delta_H$ is determined uniquely by the condition that if $\alpha \in {}_k\Delta_H$, then $\alpha_h - \alpha$ is a root and the sum of the coefficients of ${}_k\Delta_H$ in α_h is 2*

k-type	α_h	${}_k\Delta_H$	commutative roots
A_n	$\alpha_1 + \alpha_2 + ... + \alpha_n$	$\{\alpha_1, \alpha_2\}$	$\{\alpha_1, .., \alpha_n\}$
B_n	$\alpha_1 + 2\alpha_2 + ... + 2\alpha_n$	α_2	α_1
C_n	$2\alpha_1 + 2\alpha_2 + ... + 2\alpha_{n-1} + \alpha_n$	α_1	α_n
D_n	$\alpha_1 + 2\alpha_2 + ... + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	α_2	$\{\alpha_1, \alpha_{n-1}, \alpha_n\}$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$	α_2	$\{\alpha_1, \alpha_6\}$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$	α_1	α_7
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$	α_8	$\emptyset$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$	α_1	$\emptyset$
G_2	$3\alpha_1 + 2\alpha_2$	α_2	$\emptyset$
BC_n	$2\alpha_1 + 2\alpha_2 + ... + 2\alpha_{n-1} + 2\alpha_n$	α_1	$\emptyset$

1.3.2. (cf. [7, 5.5.1]) Among maximal parabolic subgroups, the adjoint representation of a Levi component on the Lie algebra of the unipotent radical is absolutely irreducible if and only if the unipotent radical is commutative. These cases are quite restricted, and the list is well-known.

List of parabolic subgroups with commutative unipotent radical.

Group	Levi component	unipotent radical
GL_n	$GL_k \times GL_{n-k}$	$\mathbb{C}^k \otimes \mathbb{C}^{n-k}$
SO_{2n+1}	$O_{2n-1} \times GL_1$	$\mathbb{C}^{2n-1}$
Sp_{2n}	GL_n	$S^2(\mathbb{C}^n)$
SO_{2n}	$O_{2n-1} \times GL_1$	$\mathbb{C}^{2(n-1)}$
SO_{2n}	GL_n	$\Lambda(\mathbb{C}^n)$
E_6	$Spin_{10} \times GL_1$	$spin_{10^+}$
E_7	$E_6 \times GL_1$	V_{27}

1.3.3. (cf. [7, 5.5.2]) The commutator quotient of the Lie algebra of a Heisenberg horospherical subgroup carries a sympletic form defined in terms of the Lie bracket. This form is preserved by the action of the Levi component. Except the groups of type A_n, the representation of the Levi component on the commutator quotient is absolutely irreducible.

List of parabolic subgroups with Heisenberg unipotent radical.

Group	Levi component	Commutator quotient
GL_n	$GL_1 \times GL_{n-2} \times GL_1$	$\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$
O_n	$O_{n-4} \times GL_2$	$\mathbb{C}^{2(n-4)}$
Sp_{2n}	$Sp_{2(n-1)} \times GL_1$	$\mathbb{C}^{2(n-1)}$
E_6	GL_6	$\Lambda^3(\mathbb{C}^6)$
E_7	$Spin_{12} \times GL_1$	$spin_{12^+}$
E_8	$E_7 \times GL_1$	V_{56}
F_4	$Sp_6 \times GL_1$	$\Lambda^3_{prim}(\mathbb{C}^6)$
G_2	GL_2	$S^3(\mathbb{C}^2)$

1.3.4. The following two theorems of Dynkin classify the maximal connected subgroups of $SL_n(\mathbb{C})$ into the three categories-reducible (as linear groups), irreducible non-simple and irreducible simple. It is known that any irreducible connected Lie subgroup in $SL_n(\mathbb{C})$ is semisimple [15, Theorem 1.1.1].

Theorem. *(cf.* [15, 3.3.1 -3.3.2]*) Let M be a maximal connected (complex) Lie subgroup of $SL_n(\mathbb{C})$.*

1. *If M is reducible, then it is a maximal parabolic subgroup of $SL_n(\mathbb{C})$.*
2. *If M is non-simple irreducible, then it is conjugate to the subgroup $SL_s(\mathbb{C}) \otimes SL_t(\mathbb{C})$ where $n = st$, s, $t \geqq 2$.*

1.3.5. Theorem. *(cf.* [15, 3.3.3]*) Let $R : M \to GL(V)$ be a non-trivial irreducible linear representation of a simply connected simple complex linear group M. If there are no nondegenerate bilinear forms in V invariant under R, then $R(M)$ is a maximal connected subgroup of $SL(V)$, and if R is orthogonal or symplectic, then R is a maximal connected subgroup of $SO(V)$ or $Sp(V)$, respectively. The only exceptions are the representations listed in Table 7 in* [15].

1.3.6. As a corollary of the above theorem and section 1.3.2−3, we obtain the following two propositions. We denote by P a parabolic $\mathbb{R}$-subgroup of an adjoint absolutely simple algebraic group $\mathbf{G}$, by $\mathbf{L}$ a Levi $\mathbb{R}$-subgroup of P and $\mathbf{H} = [\mathbf{L}, \mathbf{L}]$, and $H = \mathbf{H}(\mathbb{R})^0$. The notation Ad is as in 1.1.14.

Proposition. *If $R_u(P)$ is commutative, the representation $Ad_i : \mathbf{H} \to SL(\mathcal{U}_i)$ is absolutely irreducible and $Ad_i(H)$ is a maximal connected closed subgroup of $SL(\mathcal{U}_i(\mathbb{R}))$.*

Proof. The first statement is due to the well known fact that all the representations listed in 1.3.2 are absolutely irreducible. By proposition 1.2.5, there exist a simply connected simple $\mathbb{R}$-group $\tilde{\mathbf{H}}$ and an $\mathbb{R}$-isogeny $p : \tilde{\mathbf{H}} \to \mathbf{H}$. Consider the representation $Ad_i \circ p$ of $\tilde{\mathbf{H}}$. By the previous theorem, $Ad_i(p(\tilde{\mathbf{H}}))$, which is the same as $Ad_i(\mathbf{H})$, is a maximal subgroup of $SL(\mathcal{U}_i)$. Since $Ad_i(\mathbf{H}(\mathbb{R})) \subset SL(\mathcal{U}_i(\mathbb{R}))$, it is clear that $Ad_i(H)$ is a maximal connected closed subgroup of $SL(\mathcal{U}_i(\mathbb{R}))$. □

1.3.7. Consider the case when $R_u(P)$ is Heisenberg. Except for groups of type A_n, $\mathcal{U}_i$ has two Ad_i-invariant subspace one of which is $Z(\mathcal{U}_i)$. Denote by $\mathcal{V}_i$ the other invariant subspace. For groups of type A_n, Ad_i decomposes into the direct sum $\rho \oplus \rho^* \oplus \mathrm{id}$ where ρ is the $(n-1)$-dimensional standard representation of A_{n-2}, ρ^* is the dual of ρ and id is the one-dimensional trivial representation. Let $\mathcal{W}_i$ and ${\mathcal{W}_i}^*$ be the invariant subspaces of ρ and ρ^* respectively and in this case, let $\mathcal{V}_i$ denote $\mathcal{W}_i \oplus \mathcal{W}_i^*$. We shall denote by Ad_i' the restriction of Ad_i on $\mathcal{V}_i$. Set $Sp(\mathcal{V}_i) = \{g \in SL(\mathcal{V}_i) \mid [gv, gw] = [v, w] \text{ for all } v, w \in \mathcal{V}_i\}$.

Proposition.

(1) *If $\mathbf{G}$ is not of type A_n, Ad' is absolutely irreducible and $Ad_i'(H)$ is a maximal connected closed subgroup of $Sp(\mathcal{V}_i(\mathbb{R}))$.*

(2) *For A_n, $Ad' : \mathbf{H} \to Sp(\mathcal{V}_i)$ is equivalent to $\rho \oplus \rho^*$. Let $P = \{g \in Sp(\mathcal{V}_i) \mid g\mathcal{W}_i = \mathcal{W}_i\}$ and $P^* = \{g \in Sp(\mathcal{V}_i) \mid g\mathcal{W}_i^* = \mathcal{W}_i^*\}$ be the parabolic subgroups of $Sp(\mathcal{V}_i)$ which stabilize $\mathcal{W}_i$ and $\mathcal{W}_i^*$, respectively. Then $\mathbf{H}$ is the semisimple part of the common Levi subgroup $P \cap P^*$. Hence $Ad_i'(H)$ is the maximal connected semisimple Lie subgroup of $(P \cap P^*)(\mathbb{R})^0$.*

Proof. Part (1) can be shown by the same argument as the proof of the previous proposition. To see (2), we present an explicit description for P and P^* by realizing G as an

$\mathbb{R}$-form of $SL_{n+1}(\mathbb{C})$. We may assume, up to conjugation, that $\mathbf{G}$ and $\mathbf{H}$ are the real forms of

$$SL_{n+1}(\mathbb{C}) \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & SL_{n-1}(\mathbb{C}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively .

It follows that $Sp(\mathcal{V}_i) = Sp_{2n}(\mathbb{C}) = \{g \in SL_{2n}(\mathbb{C}) \mid {}^t gJg = J\}$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $Ad'_i \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$ for all $A \in \mathbf{H}$.

Then $P = \{\begin{pmatrix} A & C \\ 0 & {}^tA^{-1} \end{pmatrix} \mid A \in GL_n(\mathbb{C})\}$ and $P^* = \{\begin{pmatrix} A & 0 \\ C & {}^tA^{-1} \end{pmatrix} \mid A \in GL_n(\mathbb{C})\}$. Therefore $\mathbf{H}$ is the commutator subgroup of the Levi subgroup $P \cap P^*$. Since P is a maximal parabolic subgroup of $Sp(\mathcal{V}_i)$, $Ad'_i(\mathbf{H})$ is a maximal semisimple connected algebraic subgroup of $Sp(\mathcal{V}_i)$. Hence $Ad'_i(H)$ is a maximal semisimple connected closed subgroup of $Sp(\mathcal{V}_i(\mathbb{R}))$. □

1.3.8. Lemma. *Let H be a connected simple Lie group and for each $i = 1, 2$ $\phi_i : H \to G_i$ an isomorphism for some Lie group G_i. Then $\delta(H) = \{(\phi_1(h), \phi_2(h)) \mid h \in H\}$ is a maximal closed connected subgroup of $G_1 \times G_2$.*

Proof. If L is a closed connected subgroup which contains $\delta(H)$ properly, then there exists an element $(g, e) \in L$. It follows that L contains the subgroup $\{(\phi_1(h)g\phi_1(h^{-1}), e) \mid h \in H\}$. Since H is simple hence so is G_1, $\{(\phi_1(h)g\phi_1(h^{-1}), e) \mid h \in H\} = G_1 \times \{e\}$. In the same way, we can show that L contains $\{e\} \times G_2$; hence $L = G_1 \times G_2$. □

1.4. $\mathbb{Q}$-forms of algebraic groups and $\mathbb{Q}$-rational representations

We prove some basic propositions on the $\mathbb{Q}$-forms of semisimple groups and $\mathbb{Q}$-rational representations. The main reference for this section is [21] and [22].

1.4.1. (see [21, Ch 2]) Let G be a semisimple algebraic k-group, $\Gamma = Gal(K/k)$ where K is the separable closure of k, and S, T, Δ and ${}_k\Delta$ be as in 1.2.1. Denote by Δ_0 the subset of Δ which vanish on S. We define the so-called $*$-action of Γ on Δ. For $\gamma \in \Gamma$, there exists a unique element w in the Weyl group such that $w({}^\gamma\Delta) = \Delta$ and we set $\gamma(\alpha) = w({}^\gamma\alpha)$. If the $*$-action is trivial, the k-form is called *inner* and otherwise, *outer*. The orbits of Γ, whose elements do not belong to Δ_0, are called *distinguished orbits*.

The Tits index of a group G is the data consisting of Δ, together with Dynkin diagram, Δ_0, and the $*$-action of Γ. The group G is determined, up to k-isomrophsim, by its K-isomorphism class, the Tits index, the commutator subgroup of $C(S)$, called the *semisimple anisotropic kernel* of G, given up to k-isogeny.

Proposition. [21, 2.5.4] *Let P_Θ denote the parabolic subgroup generated by T and $U_\alpha(\alpha \in \Delta)$ and $U_{-\alpha}(\alpha \in \Theta)$. Then P_Θ is defined over k if and only if Θ contains Δ_0 and is invariant under the $*$-action of Γ.*

1.4.2. For an adjoint semisimple k-group G, there exists an adjoint semisimple k-split group G^d and an adjoint semisimple k-quasi-split group G^q whose $*$-action of Γ on the Dynkin diagram is the one given by the index of G, both of which are K-isomorphic to G. The k-form of G is obtained by twisting G^d (resp. G^q) by a cocycle c of Γ with values in $Aut_K(G^d)$ (resp. $Int_K(G^q)$) (see [21, 3.4.2]).

The proof of the following proposition is due to G. Prasad.

Proposition. *Let G be an adjoint semisimple algebraic $\mathbb{R}$-group. Then there exists a $\mathbb{Q}$-form on G with respect to which every parabolic $\mathbb{R}$-subgroup of G is defined over $\mathbb{Q}$.*

Proof. The group G decomposes into a direct product of adjoint $\mathbb{R}$-simple normal $\mathbb{R}$-groups by Proposition 1.2.4 and a parabolic $\mathbb{R}$-subgroup of G is a product of parabolic $\mathbb{R}$-subgroups of each $\mathbb{R}$-simple factor of G. Therefore it is enough to prove the proposition for an $\mathbb{R}$-simple group.

Case (1): G is absolutely simple.

Let G^q be the adjoint $\mathbb{Q}$-split $\mathbb{Q}$-group if the $\mathbb{R}$-form of G is of inner type, and the quasi-split $\mathbb{Q}$-form of G, splitting over $k = \mathbb{Q}(i)$, if the $\mathbb{R}$-form of G is of outer type. Let $P = MU$ be a minimal parabolic $\mathbb{R}$-subgroup of G and $P^q = M^q U^q$ be the corresponding parabolic $\mathbb{Q}$-subgroup of G^q, where M^q is a Levi $\mathbb{Q}$-subgroup of P^q and U^q is the unipotent radical. Set $M' = [M, M]$ and $M^{q\prime} = [M^{q\prime}, M^{q\prime}]$. Since the two $\mathbb{R}$-forms of G and G^q differ only by their semisimple anisotropic kernels M' and $M^{q\prime}$ and the indices of M' and $M^{q\prime}$ coincide, the $\mathbb{R}$-form of G is the twist of the $\mathbb{R}$-form of G^q by a cocycle c of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ with values in $Int_{\mathbb{C}} M^q = M^{q\prime}$ (see [21, 3.4.2]). But it is known (see [3, Theorem 1.7]) that the natural homomorphism of $H^1(\mathbb{Q}, M^{q\prime})$ to $H^1(\mathbb{R}, M^{q\prime})$ is surjective. Thus there is an $M^{q\prime}$-valued cocycle d on $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ cohomologous to c over $\mathbb{Q}$. The twist of G^q by the cocycle d is a $\mathbb{Q}$-form of G with the following properties: (i) over $\mathbb{R}$, it coincides with the $\mathbb{R}$-form of G. (ii) its distinguished orbits of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ contains all the distinguished orbits of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ of the $\mathbb{R}$-form of G. Therefore by Proposition 1.4.1, this $\mathbb{Q}$-form has the required property.

Case (2): G is $\mathbb{R}$-simple but not absolutely simple.

There exists an adjoint absolutely simple group G' such that $G = R_{\mathbb{C}/\mathbb{R}} G'$ by Proposition 1.2.6. Let ${G'}^d$ be the adjoint $\mathbb{Q}(i)$-split form of G' and $G^d = R_{\mathbb{Q}(i)/\mathbb{Q}} {G'}^d$. It is clear that G^d is a $\mathbb{Q}$-form of G with the desired property. □

1.4.3. Let D be a central simple division algebra over k of degree d. For any r, there exists a k-form of $GL_{r,D}$ of GL_{rd} such that $GL_{r,D}(k) = GL_r(D)$. Let V be a finite dimensional (left) vector space over D. A D-representation or (D,k)-representation of G in V is a k- homomorphism $G \to GL_{V,D}$. A representation of G is called D-rational if it is equivalent to a D-representation.

1.4.4. Proposition. [22, Theorem 3.3] *For any dominant weight λ fixed by Γ, there exists a division algebra D, unique up to k-isomorphism, such that an absolutely irreducible representation of G with the highest weight λ is D-rational.*

1.4.5. Proposition. *Let G be an absolutely simple algebraic k-group of type C_n. If the standard $2n$ -dimensional representation of G is k-rational, then G is split over k.*

Proof. By the classification of k-forms given in [21], $G(k)$ is $SU_{2n/d}(D,h)$ where D is a division algebra of degree d and h is a nondegenerate antihermitian sesquilinear form of index r relative to a first kind involution σ such that D^σ has dimension $1/2d(d+1)$. The canonical representation $SU_{2n/d}(D,h) \to SL_{2n,D}$ is an absolutely irreducible representation with the same highest weight as the standard $2n$-dimensional representation. Since the standard representation is rational over k, by the uniqueness of such a D, $D = k$. It implies that G split over k. □

1.4.6. Let α be a dominant weight and k_α the invariant field by the stabilizer of α in Γ. Then Proposition 1.4.4 gives a central division algebra D_α over k_α such that $\rho_\alpha : G \to GL_{n,D_\alpha}$ is an absolutely irreducible (D_α, k_α)-representation of dominant weight α. Define $rest_{D_\alpha/k_\alpha} : GL_{n,D_\alpha} \to GL_{nd^2}(K) = GL_n(D_\alpha \otimes_{k_\alpha} K)$ and similarly $rest_{k_\alpha/k}$. We put ${}^k\rho_\alpha = rest_{k_\alpha/k}(rest_{D_\alpha/k} \circ \rho_\alpha)$.

Theorem. [22, Theorem 7.2, Lemma 7.4]

(1) *For each dominant weight of G, ${}^k\rho_\alpha$ is irreducible over k.*
(2) *Each k-irreducible representation of G is k-equivalent to the representation of the form ${}^k\rho_\alpha$.*
(3) *Let α and α' be dominant weight . Then ${}^k\rho_\alpha$ and ${}^k\rho_{\alpha'}$ are k-equivalent iff there exists $\gamma \in \Gamma$ such that $\gamma(\alpha) = \alpha'$.*
(4) *If $\Gamma(\alpha) = \{\alpha_1, ..., \alpha_n\}$, then ${}^k\rho_\alpha$ is equivalent over K to the direct sum of d irreducible representations of dominant weight α_1,..., d irreducible representations of dominant weight α_n where $d^2 = [D_\alpha : k_\alpha]$.*

1.4.7. Proposition. *Let $G = SL_n(\mathbb{C})$ and $\psi : G \to SL_{2n}(\mathbb{C})$ the direct sum $\rho \oplus \rho^*$ where ρ is the standard representation of G and ρ^* the dual of ρ, i.e., $\rho^*(A) = {}^tA^{-1}$*

for $A \in G$. Then the $\mathbb{Q}$-forms of G with respect to which ψ is $\mathbb{Q}$-rational are, up to conjugation, the followings:

(1) *if ψ is reducible over $\mathbb{Q}$, $G(\mathbb{Q}) = SL_n(\mathbb{Q})$;*
(2) *otherwise $G(\mathbb{Q}) = SU(h)(\mathbb{Q}) = \{g \in SL_{n-1}(k) \mid {}^t g^\sigma h g = h\}$ or $G(\mathbb{Q}) = {}^t(SU(h)(\mathbb{Q}))^{-1} = \{g \in SL_{n-1}(k) \mid {}^t g^{-1} \in SU(h)(\mathbb{Q})\}$ where k is a quadratic extension field of $\mathbb{Q}$, σ is a non-trivial element in $Gal(k/\mathbb{Q})$ and $h \in GL_{n-1}(k)$ such that ${}^t h^\sigma = h$.*

Proof. If ψ is reducible over $\mathbb{Q}$, then each ρ and ρ^* is rational over $\mathbb{Q}$, proving (1). Assume that ψ is irreducible over $\mathbb{Q}$. Let T be the group of diagonal matrices and $\lambda_i : T \to \mathbb{C}$ such that $\lambda_i(\mathrm{diag}(a_1, ..., a_n)) = a_i$ for all $i = 1, .., n$. We may assume that for a suitable ordering, the highest weights of ρ and ρ^* are λ_1 and λ_n respectively.

Since ψ is rational over $\mathbb{Q}$, $\Gamma(\lambda_1, \lambda_n) = \{\lambda_1, \lambda_n\}$, and since ψ is irreducible over $\mathbb{Q}$, $\gamma(\lambda_1) = \lambda_n$ for some $\gamma \in \Gamma$. It follows that ψ is ${}^{\mathbb{Q}}\rho_{\lambda_1}$, $D_{\lambda_1} = \mathbb{Q}_{\lambda_1}$ and the $\mathbb{Q}$-form must be outer. Therefore by the classification given in [21], the $\mathbb{Q}$-form of G is a special unitary group $SU_{n/d}(D, h)$ or ${}^t SU_{n/d}(D, h)^{-1}$ where D is a central simple division algebra of degree d over a quadratic extension k of $\mathbb{Q}$ with an involution of the second kind σ such that $\mathbb{Q} = \{x \in k \mid x^\sigma = x\}$ and h is a nondegenerate hermitian form of index r relative to σ. It implies that $\mathbb{Q}_{\lambda_1} = k$. Since $SU_{n/d}(D, h) \to SL_{n/d} D$ is an absolutely irreducible (D, k)-representation with the same highest weight as ρ, we have $D = k$ by the uniqueness of such a D, proving the proposition. □

1.5. Extension of $\mathbb{Q}$-forms

We show that for an adjoint absolutely simple group G and its parabolic subgroup P such that $R_u(P)$ is commutative or Heisenberg, a $\mathbb{Q}$-form of $[P, P]$ extends to a $\mathbb{Q}$-form of G. The following theorem of Raghunathan plays a key role.

1.5.1. Theorem. *(Raghunathan,* [18, 3.31]*) Let $\mathcal{G}$ be a complex semisimple Lie algebra and $\mathcal{P}$ the Lie algebra corresponding to a parabolic subgroup P of G (= a semisimple Lie group with $\mathfrak{g}$ its Lie algebra). Let $\mathcal{P}_{\mathbb{Q}}$ be a Lie algebra over $\mathbb{Q}$ and $i : \mathcal{P}_{\mathbb{Q}} \to \mathcal{P}$ an injective homomorphism such that $i \otimes id : \mathcal{P}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathcal{P}$ is an isomorphism. Then there exists a $\mathbb{Q}$-Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ and injective homomorphisms $j : \mathfrak{g}_{\mathbb{Q}} \to \mathcal{G}$ and $\alpha : \mathcal{P}_{\mathbb{Q}} \to \mathfrak{g}_{\mathbb{Q}}$ such that*

(1) *$j \otimes id : \mathfrak{g}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \to \mathcal{G}$ is an isomorphism and*
(2) *$j \circ \alpha = \beta \circ i$ where $\beta : \mathcal{P} \to \mathcal{G}$ is the natural inclusion.*

1.5.2. Considering the adjoint representation of G, the following is a direct consequence of the above theorem.

Corollary. *Let G be an adjoint semisimple algebraic $\mathbb{R}$-group and P its parabolic $\mathbb{R}$-subgroup. If P is equipped with a $\mathbb{Q}$-form, then there exists a $\mathbb{Q}$-form of G which extends the $\mathbb{Q}$-form of P.*

1.5.3. It is well known that for every unipotent subgroup U, the logarithmic mapping $\ln U \to \mathrm{Lie}(U)$ is equivariant in the following sense: if f is a biregular automorphism of U, then $\ln \circ f = df \circ \ln$ (see [11, 1.1.3]).

In particular, we have the following:

Lemma. *Let G be a semsimple algebraic group and P a parabolic subgroup with a Levi subgroup L. Then $Ad(h) \circ ln = ln \circ Int(h)$ where $Ad : L \to GL(\mathcal{U})$ is the representation as in* 1.1.14.

1.5.4. Lemma. *Let G be an adjoint semisimple algebraic group and $P, L, \mathcal{U}$ be as in the above lemma. Then the representation $Ad : L \to GL(\mathcal{U})$ is faithful*

Proof. Applying the previous lemma to the conjugation map of U by each element of L, we can see that the kernel of $Ad : L \to GL(\mathcal{U})$ is equal to $C(U) \cap L$. But it is known (e.g. [10, Lemma 11.16] or [16, Proposition 11.19]) that $C(U) \subset U$. Therefore $C(U) \cap L$ is trivial, proving the lemma. □

1.5.5. Proposition. *Let G be an adjoint absolutely simple algebraic $\mathbb{R}$-group, P a parabolic $\mathbb{R}$-subgroup with an $\mathbb{R}$-Levi decomposition $LR_u(P)$ and $H = [L, L]$. Assume that H and $R_u(P)$ have $\mathbb{Q}$-forms such that $H(\mathbb{Q})$ normalizes $R_u(P)(\mathbb{Q})$. If $R_u(P)$ is commutative or Heisenberg and H is non-trivial, there exists a $\mathbb{Q}$-form of G which extends the $\mathbb{Q}$-forms of H and $R_u(P)$.*

Proof. By Corollary 1.5.2, it is enough to extend a $\mathbb{Q}$-forms of H and $R_u(P)$ to a $\mathbb{Q}$-form of P. Consider the representation $Ad : L \to GL(\mathcal{U})$, $\mathcal{U} = \mathrm{Lie}(R_u(P))$ defined in 1.1.12. The $\mathbb{Q}$-form on $R_u(P)$ defines a $\mathbb{Q}$-structure on the vector space $\mathcal{U}$, through the logarithm map, and hence a $\mathbb{Q}$-form on $GL(\mathcal{U})$. Since $H(\mathbb{Q})$ normalizes $R_u(P)(\mathbb{Q})$ and the logarithm map is $\mathbb{Q}$-rational, $Ad(H(\mathbb{Q}))$ preserves $\mathcal{U}(\mathbb{Q})$. Since H is defined over $\mathbb{Q}$, and hence $H(\mathbb{Q})$ is Zariski dense in H, it implies that $Ad(H)$ is a $\mathbb{Q}$-subgroup of $GL(\mathcal{U})$.

Case (1): Ad is absolutely irreducible, or equivalently, $R_u(P)$ is commutative.

Since Ad is absolutely irreducible, it follows from Schur's lemma that the centralizer $C(Ad(H))$ of $Ad(H)$ coincides with $Z(GL(\mathcal{U}))$. Since $Ad(Z(L)) \subset C(Ad(H))$ and $Ad(Z(L))$ has dimension 1, $Ad(Z(L)) = Z(GL(\mathcal{U}))$. It follows that $Ad(Z(L))$ is a $\mathbb{Q}$-subgroup of $GL(\mathcal{U})$. Since $Ad(H) \cap GL(\mathcal{U}(\mathbb{Q}))$ and $Z(L) \cap GL(\mathcal{U}(\mathbb{Q}))$ are Zariski dense in $Ad(H)$ and $Z(L)$, respectively, $Ad(H)Ad(Z(L)) \cap GL(\mathcal{U}(\mathbb{Q}))$ is Zariski dense in $Ad(H)Ad(Z(L))$. Since $L = HZ(L)$, we obtain that $Ad(L)$ is a $\mathbb{Q}$-subgroup of $GL(\mathcal{U})$.

Since Ad is faithful, $(Ad, Ad(L))$ now provides a $\mathbb{Q}$-form of L, which extends the given $\mathbb{Q}$-form of H. Since $L(\mathbb{Q})$ normalizes $R_u(\mathbb{Q})$ with respect to this $\mathbb{Q}$-form, we have a $\mathbb{Q}$-form of P.

Case (2): $R_u(P)$ is Heisenberg and G is not of type A_n.

Then $\mathcal{U}$ decomposes into $\mathcal{V} \oplus Z(\mathcal{U})$ where $\mathcal{V}$ is the $Ad(H)$-invariant subspace other than $Z(\mathcal{U})$. Since $Z(\mathcal{U})$ has dimension one and H has no rational characters, $Ad(H)$ acts trivially on $Z(\mathcal{U})$. Denote by Ad' the restriction of Ad on $\mathcal{V}$ so that $Ad = Ad' \oplus Id$. We note that $\mathcal{V}$ is a $\mathbb{Q}$-subspace of $\mathcal{U}$; In fact, let $(v, z) \in \mathcal{U}(\mathbb{Q})$ where $v \in \mathcal{V}$ and $z \in Z(\mathcal{U})$, and $h \in H(\mathbb{Q})$. Then $Ad(h)(v, z) = (Ad'(h)v, z) \in \mathcal{U}(\mathbb{Q})$ and hence $(Ad'(h)v - v, 0) \in \mathcal{U}(\mathbb{Q})$, giving a non-trival element in $\mathcal{V} \cap \mathcal{U}(\mathbb{Q})$. It follows from the fact that Ad' acts absolutely irreducibly on $\mathcal{V}$ that $Ad(H)(\mathbb{Q})(\mathcal{V} \cap \mathcal{U}(\mathbb{Q}))$ generates a Zariski dense subspace of $\mathcal{V}$, proving the claim.

Since the restriction of $Ad(H)$ on $\mathcal{V}$ is absolutely irreducible, by the same argument as the previous case, we can extend the $\mathbb{Q}$-form of H to L with respect to which $Ad(L(\mathbb{Q}))$ preserves $\mathcal{V}(\mathbb{Q})$. Since $Z(\mathcal{U}(\mathbb{Q})) = [\mathcal{V}(\mathbb{Q}), \mathcal{V}(\mathbb{Q})]$, $Ad(L(\mathbb{Q}))$ also preserves $Z(\mathcal{U}(\mathbb{Q}))$. It follows that $L(\mathbb{Q})$ normalizes $R_u(P)(\mathbb{Q})$, yielding the desired $\mathbb{Q}$-form of P.

Case (3): $R_u(P)$ is Heisenberg in a group of type A_n, $n \geq 3$.

In this case, H is of type A_{n-2} and Ad is equivalent to the direct sum $\rho \oplus \rho^* \oplus id$ where ρ (resp. ρ^*) denotes the (resp. dual of) standard representation of A_{n-2}. By Proposition 1.4.7, the $\mathbb{Q}$-forms of H with respect to which Ad is $\mathbb{Q}$-rational are, up to conjugation and isogeny, such that $H(\mathbb{Q}) = SL_{n-1}(\mathbb{Q})$, $SU(h)(\mathbb{Q}) = \{g \in SL_{n-1}(k) \mid {}^t g^\sigma h g = h\}$ or ${}^t(SU(h)(\mathbb{Q}))^{-1} = \{g \in SL_{n-1}(k) \mid {}^t g^{-1} \in SU(h)(\mathbb{Q})\}$ where k is a real quadratic extension field of $\mathbb{Q}$, σ is a non-trivial element in $Gal(k/\mathbb{Q})$ and $h \in GL_{n-1}(k)$ such that ${}^t h^\sigma = h$. We observe that each of those $\mathbb{Q}$-forms of H extends to a $\mathbb{Q}$-form of G with respect to which $G(\mathbb{Q})$ is isogenous to $SL_{n+1}(\mathbb{Q})$, $SU(h')(\mathbb{Q}) = \{g \in SL_{n+1}(k) \mid {}^t g^{-1} h' g = h'\}$ or ${}^t(SU(h')(\mathbb{Q}))^{-1}$ for $h' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h & 0 \\ 1 & 0 & 0 \end{pmatrix}$ respectively. $\square$

1.5.6 Remark. If the rank of G is one, H is trivial. For the groups with rank at least 2, H is trivial only if G is of type A_2 and P is a Borel subgroup, or equivalently, $R_u(P)$ is Heisenberg.

2. THE SUBGROUPS OF THE FORM Γ_{F_1,F_2} and $\mathbb{Q}$-FORMS

2.1. Discrete subgroups in algebraic groups

2.1.1. The following two theorems are the well known Borel density theorem and a theorem of Borel and Harish-Chandra, respectively.

Theorem. *(Borel density theorem,* [2]*) Any lattice in a connected semisimple Lie group without compact factors is Zariski dense.*

2.1.2. Theorem. [4] *Let G be a semisimple linear algebraic group defined over $\mathbb{Q}$. Then $G(\mathbb{Z})$ is a lattice in $G(\mathbb{R})$.*

2.1.3. Lemma. *Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{R}$-group, $G = \mathbf{G}(\mathbb{R})^0$ and Γ a discrete and Zariski dense subgroup in G. Then the normalizer $N(\Gamma)$ of Γ in G is discrete.*

Proof. Let N^0 be the connected component of the identity of $N(\Gamma)$. Since N^0 normalizes Γ and Γ is discrete, N^0 centralizes Γ. That is, $N^0 \subset C(\Gamma)$ and therefore $\Gamma \subset C(N^0)$. Since $C(N^0)$ is algebraic and Γ is Zariski dense, $G \subset C(N^0)$ and hence N^0 is contained in the center of G, which is finite. Thus $N^0 = \{e\}$, implying that $N(\Gamma)$ is discrete. □

2.1.4. Lemma.

(1) *If G is a locally compact group and Γ is a discrete subgroup of G containing a lattice in G, then Γ is a lattice in G.*
(2) *If $\mathbf{G}$ is a semisimple algebraic $\mathbb{Q}$-group, $G = \mathbf{G}(\mathbb{R})^0$ and Γ is a discrete subgroup containing an arithmetic subgroup $G(\mathbb{Z})$, then Γ is also an arithmetic subgroup of G.*
(3) *Let G be as in (2) and in addition, suppose that G has no compact factors. Then the normalizer of an arithmetic subgroup of G is again an arithmetic subgroup of G.*

Proof. (1) is an immediate corollary of Lemma 1.6 of [16]. By Theorem 2.1.2, $G(\mathbb{Z})$ is a lattice in G. Again by Lemma 1.6 of [16], the subgroup $G(\mathbb{Z})$ is a subgroup of a finite index in Γ, proving (2). (3) is a direct consequence of Borel density theorem and Lemma 2.1.3. □

2.1.5. We state a well known arithmeticity theorem of Margulis for a real field case, followed by his finiteness theorem.

Theorem. *(Margulis' arithmeticity theorem, see* [13] *or* [27, Theorem 6.1.2]*) Let G be a connected semisimple Lie group with trivial center and no compact factors. Suppose that the rank of G is at least 2 and that Γ is an irreducible lattice in G. Then there exists a semisimple algebraic $\mathbb{Q}$-group H and an epimorphism $p : H(\mathbb{R})^0 \to G$ with compact kernel such that $p(H(\mathbb{Z}) \cap H(\mathbb{R})^0)$ is commensurable to Γ. Furthermore if Γ is a non-uniform lattice, p can be taken as an isomorphism.*

2.1.6. Theorem. *(Margulis' finitenss theorem, see* [11, Ch 8] *or* [27, Theorem 8.1.2]*) Let G and Γ be as in Theorem* 2.1.5. *Then any non-central normal subgroup of Γ has a finite index in Γ.*

2.1.7. Let G be a locally compact group and S_n a sequence of subsets of G. We say that S_n converges S if for every compact subset $K \subset G$ and a neighborhood U of e in G, there exists an integer $r = r(K, U)$ such that for all $n \geqq r$ and $x \in S_n \cap K$, $xU \cap S \neq \emptyset$ and for all $y \in S \cap K$, $yU \cap S_n \neq \emptyset$.

Theorem. *(Chabauty)* [16, Theorem 1.20] *Let G be a Lie group and Γ_n a sequence of lattices in G such that for some open set W of G with $e \in W$, $W \cap \Gamma_n = \{e\}$ for all n. Then a subsequence Γ_{i_n} of Γ_n converges to Γ and Γ is a discrete subgroup. Furthermore, if μ is a right Haar measure on G, $\mu(G/\Gamma) \leqq Lim\ inf\ \mu(G/\Gamma_{i_n})$.*

2.1.8. A subgroup H of a Lie group G is said to have property (P) if every $Ad(H)$-stable subspace of $\mathfrak{g}_{\mathbb{C}}$ is $Ad(G)$-stable where Ad denotes the adjoint representation of G in the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$.

Theorem. *(Wang)* [16, Lemma 9.5] *Let G be a semisimple Lie group and K a compact set of G. Then there is a neighborhood U of e in G such that the following holds: if Γ is any discrete subgroup of G such that $\Gamma \cap U$ generates a subgroup with property (P), then $\Gamma \cap U = \{e\}$.*

For example, a Zariski dense subgroup in an algebraic group has property (P) since the adjoint representation is algebraic.

2.1.9. In the following, we list well known lemmas on lattices on unipotent algebraic groups (see [12, Ch 3]).

Lemma. *Let U be a unipotent algebraic $\mathbb{R}$-group and F a discrete subgroup of $U(\mathbb{R})$. Then F is Zariski dense in U if and only if the factor space $U(\mathbb{R})/F$ is compact.*

2.1.10. Lemma. *Assume further that F is a lattice in $U(\mathbb{R})$. Then*

(1) *$U(\mathbb{R})/F$ is compact.*
(2) *F is an arithmetic subgroup of $U(\mathbb{R})$.*
(3) *$F \cap Z(U(\mathbb{R}))$ is a lattice in $Z(U(\mathbb{R}))$.*

2.1.11. Lemma. *Let U be a unipotent $\mathbb{Q}$-group.*

(1) *For any $n \in \mathbb{N}$, $U(n\mathbb{Z})$ is a lattice in $U(\mathbb{R})$;*
(2) *if $F \subset U(\mathbb{Q})$ and F is a lattice in $U(\mathbb{R})$, then F is commensurable to $U(\mathbb{Z})$;*
(3) *every subgroup of finite index in $U(\mathbb{Z})$ contains $U(n\mathbb{Z})$ for some $n \in \mathbb{N}$.*

2.2. Generators of arithmetic subgroups

2.2.1. Let k be a number field and G a simply connected and absolutely almost simple algebraic group defined over k with respect to which U_1 and U_2 are opposite horospherical k-subgroups. Let S be a fine set of valuations of k containing all archimedean valuations and Λ be the ring of S-integers in k and the S-rank of G is at least 2. If U_1 and NU_2 are maximal horospherical k-subgroups, then for any ideal A of Λ, the subgroup generated by $U_1(A)$ and $U_2(A)$ is of fine index in $G(\Lambda)$ (for k-rank $G \geq 2$ by Raghunathan [17] and for k-rank $G = 1$ by Venkataramana [25]). This result was known for Chevalley groups for k-rank $G \geqq 2$ by Tits [23] and for arbitrary groups of k-rank $G \geqq 2$ by Vaserstein [24].

In fact, this result holds for arbitrary pair of opposite horospherical k-subgroups. The argument for this was explained to the author by T. N. Venkataramana.

2.2.2. Corollary. *Let G be as above and U_1, U_2 a pair of opposite horospherical k-subgroups. Then for any ideal A of Λ, the subgroup generated by $U_1(A)$ and $U_2(A)$ is of finte index in $G(\Lambda)$.*

Proof. Let $P_1' \subset N(U_1)$ and $P_2' \subset N(U_2)$ be a pair of opposite minimal parabolic k-subgroups. This can be done as follows: if the pair $N(U_1)$, $N(U_2)$ is conjugate to ${}_kP_\Theta$, ${}_kP_\Theta^-$ by $g \in G(k)$ as in Lemma 1.2.2, set $P_i' = gP_\emptyset g^{-1}$ for each $i = 1, 2$. Set $U_i' = R_u(P_i')$ for each $i = 1, 2$ and $L = N(U_1) \cap N(U_2)$. Then U_i' is the semi-direct product of $(L \cap U_i')$ and U_i. Denote by Γ_A the subgroup generated by $U_1(A)$ and $U_2(A)$, and by Γ_B' the subgroup generated by $U_1'(B)$ and $U_2'(B)$. By the result mentioned above for maximal horospherical k-subgroups, Γ_B' is an arithmetic subgroup. We note that $(L \cap U_i')(A)U_i(A)$ has a finite index in $U_i'(A)$. Therefore there exists an ideal $B \subset A$ such that $U_i'(B) \subset (L \cap U_i')(A)U_i(A)$. We claim that Γ_B' normalizes Γ_A. Let $x \in U_1'(B)$ and write it as ly where $l \in (L \cap U_1')(A)$, $y \in U_1(A)$. For $u_1 \in U_1(A)$, $yu_1y^{-1} \in U_1(A)$ and hence $lyu_1l^{-1}y^{-1} \in U_1(A)$ since $L(A)$ normalizes $U_1(A)$. So, $xU_1(A)x^{-1} = U_1(A)$. For $u_2 \in U_2(A)$, since $xu_2x^{-1} = (lyl^{-1})(lu_2l^{-1})(lyl^{-1})^{-1}$, $lyl^{-1} \in U_1(A)$ and $lu_2l^{-1} \in U_2(A)$, we have $xu_2x^{-1} \in \Gamma_A$. It shows that $U_1'(B)$ normalizes Γ_A and similarly we can show that $U_2'(B)$ normalizes Γ_A. Therefore $\Gamma_A \subset N(\Gamma_B')$.

By the finiteness theorem of Margulis (Theorem 2.1.6), Γ_A has a finite index in $N(\Gamma_B')$ and hence an arithmetic subgroup. It is now clear that Γ_A has a finite index in $G(A)$. □

2.2.3. Corollary. *Let G be a connected semisimple $\mathbb{Q}$-group such that each $\mathbb{Q}$-simple factor has $\mathbb{Q}$-rank at least 1 and $\mathbb{R}$-rank at least 2. Let U_1, U_2 be a pair of opposite horospherical $\mathbb{Q}$-subgroups and F_1, F_2 lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ which are commen-*

surable to $U_1(\mathbb{Z})$ and $U_2(\mathbb{Z})$ respectively. If Γ_{F_1,F_2} is discrete, then it is an arithmetic subgroup of $G(\mathbb{R})$.

Proof. Suppose that G is simply connected. Using Proposition 1.2.4, we may assume that G is almost $\mathbb{Q}$-simple. By Proposition 1.2.6, there exist a finite separable extension k of $\mathbb{Q}$, a connected simply connected absolutely almost simple k-group G' and a pair of opposite horospherical k-subgroups U_1', U_2' of G' such that $G = R_{k/\mathbb{Q}}G'$ and $U_i = R_{k/\mathbb{Q}}U_i'$ for each $i = 1, 2$. We also note that $\mathbb{R}$-rank of G is equal to the S-rank of G' where S is the set of archimedean valuations of k. In fact, if k has r real embeddings and $2s$ imaginary embeddings, then S has $r + s$ elements. Both the $\mathbb{R}$-rank of G and the S-rank of G' are equal to $(r + s)$(k-rank of (G')). Since there exists $n \in \mathbb{N}$ such that $U_i(n\mathbb{Z}) \subset F_i$ by Lemma 2.1.11, we may assume that $F_i = U_i(n\mathbb{Z})$ for each $i = 1, 2$. On the other hand, we can find an ideal A of Λ, the ring of the integers of k, such that $R_{k/\mathbb{Q}}(U_i'(A)) \subset U_i(n\mathbb{Z})$. By the previous corollary, the subgroup of G' generated by $U_1'(A)$ and $U_2'(A)$ has a finite index in $G'(\Lambda)$. Since Γ_{F_1,F_2} contains a subgroup generated by $R_{k/\mathbb{Q}}(U_1'(A))$ and $R_{k/\mathbb{Q}}(U_2'(A))$ and $G(\mathbb{Z}) = R_{k/\mathbb{Q}}(G'(\Lambda))$, Γ_{F_1,F_2} contains a subgroup of finite index in $G(\mathbb{Z})$.

In general, there exists a simply connected semisimple $\mathbb{Q}$-group $\tilde{G}$ and a central $\mathbb{Q}$-isogeny $p : \tilde{G} \to G$ by Proposition 1.2.5. Then each $\tilde{U}_i = p^{-1}(U_i)$ is a horospherical $\mathbb{Q}$-subgroup of $\tilde{G}$ and there exists $m \in \mathbb{N}$ such that $p(\tilde{U}_i(m\mathbb{Z})) \subset U_i(n\mathbb{Z})$ for each $i = 1, 2$. Since $p(\Gamma_{\tilde{U_1}(m\mathbb{Z}),\tilde{U_2}(m\mathbb{Z})}) \subset \Gamma_{U_1(n\mathbb{Z}),U_2(n\mathbb{Z})}$ and the image of an arithmetic subgroup under an isogeny map is an arithmetic subgroup, it only remains to apply the previous case to $\Gamma_{\tilde{U_1}(m\mathbb{Z}),\tilde{U_2}(m\mathbb{Z})}$ to obtain that $\Gamma_{U_1(n\mathbb{Z}),U_2(n\mathbb{Z})}$ is an arithmetic subgroup. $\square$

2.2.4. Example. We present an example which shows that the assumption on the discreteness of Γ_{F_1,F_2} in the above corollary is essential. Also see Remark in the introduction.

Let U_1 and U_2 be the subgroups of the form $\begin{pmatrix} I_m & M_{m\times k}(\mathbb{R}) \\ 0 & I_k \end{pmatrix}$ and $\begin{pmatrix} I_m & 0 \\ M_{k\times m}(\mathbb{R}) & I_k \end{pmatrix}$ for some $m, k \in \mathbb{N}$, respectively. By $M_{m\times k}(\mathbb{R})$, we mean the set of all $m \times k$ matrices whose coefficients lie in $\mathbb{R}$. In fact any pair of minimal opposite horospherical subgroups in $SL_n(\mathbb{R})$, $n = m + k$, is conjugate to such a pair U_1, U_2. Let $F_1 = M_{m\times k}\mathbb{Z}$ and $F_2 = (1/p)M_{k\times m}\mathbb{Z}$ for some $p \in \mathbb{N}$. Obviously F_2 is commensurable to $M_{k\times m}\mathbb{Z}$.

We claim that unless $p = 1$, then Γ_{F_1,F_2} is not discrete. The subgroup Γ_{F_1,F_2} contains a subgroup, which is isomorphic to the subgroup Γ_p of $SL_2(\mathbb{R})$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1/p & 1 \end{pmatrix}$. Applying the below Jorgenson's inequality to Γ_p, we obtain that if Γ_p is discrete, then $p \leqq 1$, yielding $p = 1$ since $p \in \mathbb{N}$.

Jorgenson's Inequality. *(cf. [7]) Let A and B be matrices in $SL_2(\mathbb{C})$ which generate an infinite subgroup. If the subgroup generated by A and B is discrete, then* $|tr^2(A) - 4| + |tr(ABA^{-1}B^{-1}) - 2| \geqq 1$.

2.2.5. Example. We close this section by giving two examples which demonstrate how the discretness assumption on Γ_{F_1,F_2} restricts the choice of lattices F_1 and F_2 in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$, respectively.

(1) Let $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ and $\Gamma_{x,y}$ the subgroup of $SL_2(\mathbb{R})$ generated by g and h. The Jorgesons's inequality says that if $\Gamma_{x,y}$ is discrete, then $x^2y^2 \geqq 1$.

(2) Let U_1 and U_2 be as in the remark 2.2.4. Let $F_1 = M_{m\times k}\mathbb{Z}$ and $F_2 = \alpha M_{k\times m}\mathbb{Z}$ for some $\alpha \in \mathbb{R}$.

We claim that if α is an irrational number, then Γ_{F_1,F_2} is not discrete. Let E_{ij} denote the elementary matrix whose entries are all 0 but 1 in (i,j)-entry. Note that the commutator $[I + \alpha E_{m+1,1}, I + E_{1,n}]$ of two elements $I + \alpha E_{m+1,1}$ and $I + E_{1,n}$ is the element $I + \alpha E_{m+1,n}$. Let's denote this element by g. Then

$$g \begin{pmatrix} I_m & M_{m\times k}\mathbb{Z} \\ 0 & I_k \end{pmatrix} g^{-1} = \begin{pmatrix} I_m & (M_{m\times k}\mathbb{Z})(I + \alpha E_{1,k}) \\ 0 & I_k \end{pmatrix}$$

Therefore $\Gamma \cap U_1$ is not discrete if α is an irrational number.

2.3. REFLEXIVE HOROSPHERICAL SUBGROUPS

2.3.1. Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{R}$-group without any $\mathbb{R}$-anisotropic factors and $G = \mathbf{G}(\mathbb{R})^0$. Let U_1, U_2 be a pair of opposite horospherical $\mathbb{R}$-subgroups and F_1 and F_2 lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively.

Lemma.

(1) *G is a normal subgroup of finite index in $G(\mathbb{R})$.*
(2) *$U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ generates G. In particular G is generated by one-parameter unipotent subgroups.*
(3) *The subgroup Γ_{F_1,F_2} is Zariski dense in $G(\mathbb{R})$.*

Proof. Part (1) is well known. Since $\mathbf{G}$ has no $\mathbb{R}$-anisotropic factors, (2) follows from [5, 6.14-15] (or see [11, Ch 1, Theorem 2.3.1]). Since the Zariski closure of Γ_{F_1,F_2} contains U_1 and U_2, (3) is a direct corollary of (2). □

2.3.2. Lemma. *Suppose that P_1 is a reflexive parabolic k-subgroup of a connected semsimple k-group G and that P_2 is parabolic k-subgroup opposite to P_1. Then the set*

$M(P_1, P_2) = \{h \in R_u(P_2) \mid hP_1h^{-1}$ *and* P_1 *are opposite*$\}$ *is Zariski dense and open in* $R_u(P_2)$. *In particular,* $M(P_1, P_2) \cap R_u(P_2)(k)$ *is non-empty.*

Proof. It follows from Proposition 4.10 and Lemma 4.12 in [5] that the set $M = \{h \in G \mid hP_1h^{-1}$ and P_1 are opposite$\}$ is Zariski dense and open in G. Since $R_u(P_2)P_1$ is open and M is right P_1-invariant, we have that $R_u(P_2) \cap M$ is open in $R_u(P_2)$. □

2.3.3. Lemma. *Suppose that* U_1 *is reflexive, i.e.,* $wU_1w^{-1} = U_2$ *for some* $w \in G(\mathbb{R})$. *If* U_1 *is commutative (resp. Heisenberg), there exist* $u \in U_1(\mathbb{R})$, $x \in (N(U_1) \cap N(U_2))(\mathbb{R})$ *(independent of* F_1 *and* F_2*) such that* $u\Gamma_{F_1,F_2}u^{-1}$ *contains* $\Gamma_{F_1,x(wF_1w^{-1})x^{-1}}$ *(resp.* $\Gamma_{F_1',xF_2'x^{-1}}$ *where* F_1' *and* F_2' *are lattices whose quotients by commutators are the same as those of* F_1 *and* wF_1w^{-1} *respectively).*

Proof. Since Γ_{F_1,F_2} is Zariski dense, it follows from Lemma 2.3.2 that there exists $\gamma \in \Gamma_{F_1,F_2}$ such that $\gamma N(U_1)\gamma^{-1}$ is opposite to $N(U_1)$. Then there exists a unique element $u \in U_1(\mathbb{R})$ such that $u\gamma N(U_1)\gamma^{-1}u^{-1} = N(U_2)$. Since $wN(U_1)w^{-1} = N(U_2)$, it follows that $u\gamma w^{-1}$ is in $N(U_2)$. By Levi decomposition we can write $u\gamma w^{-1} = xy$ for $x \in N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$ and $y \in U_2(\mathbb{R})$. Now $(u\gamma)F_1(u\gamma)^{-1} = (xy)wF_1w^{-1}(xy)^{-1}$. Set $F_1' = uF_1u^{-1}$ and $F_2' = ywF_1w^{-1}y^{-1}$. Then $u\Gamma u^{-1} = u\gamma\Gamma\gamma^{-1}u^{-1}$ contains the subgroup generated by $uF_1u^{-1} = F_1'$ and $(u\gamma)F_1(u\gamma)^{-1} = xF_2'x^{-1}$. If U_1 is commutative, then $F_1' = F_1$ and $xF_2'x^{-1} = xF_2x^{-1}$. If U_1 is Heisenberg, it is clear that F_1' and F_2' satisfy the desired properties. □

2.3.4. Remark. For an adjoint absolutely simple group G and its root system Φ with a choice of basis Δ, there exists a unique involutory permutation i, so-called *opposition involution* of Δ such that $\alpha \to -i(\alpha)$ extends to an operation of an Weyl element, say n. Then $nP_\Theta n^{-1} = P_{i(\Theta)}{}^{-}$ for each subset $\Theta \subset \Delta$. Therefore if $i(\Theta) = \Theta$, then P_Θ (resp. V_Θ) is conjugate to P_Θ^- (resp. V_Θ^-).

It is known that i induces a non-trivial automorphsim on Δ if and only if G is type of A_n, D_{2n+1} or E_6 (cf. [21, 1.5.1]). In such cases, we have non-reflexive commutative horospherical subgroups. On the other hand, it follows from the classification given in 1.3.1 that $i(\Delta_H) = \Delta_H$. Therefore every Heisenberg horospherical subgroup is reflexive.

2.4. Margulis' theorem on representation theory and extension of $\mathbb{Q}$-structures

2.4.1. Let K be an algebraically closed field, H a connected algebraic K-group, $\Lambda \subset H$ a Zariski dense subgroup, T a faithful linear representation of H into a finite dimensional linear space L over K, M and N linear subspaces of L, and W a non-empty Zariski-open subset of H. We put $d = \min_{h\in H}\dim(N\cap T(h)M)$ and assume that $\dim(M\cap T(w)N) = \dim(N\cap T(w^{-1})M) = d$ for all $w \in W$. In addition, we assume that M, N and L are generated as linear subspaces by the unions $\bigcup_{w\in W}(M\cap T(w)N)$, $\bigcup_{w^{-1}\in W}(N\cap T(w)M)$ and $\bigcup_{h\in H} T(h)M$, respectively.

For a subfield k of K, any k-structure on M (resp. N) induces in a natural way a k-structure on $T(\lambda)M$ (resp. $T(\lambda)N$)) for any $\lambda \in \Lambda$. We say that two k-structures on M and N are compatible if $M \cap T(\lambda)N$ for any $\lambda \in \Lambda\cap W$ is a k-subspace of both M and $T(\lambda)N$.

Theorem. *(Margulis,* [12, Lemma 8.6.2]*) If there exist compatible k-structures on M and N, then H can be given a k-structure such that $\Lambda \subset H_k$.*

2.4.2. Proposition. *Let G be an adjoint semisimple algebraic $\mathbb{R}$-group, U_1, U_2 a pair of opposite horospherical $\mathbb{R}$-subgroups and F_1, F_2 lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively. Suppose that there exists a non-trivial connected semisimple normal $\mathbb{R}$-algebraic subgroup S of $N(U_1)\cap N(U_2)$ with a $\mathbb{Q}$-form with respect to which $S\cap\Gamma_{F_1,F_2}$ is a Zariski dense arithmetic subgroup of $S(\mathbb{R})$ and the projection of S onto each simple factor of G is infinite. Then there exists a $\mathbb{Q}$-form of G such that $\Gamma_{F_1,F_2} \subset G(\mathbb{Q})$.*

Proof. Set $\Gamma = \Gamma_{F_1,F_2}$, $B = \{g \in G \mid gN(U_2)g^{-1}$ and $N(U_1)$ are opposite $\}$, $A_i = S\ltimes U_i$ and $\mathfrak{A}_i = \mathrm{Lie}(A_i)$ for each $i = 1, 2$. Denote by M and N the subspaces of the Lie algebra $\mathfrak{g}$ of G generated, respectively, by $\bigcup_{\gamma\in B\cap\Gamma}(\mathfrak{A}_1\cap Ad(\gamma)\mathfrak{A}_2)$ and $\bigcup_{\gamma^{-1}\in B\cap\Gamma}(\mathfrak{A}_2\cap Ad(\gamma)\mathfrak{A}_1)$ and by V the set of $g_0 \in G$ for which $\dim(M \cap Ad(g_0)N)= \min_{g\in G}\dim(M \cap Ad(g)N)$. We set $W = B \cap V$.

We claim that the assumptions of Lemma 2.4.1 hold for $G = H$, $\Gamma = \Lambda$, $Ad = T$, $L = \mathfrak{g}$, M, N and W. First of all, since G is adjoint, equivalently $Z(G)$ is trivial, the adjoint representation Ad of G is faithful. Since $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is the Lie algebra of S, $\mathfrak{A}_1 \cap \mathfrak{A}_2 \subset M$ is non-trivial. Since $\mathfrak{g}$ is simple and the projection of S onto each simple factor of G is infinite, $\mathfrak{g}$ is generated by $Ad(g)M$, $g \in G$. We observe that $M\cap Ad(\gamma)N = \mathfrak{A}_1\cap Ad(\gamma)\mathfrak{A}_2$ and $N\cap Ad(\gamma)M = \mathfrak{A}_2\cap Ad(\gamma)\mathfrak{A}_1$ for all $\gamma \in B\cap\Gamma$. Since the set B is of the form $N(U_1)N(U_2)$ [5, Lemma 4.1.2] and A_i is a normal subgroup of P_i, we have that $A_1 = (A_1\cap\gamma A_2\gamma^{-1})\ltimes U_1$ for all $\gamma \in B$. This implies that $\dim(M\cap Ad(\gamma)N) = \dim(\mathfrak{A}_1 \cap Ad(\gamma)\mathfrak{A}_2) = \dim(A_1 \cap \gamma A_2\gamma^{-1}) = \dim(A_1) - \dim(U_1)$ for all $\gamma \in B \cap \Gamma$. But $W \subset B$, W is open and Γ is Zariski dense. Therefore $d = \dim(A_1) - \dim(U_1)$ and hence $B \cap \Gamma = W \cap \Gamma$. It follows that M and N are generated by $\bigcup_{\gamma\in W}(\mathfrak{A}_1 \cap Ad(\gamma)\mathfrak{A}_2)$ and $\bigcup_{\gamma^{-1}\in W}(\mathfrak{A}_2 \cap Ad(\gamma)\mathfrak{A}_1)$ respectively, proving the claim.

Since F_i is a Zariski dense arithmetic subgroup of $U_i(\mathbb{R})$ for $i = 1, 2$ and $S \cap \Gamma$ is a Zariski dense arithmetic subgroup of $S(\mathbb{R})$ normalizing F_1 and F_2, we can give $\mathbb{Q}$-forms on A_1 and A_2 with respect to which $A_1 \cap \Gamma$ and $A_2 \cap \Gamma$ are arithmetic subgroups of

$A_1(\mathbb{R})$ and $A_2(\mathbb{R})$ (e.g. Lemma 3.3.11). In a natural way, these $\mathbb{Q}$-forms induce $\mathbb{Q}$-structures on $\mathfrak{A}_1$ and $\mathfrak{A}_2$ and also on M and N. Since $A_i \cap \Gamma$ is a lattice in $A_i(\mathbb{R})$ for each $i = 1, 2$, we have that for any $\gamma \in \Gamma$, $(A_1 \cap \gamma A_2 \gamma^{-1} \cap \Gamma) \ltimes F_1$ has a finite index in $A_1 \cap \Gamma$ (see [Lemma 2.15, Mar4]). Since $A_1 \cap \Gamma$ is Zariski dense in A_1, the Zariski closure of $A_1 \cap \gamma A_2 \gamma^{-1} \cap \Gamma$ is $A_1 \cap \gamma A_2 \gamma^{-1}$ and hence $A_1 \cap \gamma A_2 \gamma^{-1}$ is a $\mathbb{Q}$-subgroup of A_1. Since $M \cap Ad(\gamma)N = \mathfrak{A}_1 \cap Ad(\gamma)\mathfrak{A}_2$ for all $\gamma \in B \cap \Gamma$, it follows that the $\mathbb{Q}$-structures on M and N are compatible. Now by Lemma 2.3.1, there exists a $\mathbb{Q}$-form of G such that $\Gamma_{F_1,F_2} \subset G(\mathbb{Q})$. $\square$

2.4.3. Lemma. *If $\Gamma_{F_1,F_2} \subset G(\mathbb{Q})$, then U_i is defined over $\mathbb{Q}$ and there exists $n \in \mathbb{N}$ such that $U_i(n\mathbb{Z}) \subset F_i$ for each $i = 1, 2$.*

Proof. Since $F_i \subset U_i \cap G(\mathbb{Q})$ and F_i is Zariski dense in U_i by Lemma 2.1.8, U_i is defined over $\mathbb{Q}$ by Proposition 1.2.7. The second claim follows from Lemma 2.1.8. $\square$

2.4.4. Lemma.

(1) *If $\Gamma_{F_1,F_2} \subset G(\mathbb{Q})$ and for every infinite proper normal $\mathbb{R}$-subgroup G' of G, $F_i \cap G'(\mathbb{R})$ is finite for each $i = 1, 2$ then G is $\mathbb{Q}$-simple.*

(2) *If G is $\mathbb{Q}$-simple and U_i is defined over $\mathbb{Q}$, then for every infinite proper normal $\mathbb{R}$-subgroup G' of G, $U_i \cap G'(\mathbb{R})$ is finite.*

Proof. (1): If G is not $\mathbb{Q}$-simple, there exist non-trivial semisimple normal $\mathbb{Q}$-subgroups G' and G'' such that G is the direct product of G' and G'' by Proposition 1.2.4. By Lemma 2.4.3, F_i contains $U_i(n\mathbb{Z})$ for some $n \in \mathbb{N}$. Since $U_i \cap G'$ is $\mathbb{Q}$-subgroup, $U_i \cap G'(n\mathbb{Z})$ is infinite. But $U_i(n\mathbb{Z}) = (U_i \cap G'(n\mathbb{Z}))(U_i \cap G''(n\mathbb{Z}))$, yielding that $F_i \cap G'(\mathbb{R})$ is infinite. This contradicts the assumption.

(2): There exist a finite extension field k of $\mathbb{Q}$, an adjoint absolutely simple k-group G_0 and a horospherical k-group U_i' such that $G = R_{k/\mathbb{Q}}(G_0)$ and $U_i = R_{k/\mathbb{Q}}(U_i')$ by Proposition 1.2.6. Any infinite normal algebraic $\mathbb{R}$-subgroup G' is then isomorphic to the product of suitable n-copies of G_0. Since $U_i(\mathbb{Z}) = R_{k/\mathbb{Q}}(U_i')(J)$, J the ring of the integers of k, the claim follows. $\square$

2.4.5. Corollary. *Let G, U_1, U_2, F_1, F_2 and S be as in Proposition 2.4.2. Assume that the real rank of each $\mathbb{R}$-simple factor of G is at least 2 and $F_i \cap G'(\mathbb{R})$ is finite for every infinite proper normal algebraic $\mathbb{R}$-subgroup G' of G and each $i = 1, 2$. If Γ_{F_1,F_2} is discrete, then it is an arithmetic subgroup of $G(\mathbb{R})$.*

Proof. By Proposition 2.4.2 and Lemma 2.4.3, there exists a $\mathbb{Q}$-form of G such that $\Gamma_{F_1,F_2} \subset G(\mathbb{Q})$ and $U_i(n\mathbb{Z}) \subset F_i$ for some $n \in \mathbb{N}$. By Lemma 2.4.4, G is almost $\mathbb{Q}$-simple. It follows from Corollary 2.2.3 that the subgroup Γ generated by $U_1(n\mathbb{Z})$ and

$U_2(n\mathbb{Z})$ is an arithmetic subgroup of $G(\mathbb{R})$. By Lemma 2.1.4, Γ_{F_1,F_2} is an arithmetic subgroup of $G(\mathbb{R})$. $\square$

2.4.6. Proposition. *Let G be a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group with real rank at least 2 and U_1, U_2 be defined over $\mathbb{Q}$. Suppose that the semisimple part of $N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$ does not have compact factors. If F_1 and F_2 are commensurable to $U_1(\mathbb{Z})$ and $wU_2(\mathbb{Z})w^{-1}$ for some $w \in Z(L)(\mathbb{R})$ respectively and Γ_{F_1,F_2} is discrete, Γ_{F_1,F_2} is an arithmetic subgroup of $G(\mathbb{R})$.*

Proof. We may assume that $F_1 = U_1(\mathbb{Z})$ and $F_2 = wU_2(\mathbb{Z})w^{-1}$. Let $N = N(\Gamma_{F_1,F_2})$ and H the semisimple part of $N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$. Then N is discrete by Lemma 2.1.3 and H is a semisimple subgroup defined over $\mathbb{Q}$. Since $w \in Z(L)$, $H(\mathbb{Z})$ normalizes F_1 and F_2 and hence $H(\mathbb{Z}) \subset N$. Since $H(\mathbb{Z})$ is a Zariski dense arithmetic subgroup of $H(\mathbb{R})$, it follows from Proposition 2.4.2 that there exists a $\mathbb{Q}$-form of G such that $N \subset G(\mathbb{Q})$. Since $\Gamma_{F_1,F_2} \subset G(\mathbb{Q})$, we can show the rest of claim by the same argument as the proof of Corollary 2.4.5. $\square$

2.5. The subgroup generated by $Z(U_1)$ and $Z(U_2)$

We prove some algebraic lemmas which we will need later in applying Proposition 2.4.2.

2.5.1. Let G be an adjoint k-simple algebraic k-group. This section has content only when the k-rank of G is at least 2. We continue the notation from section 1.3.1 for S, T, $\Phi = \Phi(T,G)$, ${}_k\Phi = \Phi(S,G)$, $j : \Phi \to {}_k\Phi \cup \{0\}$, Δ, ${}_k\Delta$, ${}_kV_\Theta$, ${}_kV_\Theta^-$ for $\Theta \subset {}_k\Delta$ and so on. We recall that α_h denotes the highest k-root in ${}_k\Phi$ and ${}_k\Delta_H$ the unique set of k-Heisenberg roots in ${}_k\Delta$.

The following lemma can be checked case by case in each irreducible root system.

Lemma.

(1) *The only positive k-root the sum of whose coefficients with respect to which the roots in ${}_k\Delta_H$ is 2 is α_h.*
(2) *If $\beta \in {}_k\Phi^+$ is such that the sum of the coefficients with respect to ${}_k\Delta_H$ is 1 and there exists a simple root in ${}_k\Delta$ with respect to which the coefficient of β is 0, then $\alpha_h - \beta \in {}_k\Phi^+$.*

2.5.2. The commutator law over an algebraically closed field is well known. The following is a result of Vinberg which says that the commutator law holds for an arbitrary field. Hereafter, we shall also refer to this lemma as the commutator law.

Lemma. *(e.g. [12, 4.5.1]) If $a, b, a+b \in {}_k\Phi$, then $[{}_k\mathcal{U}_a, {}_k\mathcal{U}_b] = {}_k\mathcal{U}_{a+b}$ and moreover, for any non-zero $x \in ({}_k\mathcal{U}_a)$, we have $[x, {}_k\mathcal{U}_b] \neq 0$.*

2.5.3. We recall that the notation U_{α_h} denotes the k-root group corresponing to the highest k-root α_h and U_H denotes the k- Heisenberg horospherical k-subgroup ${}_kV_{{}_k\Delta - {}_k\Delta_H}$.

We observe that $Z({}_kV_\Theta) = U_\psi$ where $\psi = \{\beta \in {}_k\Phi^+ - [\Theta] \mid \beta + \alpha \notin {}_k\Phi$ for any $\alpha \in {}_k\Phi^+ - [\Theta]\}$. In particular $\alpha_h \in \psi$.

Lemma. *$Z({}_kV_\Theta) = U_{\alpha_h}$ if and only if ${}_k\Delta_H \subset {}_k\Delta - \Theta$.*

Proof. If $\gamma \in {}_k\Delta_H \cap \Theta$, $\alpha_h - \gamma$ is a k-root by Lemma 2.5.1; hence $\alpha_h - \gamma \in {}_k\Phi^+ - [\Theta]$. We claim that $(\alpha_h - \gamma) + \beta \notin {}_k\Phi^+$ for any $\beta \in {}_k\Phi^+ - [\Theta]$. Suppose not. Then by the height calculation, we have $(\alpha_h - \gamma) + \beta = \alpha_h$. It implies that $\gamma = \beta$, contradicting the assumption $\gamma \in \Theta$. Therefore if $Z(U) = U_{\alpha_h}$, or equivalently $\psi = \{\alpha_h\}$ as the above notation, then ${}_k\Delta_H \cap \Theta$ is empty.

To see the converse, suppose that ${}_k\Delta_H \subset \Theta^c$. Then U_H is contained in ${}_kV_\Theta$. Since the centralizer of U_H in ${}_kV_\Theta$ is contained in $R_u(N(U_H)) = U_H$ and hence coincides with $Z(U_H) = U_{\alpha_h}$, we have $Z({}_kV_\Theta) \subset U_{\alpha_h}$. □

2.5.4. For $a \in {}_k\Phi$, denote by ${}_k\mathcal{U}_a$ the root space corresponding to a and put ${}_kG_a = \exp({}_k\mathcal{U}_a)$. Note that for any subset $\psi \subset {}_k\Phi$, the subgroup ${G_{[\psi]}}^*$ (as in the notation of section 1.2.1) coincides with the subgroup generated by all the ${}_kG_a$ $(a \in \psi)$.

A closed subset Φ_0 of Φ is called an ideal if the following holds: for all $\alpha \in \Phi_0$ and $\beta \in \Phi$ such that $\alpha + \beta \in \Phi$, $\alpha + \beta \in \Phi_0$. It follows from the commutator law that if Φ_0 is an ideal of Φ, then the subgroup $G^*_{\Phi_0}$ is a normal subgroup of G.

In proving the following proposition, we use the fact (e.g. [26 , Proposition 1.1.10]) that if ϕ is a closed subset of Φ such that if $\alpha \in \phi$, then $-\alpha \in \phi$, then the subalgebra generated by $\mathcal{U}_b$, $b \in \phi$ is semi-simple.

Proposition. *Let $U_1 = {}_kV_\Theta$ and $U_2 = {}_kV_\Theta^-$ for some $\Theta \subset \Delta$. Denote by G_0 the subgroup of G generated by $Z(U_1)$ and $Z(U_2)$, $\mathbf{H}$ the commutator subgroup of $N(U_1) \cap N(U_2)$ and $S = \mathbf{H} \cap G_0$. Then*

1. *G_0 is a connected almost k-simple algebraic k-subgroup, $Z(U_1)$ and $Z(U_2)$ are opposite commutative horospherical k-subgroups of G_0 and S is a connected semisimple normal algebraic k-subgroup of $\mathbf{H}$. Moreover, if G is absolutely almost simple, so is G_0.*
2. *If $Z(U_1)$ is strictly bigger than U_{α_h}, then S is isotropic over k and the k-rank of G_0 is at least 2.*
3. *If $\mathbf{H}$ does not have any k-anisotropic factors, neither does S.*

Proof. Set $\psi = \{\beta \in {}_k\Phi^+ - [\Theta] \mid \beta + \alpha \notin {}_k\Phi$ for any $\alpha \in {}_k\Phi^+ - [\Theta]\}$. Note that ψ is a closed subset of ${}_k\Phi^+$ and $Z(U_1) = U_\psi$ and $Z(U_2) = U_{-\psi}$.

(1): note that $[\psi]$ is a closed set of ${}_k\Phi$ and the subgroup $G^*_{[\psi]}$ generated by U_b, $b \in j^{-1}([\psi])$ (or equivalently, $b \in [j^{-1}(\psi)]$) coincides with G_0. This group is algebraic, connected (any algebraic group generated by unipotent subgroups is connected) and semisimple (by the fact mentioned preceding the statement of this proposition). On the other hand, ψ is invariant under $\mathrm{Gal}(K/k)$ since $Z(U_1)$ is defined over k (see 1.2.1). Therefore $[\psi]$ is also invariant under $\mathrm{Gal}(K/k)$ and hence G_0 is defined over k.

To show that G_0 is almost k-simple, assume that $\Psi = [\psi]$ is the disjoint union $\Psi_1 \cup \Psi_2 \cup ... \cup \Psi_s$ where each Ψ_i is a non-empty irreducible ideal of Ψ. We will show that each Ψ_i contains the unique highest k-root α_h, which is an obvious contradiction if $s \geqq 2$. Note that $\Psi_i \cap \psi$ is non-empty for each i. For any $\alpha \in \Psi_i \cap j^{-1}(\psi)$, there exists simple k-roots $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_k}$ (not necessarily different) such that $\alpha + \alpha_{i_1} + \cdots + \alpha_{i_j} \in {}_k\Phi^+$ for each $j = 1, 2, ..., k$ and $\alpha + \alpha_{i_1} + \cdots + \alpha_{i_k} = \alpha_h$.

Since $\alpha + \alpha_{i_1} \in {}_k\Phi^+$, $\alpha + \alpha_{i_1} \in \psi$. It follows that $\alpha_{i_1} \in \Psi$. Since Ψ_i is an ideal of Ψ, we have $\alpha + \alpha_{i_1} \in \Psi_i$. Therefore by induction $\alpha + \alpha_{i_1} + \cdots + \alpha_{i_j} \in \Psi_i$ for each $j = 1, 2, ..., k$; hence $\alpha_h \in \Psi_i$. This proves that G_0 is almost k-simple.

If G is absolutely almost simple, then Φ has the unique highest root. Then using the same argument as above we can show that $[j^{-1}(\psi)]$ cannot be a disjoint union of ideals. It follows that G_0 is absolutely almost simple.

We note that Lie G_0 is decomposed into (Lie $L \cap$ Lie G_0) $\oplus$ Lie $Z(U_1) \oplus$ Lie$Z(U_2)$. Clearly $Z(U_1)$ are $Z(U_2)$ are opposite horospherical subgroups of G_0.

By the same argument we made for G_0, we can show that S is a semisimple algebraic k-group.

We now show that S is a normal subgroup of H. Let $s \in S$ and $h \in H$. Then $S = a_1 a_2 ... a_k$ for some $a_i \in Z(U_1) \cup Z(U_2)$. Since $H \subset N(U_1) \cap N(U_2)$, $h a_i h^{-1} \in U_1 \cup U_2$ for each i. Suppose that $a_i \in Z(U_1)$; hence $h a_i h^{-1} \in U_1$. To see $h a_i h^{-1} \in Z(U_1)$, let $u \in U_1$ and consider $u(h a_i h^{-1}) = h(h^{-1} u h) a_i h^{-1}$, which is equal to $h a_i (h^{-1} u h) h^{-1}$ since $\alpha_i \in Z(U_i)$. So $u(h a_i h^{-1}) = (h a_i h^{-1})u$. It proves that H normalizes $Z(U_1)$ and similarly $Z(U_2)$. Therefore $hSh^{-1} \in G_0$; hence $hSh^{-1} \in S$, proving H normalizes S.

(2): Suppose that U_{α_h} is a proper subgroup of $Z(U_1)$. By Lemma 2.5.3, we have an element $\alpha \in \Theta \cap {}_k\Delta_H$. Then $\alpha_h - \alpha \in {}_k\Phi^+$ by Lemma 2.5.1. We claim that this root belongs to ψ. Otherwise, for some element $\beta \in {}_k\Phi^+ - [\Theta]$, $(\alpha_h - \alpha) + \beta$ is an k-root. By comparing height, it follows that $(\alpha_h - a) + \beta = \alpha_h$ hence $\alpha = \beta$, contradicting $\alpha \in \Theta$. Therefore $\alpha = (\alpha_h) - (\alpha_h - \alpha) \in [\psi]$, showing that S is isotropic over k, since Lie S clearly contains $\mathcal{U}_\alpha$ for all $\alpha \in [\psi] \cap \Theta$.

(3): Since S is a normal algebraic k-subgroup of $\mathbf{H}$, S has no k-anisotropic factors unless $\mathbf{H}$ does. □

2.5.5. Example. The following example illustrates the power of Proposition 2.4.2, which will be also one of the most fundamental ideas of the proof of the main theorem.

Let U_1, U_2 be the pair of opposite horospherical subgroups in $SL_n(\mathbb{R})$, consisting of the elements of the form $\begin{pmatrix} I_l & X_1 & Z_1 \\ 0 & I_m & Y_1 \\ 0 & 0 & I_k \end{pmatrix}$ and $\begin{pmatrix} I_l & 0 & 0 \\ X_2 & I_m & 0 \\ Z_2 & Y_2 & I_k \end{pmatrix}$, respectively, where $l+m+k=n$. Note that U_1 and U_2 are two step nilpotent groups, i.e., $[U_i, U_i] = Z(U_i)$ for each $i = 1, 2$.

If $lk \geqq 2$ and F_1 and F_2 are lattices in U_1 and U_2 such that $F_1 \cap Z(U_1) = M_{l\times k}(\mathbb{Z})$ and $F_2 \cap Z(U_2) = M_{k\times l}(\mathbb{Z})$, there exists a $\mathbb{Q}$-form of $SL_n(\mathbb{R})$ such that $\Gamma_{F_1,F_2} \subset G(\mathbb{Q})$. To see this, note that the subgroup G_0 generated by $Z(U_1)$ and $Z(U_2)$ consists of the elements $\begin{pmatrix} X & 0 & X' \\ 0 & I_m & 0 \\ Z' & 0 & Z \end{pmatrix}$ in $SL_n(\mathbb{R})$ and $S = H \cap G_0$ consists of the elements $\begin{pmatrix} X & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & Z \end{pmatrix}$ where H is the commutator subgroup of $N(U_1) \cap N(U_2)$. Observe that $S \cap \Gamma_{F_1,F_2}$ is $\begin{pmatrix} M_{l\times l}(\mathbb{Z}) & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & M_{k\times k}(\mathbb{Z}) \end{pmatrix}$. It only remains to apply Proposition 2.4.2.

Since $S(\mathbb{Q})$ is split in this case and hence at least one of α_1 and α_{n-1} belongs to $\Delta - \Delta_0$ (see the notation 1.4.1), it follows from the classification of $\mathbb{Q}$-forms [21] that $G(\mathbb{Q})$ is conjugate to $SL_n(\mathbb{Q})$.

3. ADJOINT ACTION ON THE SPACE OF LATTICES

3.1. The space of lattices in algebraic unipotent groups

We collect some well known lemmas.

3.1.1. Proposition. *(e.g.* [10, Lemma 1.3.1]*) The group of real points of a unipotent algebraic $\mathbb{R}$-group is a topologically connected, simply connected, nilpotent Lie group, and every unipotent subgroup of an algebraic group is nilpotent.*

3.1.2. We state a well known criterion of compactness due to Mahler.

Lemma. *(Mahler,* [6]*) A set A of lattices of some Euclidean space is relatively compact if and only if there are positive constant d and ϵ such that the ϵ-neighborhood in the Euclidean space does not contain any non-zero element from any lattice belonging to A and det $x < d$ if $x \in A$.*

3.1.3. Lemma. *(Minkowski, [6]) For any n, there is a constant $c(n)$ such that in any quasi-lattice Δ lying in an euclidean space of dimension not exceeding n, there exists a basis $e_1, e_2, ..., e_k$ of Δ ($k = dim\Delta$) such that for any $l (1 \leq l \leq k)$, $\prod_{i=1}^{l} \|e_i\| < c(n)d(\Delta)^{\frac{1}{k}}$.*

3.1.4. Lemma. *Let W be the space of lattices of some fixed Euclidean space. For any relatively compact subset $M \subset W$, there are positive constants d_1 and d_2 such that in any lattice M, there is a basis consisting of elements of norm less than d_1 and greater than d_2.*

Proof. It is easily deduced from Mahler's compactness criterion and the previous lemma. □

3.1.5. Proposition. [10, Lemma 5.2] *Let Z be a simply connected nilpotent Lie group and $\mathfrak{Z}$ its Lie algebra. Then there exists an integer b such that if Γ is a subgroup of Z and $\overline{ln\Gamma}$ is the subring in $\mathfrak{Z}$ generated by $ln\Gamma$, then $b(\overline{ln\Gamma}) \subset ln\Gamma$. Furthermore b depends only on the length of the lower central series.*

3.1.6. Let $\mathbf{G}$ be an algebraic $\mathbb{R}$-group, $G = \mathbf{G}(\mathbb{R})^0$ and γ the Lie algebra of G. We choose a Euclidean metric $\rho_{\mathfrak{g}}$ on $\mathfrak{g}$ and a left invariant metric ρ_G on $G(\mathbb{R})$ corresponding $\rho_{\mathfrak{g}}$. Denote by X the set of all discrete unipotent subgroups of G. Since in an algebraic $\mathbb{R}$-group, any two maximal unipotent $\mathbb{R}$-groups are conjugate to each other, there exists an integer $N \in \mathbb{N}$ such that for any subgroup $F \in X$, the length of the lower central series of F is at most N. Therefore there exists an integer $b \in \mathbb{N}$ such that if $F \in X$, then $b(\overline{lnF}) \subset lnF$. We denote $b(\overline{lnF})$ by Δ_F. We introduce in the following topology on the space of quasi-lattices Ω in γ: a subset $B \subset \Omega$ is open if, for any $F \in B$, there exists some $\epsilon > 0$ such that any quasi-lattice of the form $T(F)$ belongs to B for each linear transformation T with $\|T - Id\| \leqq \epsilon$. Let $f : X \to \Omega$ be the mapping defined by $f(F) = \Delta_F$.

Lemma. [10, Lemma 5.7] *For any subset $L \subset X$ for which $f(L)$ is relatively compact in Ω, there are constants $\epsilon > 0$ and c such that if $F \in L$, then*

(1) *$\rho_G(e, x) > \epsilon$ for all non-trivial $x \in F$;*
(2) *there exists x_1,...,$x_k \in F$ such that $\rho_G(e, x_i) < c$ for any i and any algebraic subgroup of G containing $x_1, ..., x_k$ also contains the Zariski closure of F in G.*

3.2. Adjoint action

3.2.1. Let $\mathbf{G}$ be a connected adjoint semisimple algebraic $\mathbb{R}$-group with no $\mathbb{R}$-anisotropic factors and U_1, U_2 a pair of opposite horospherical $\mathbb{R}$-subgroups. Set $\mathbf{L} = N(U_1) \cap N(U_2)$,

$\mathbf{H} = [\mathbf{L}, \mathbf{L}]$ and $H = \mathbf{H}(\mathbb{R})^0$ as well as $G = \mathbf{G}(\mathbb{R})^0$. As before set $\mathcal{U}_i = \mathrm{Lie}(U_i)$ and denote by $Ad_i : \mathbf{L} \to GL(\mathcal{U}_i)$ the restriction of the adjoint representation of $N(U_i)$. We recall that Ad_i is faithful.

We mention that this section has no content if H is trivial.

3.2.2. Denote by Ω_i the space of lattices in the Euclidean space $(\mathcal{U}_i)(\mathbb{R})$. The group H acts through Ad_i on $(\mathcal{U}_i)(\mathbb{R})$ and hence on Ω_i. The notation $H.J_i$ denotes the orbit $\{Ad_i(h)J_i \in \Omega_i \mid h \in H\}$ of J_i under H for a lattice J_i in $\mathcal{U}_i(\mathbb{R})$. We also diagonalize the action of H on Ω_i described above to the space $\Omega_1 \times \Omega_2$ of pairs of lattices in $\mathcal{U}_1(\mathbb{R})$ and $\mathcal{U}_2(\mathbb{R})$ by $h(J_1, J_2) = (Ad_1(h)J_1, Ad_2(h)J_2)$. The image of H under this action is the subgroup $\{(Ad_1(h), Ad_2(h)) \in SL(\mathcal{U}_1(\mathbb{R})) \times SL(\mathcal{U}_2(\mathbb{R})) \mid h \in H\}$ and will be denoted by $\delta(H)$. The notation $\delta(H).(J_1, J_2)$ denotes the H-orbit of the pair (J_1, J_2) via this action.

3.2.3. Let F_i be a lattice in $U_i(\mathbb{R})$ and Δ_{F_i} be as in 3.1.6 for $i = 1, 2$. Denote by Λ_{F_i} the stabilizer $\{g \in H \mid Ad_i(g)\Delta_{F_i} = \Delta_{F_i}\}$ of Δ_{F_i} and by Λ_{F_1,F_2} the stabilizer $\{g \in H \mid Ad_1(g)\Delta_{F_1} = \Delta_{F_1},\ Ad_2(g)\Delta_{F_2} = \Delta_{F_2}\}$ of $(\Delta_{F_1}, \Delta_{F_2})$.

The following lemma then immediately follows from the definition of Δ_{F_i} and Lemma 1.5.3.

Lemma.

(1) $\Delta_{hF_ih^{-1}} = Ad_i(h)\Delta_{F_i}$ *for any* $h \in H$ *and any lattice* F_i *in* $U_i(\mathbb{R})$.
(2) $\Lambda_{F_i} = \{g \in H \mid gF_ig^{-1} = F_i\}$.
(3) $\Lambda_{F_1,F_2} = \{g \in H \mid gF_1g^{-1} = F_1,\ gF_2g^{-1} = F_2\}$.

3.2.4. We choose a Euclidean metric $\rho_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}$ of G and a left invariant metric ρ_G on G of corresponding to this metric. We define a topology on Ω_i and $\Omega_1 \times \Omega_2$ in the following way: a sequence of lattices $\{\Delta_k \in \Omega_i \mid k = 1, 2, ...\}$ converges to a lattice Δ if and only if there is $\epsilon > 0$ such that $\rho_{\mathfrak{g}}(0, x) > \epsilon$ for all non-zero $x \in \Delta_k$ and for all k and there exist bases in the lattices Δ_k which converge to some basis in Δ. It is not difficult to see that this topology coincides with the topology defined in section 3.1.6.

We note that not every lattice in the Euclidean space $\mathcal{U}_i(\mathbb{R})$ is of the form Δ_{F_i} for a lattice F_i in $U_i(\mathbb{R})$ unless U_i is commutative. In fact any $\mathbb{Z}$-linear span of n-linearly independent vectors forms a lattice in $\mathcal{U}_i(\mathbb{R})$ where $n = \dim(\mathcal{U}_i)$; while a lattice in $U_i(\mathbb{R})$ should satisfy certain relations from the group structure of U_i since by definition a lattice in $U_i(\mathbb{R})$ is a subgroup. On the other hand, the following lemma says that any lattice in $\overline{H.\Delta_{F_i}}$ is of the form Δ_{E_i} for some lattice E_i in $U_i(\mathbb{R})$.

For a subset M of the space of lattices, we denote by $\overline{M}$ the closure of M.

Lemma. *For any lattice $J_i \in \overline{H.\Delta_{F_i}}$, there exists a lattice E_i in $U_i(\mathbb{R})$ such that $\Delta_{E_i} = J_i$ and the determinant of J_i is equal to the determinant of Δ_{F_i}.*

Proof. Let $x_n \in H$ be a sequence such that $Ad_i(x_n)\Delta_{F_i}$ converges to J_i. Since $\{Ad_i(x_n)\Delta_{F_i} \mid n \in \mathbb{N}\}$ is relatively compact, by Lemma 3.1.6, there is a neighborhood W of e such that $W \cap x_n F_i x_n^{-1} = \{e\}$ for each $n \in \mathbb{N}$. By Chabauty's theorem (see Theorem 2.1.7), there exists a discrete subgroup E_i in $U_i(\mathbb{R})$ to which a subsequence of $x_n F_i x_n^{-1}$ converges. Since H is a semisimple group, the volume of $U_i(\mathbb{R})/x_n F_i x_n^{-1}$ coincides with the volume of $U_i(\mathbb{R})/F_i$ for all n. It follows from Chabauty's theorem (Theorem 2.1.7) that E_i is a lattice in $U_i(\mathbb{R})$. Since a subsequence of $Ad_i(x_n)\Delta_{F_i}$ converges to Δ_{E_i}, $\Delta_{E_i} = J_i$, proving the lemma. □

3.2.5. Proposition. *Suppose that Γ_{F_1,F_2} is discrete. Then for any pair $(\Delta_{E_1}, \Delta_{E_2})$ lying in $\overline{\delta(H(\mathbb{R})).(\Delta_{F_1}, \Delta_{F_2})}$, the subgroup Γ_{E_1,E_2} is discrete.*

Proof. Consider a sequence $x_i \in H$ such that $Ad(x_i)((\Delta_{F_1}, \Delta_{F_2}))$ converges to (E_1, E_2). Since the set $\{Ad_1(x_i)\Delta_{F_1}, Ad_2(x_i)\Delta_{F_2}, \Delta_{E_1}, \Delta_{E_2} | i \geq 1\}$ is relatively compact in the space of lattices, there exist $d_1 > 0$ and $d_2 > 0$ such that for any lattice in this set, there exists a basis consisting of elements of norm less than d_1 and greater than d_2 by Lemma 3.1.4. Let d_1' and d_2' be such that $d_2' \leq \rho_G(e, x) \leq d_1'$ implies that $d_2 \leq \rho_{\mathfrak{g}}(0, \ln x) \leq d_1$ for all $x \in U_1 \cup U_2$. Set $K = \{g \in G \mid d_2' \leq \rho_G(e, g) \leq d_1'\}$. For each $x_i \in H$, the set $x_i \Gamma_{F_1,F_2} x_i^{-1} \cap K$ generates the subgroup $x_i \Gamma_{F_1,F_2} x_i^{-1}$. Since Γ_{F_1,F_2} and hence $x_i \Gamma_{F_1,F_2} {x_i}^{-1}$ are Zariski dense and hence have property (P), and K is compact, there exists a neighborhood V of e such that $V \cap x_i \Gamma_{F_1,F_2} x_i^{-1} = \{e\}$ for each x_i by Theorem 2.1.8. Theorem 2.1.7 implies that the limiting subgroup Γ_{E_1,E_2} of the sequence $x_i \Gamma_{F_1,F_2} x_i^{-1}$ is discrete. □

3.2.6. We will need the following lemma for the proof of Proposition 3.2.7.

Lemma. *(e.g.* [10, Lemma 1.6.1]*) There is an ϵ-neighborhood V of the identity e in G such that for any $x, y \in V$, we have $\rho_G(e, xyx^{-1}y^{-1}) < \frac{1}{2} min(\rho_G(e, x), \rho_G(e, y))$.*

3.2.7. Proposition. *Suppose that Γ_{F_1,F_2} is discrete. Then for a sequence $x_i \in H$, the set $M_1 = \{(Ad_1(x_i)\Delta_{F_1} \in H.\Delta_{F_1} \mid i \geq 1\}$ is relatively compact if and only if the set $M_2 = \{Ad_2(x_i)\Delta_{F_2} \in H.\Delta_{F_2} \mid i \geq 1\}$ is relatively compact.*

Proof. Suppose that M_1 is relatively compact and M_2 is not. Choose a positive ϵ and V as in Lemma 3.2.6. Since the set M_1 is relatively compact, by Lemma 3.1.3, there exists a $d > 0$ such that for any lattice $Ad_1(x_i)F_1$ in M_1, there exists a basis $\mathfrak{B}_i$ such that $\rho_G(e, \exp x) < d$ for all $x \in \mathfrak{B}_i$. Let $\mathfrak{B}$ be the union of all $\mathfrak{B}_i$'s. It is not difficult to see that we can find $g \in C(H)$ such that $\rho_G(e, gug^{-1}) < \epsilon$ for all $u \in U_1(\mathbb{R})$

such that $\rho_G(e,u) < d$. By Mahler's criterion of compactness, there exists $\epsilon_0 > 0$ such that $\epsilon_0 < \epsilon$ and $Ad_1(g)M_1$ does not contain any non-trivial element x such that $\rho_G(e, \exp x) < \epsilon_0$. Since $Ad_2(g)M_2$ is not relatively compact and the determinants of the lattices in $Ad_2(g)M_2$ are equal, there exists some $j \in \mathbb{N}$ such that $Ad_2(g)Ad_2(x_j)\Delta_{F_2}$ contains a non-trivial element y with $\rho_G(e, \exp y) \leq \epsilon_0$. Set $n = x_j$. Set $\tilde{U} = \{x \in U_1 \mid \ln x \in \mathfrak{B} \cap Ad_2(g)Ad_2(n)\Delta_{F_1}\}$. With $U_{-1} = g\tilde{U}g^{-1}$, $U_0 = \{\exp y\}$ and $U_i = \{sts^{-1}t^{-1} | s \in U_i, t \in U_{i-1}\}$, $\sup_{u \in U_i} \rho_G(e,u) = d_i$ tends to 0 as i goes to ∞, as U_{-1} and U_0 lie in V. On the other hand, both U_0 and U_{-1} belong to the discrete subgroup $gn\Gamma_{F_1,F_2}n^{-1}g^{-1}$ and hence $U_i \subset gn\Gamma_{F_1,F_2}n^{-1}g^{-1}$ for all $i \geqq 0$. Thus there exists $k > 0$ such that U_k is not trivial and U_{k+1} is trivial. Let z be a nontrivial element of U_k. Then $\rho_G(e,z) \leq \epsilon_0$ and z belongs to $C(U_{-1})$. Since the centralizer of any subset is algebraic, the elements of $gnU_1n^{-1}g^{-1}$ commute with z since U_1 is the Zariski closure of $\exp(\mathfrak{B})$. Therefore, $n^{-1}g^{-1}zgn \in C(U_1) \cap \Gamma_{F_1,F_2}$. Since $C(U_1) \subset U_1$ (see the proof of Lemma 1.5.4), $C(U_1) \cap \Gamma_{F_1,F_2} = F_1$. But $\rho_G(e,z) < \epsilon_0$, contradicting that the ϵ_0-neighborhood intersects $Ad(g)M_1$ trivially. This proves the proposition. □

In fact, we have proved that if the sequence $Ad_1(x_i)\Delta_{F_1}$ converges, then we can find a convergent subsequence of $Ad_2(x_i)\Delta_{F_2}$. The following is an immediate corollary of the propositions 3.2.5 and 3.2.7.

3.2.8. Proposition. *If Γ_{F_1,F_2} is discrete, then for any lattice Δ_{E_1} in $\overline{H.\Delta_{F_1}}$, there exists a lattice E_2 in $U_2(\mathbb{R})$ such that $(\Delta_{E_1}, \Delta_{E_2})$ lies in $\overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$ and the subgroup Γ_{E_1,E_2} is discrete.*

3.3. Ratner's theorem, closedness of some orbits and $\mathbb{Q}$-structures

3.3.1. Let G be a connected Lie group and Γ a lattice in G. Consider the natural action of G by left translation on the homogeneous space G/Γ. For $x \in G/\Gamma$, $G_x = \{g \in G \mid g.x = x\}$ is the stabilizer of x. M. Ratner proved the following theorem which was known as Raghunathan's conjecture.

Theorem. *(Ratner,* [19]*) Let H be a connected closed subgroup of G generated by unipotent one-parameter subgroups in it. Then for $x \in G/\Gamma$, there exists a connected and closed subgroup L of G containing H such that $\overline{H.x}$ coincides with $L.x$ and $L \cap G_x$ is a lattice in L.*

3.3.2. Theorem. *(see* [16, Theorem 1.13]*) Let G and Γ be as above and H be a closed subgroup of G. Then if $H \cap G_x$ is a lattice for some $x \in G/\Gamma$, then $H.x$ is closed.*

3.3.3. We keep the notation for $\mathbf{G}$, $\mathbf{L}$, G and H from section 3.2.1. In addition, we assume that $\mathbb{R}$-rank of $\mathbf{G}$ is at least 2 and H has no compact factors or equivalently,

$\mathbf{H}$ has no $\mathbb{R}$-anisotropic factors. This assures that H is generated by unipotent one-parameter subgroups by Lemma 2.3.1. For instance, if G is split over $\mathbb{R}$, then it is always the case.

In this section, we fix lattices F_1 and F_2 in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$, respectively. Let Ω_i denote the space of lattices in $\mathcal{U}_i(\mathbb{R})$ of the same determinant as Δ_{F_i} for each $i = 1, 2$. Since $SL(\mathcal{U}_i(\mathbb{R}))$ acts transitively on Ω_i, Ω_i can be identified with the homogeneous space $SL_l(\mathbb{R})/SL_l(\mathbb{Z})$, through the isomorphism of $\mathcal{U}_i$ with the Euclidean space $\mathbb{R}^l$ where $l = \dim\,(\mathcal{U}_i(\mathbb{R}))$.

3.3.4. Proposition. *The orbit $H.\Delta_{F_i}$ (resp. $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$) is closed if and only if Λ_{F_i} (resp. Λ_{F_1,F_2}) is a Zariski dense arithmetic subgroup of H.*

Proof. Suppose that $H.\Delta_{F_i}$ is closed. Choose a representative of Δ_{F_i} in $SL_l(\mathbb{R})/SL_l(\mathbb{Z})$, say $g_i SL_l(\mathbb{Z})$ so that $Ad_i(\Lambda_{F_i}) \subset g_i SL_l(\mathbb{Z}) g_i^{-1}$. By Ratner's theorem, Λ_{F_i} is a lattice in H. Since H has no compact factors by the assumption, it follows from Borel density theorem that $g_i^{-1} Ad_i(\Lambda_{F_i}) g_i \subset SL(\mathbb{Q})$ is Zariski dense in $g_i^{-1} Ad_i(H) g_i$. Therefore by Proposition 1.2.7, $g_i^{-1} Ad_i(\mathbf{H}) g_i$ is a $\mathbb{Q}$-subgroup of $SL_n(\mathbb{C})$, providing a $\mathbb{Q}$-form of H such that $H(\mathbb{Z})$ is commensurable to Λ_{F_i}. If $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$ is closed and $\Delta_{F_i} = g_i SL_l(\mathbb{Z})$ for $i = 1, 2$, by the same argument as above, $(g_1^{-1}, g_2^{-1})\{(Ad_1(h), Ad_2(h)) \mid h \in H\}(g_1, g_2)$ is a $\mathbb{Q}$-subgroup of $SL_l(\mathbb{C}) \times SL_l(\mathbb{C})$, providing a $\mathbb{Q}$-form of $\delta(H)$ and hence of H such that $H(\mathbb{Z})$ is commensurable to $(\Delta_{F_1}, \Delta_{F_2})$. The converse is a direct consequence of Theorem 2.2.2 and Theorem 3.3.2. □

3.3.5. Corollary. *Let F_i and E_i be commensurable lattices in $U_i(\mathbb{R})$. Then $H.\Delta_{F_i}$ is closed if and only if $H.\Delta_{E_i}$ is closed.*

Proof. If F_i and E_i are commensurable, then Λ_{F_i} and Λ_{E_i} are commensurable. Therefore Λ_{F_i} is a lattice in H if and only if Λ_{E_i} is a lattice in H. The corollary now follows from Theorem 3.3.2. □

3.3.6. Proposition. *Assume that for any infinite proper normal $\mathbb{R}$-subgroup G' of G, $F_i \cap G'(\mathbb{R})$ is finite and that Γ_{F_1,F_2} is discrete. Then $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$ is closed if and only if Γ_{F_1,F_2} is an arithmetic subgroup (also, irreducible lattice) of G.*

Proof. Suppose that $\delta(H).(F_1, F_2)$ is closed. Then the stabilizer Λ_{F_1,F_2} is a Zariski dense arithmetic subgroup in H by Proposition 3.3.4. Let N be the normalizer of Γ_{F_1,F_2}. Then N contains Λ_{F_1,F_2} by Lemma 3.2.3. By Corollary 2.4.5, there exists a $\mathbb{Q}$-form of G such that Γ_{F_1,F_2} is commensurable to $G(\mathbb{Z})$. Since G is $\mathbb{Q}$-simple by the assumption on F_i (Lemma 2.4.4), it follows that Γ_{F_1,F_2} is an irreducible lattice.

Suppose that Γ_{F_1,F_2} is an arithmetic subgroup of G, that is, there exists a $\mathbb{Q}$-form of G such that Γ_{F_1,F_2} is commensurable to $G(\mathbb{Z})$. It follows that U_1 and U_2 are defined over

$\mathbb{Q}$ and F_i is commensurable to $U_i(\mathbb{Z})$ for each $i = 1, 2$. Therefore H is also defined over $\mathbb{Q}$ and Λ_{F_1,F_2} is commensurable to $H(\mathbb{Z})$. By Theorem 3.3.2, $\delta(H).(F_1, F_2)$ is closed. □

3.3.7. Corollary. *Suppose that* $\mathbf{G}$ *is a* $\mathbb{Q}$*-simple group with respect to which* U_1 *and* U_2 *are* $\mathbb{Q}$*-subgroups and that* $E_1 = U_1(\mathbb{Z})$ *and* $E_2 = zU_2(\mathbb{Z})z^{-1}$ *for some* $z \in Z(L(\mathbb{R}))$. *If* Γ_{E_1,E_2} *is discrete, the orbit* $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ *is closed.*

Proof. Since $z \in Z(L(\mathbb{R}))$, $\Lambda_{U_1(\mathbb{Z}),U_2(\mathbb{Z})} = \Lambda_{U_1(\mathbb{Z}),zU_2(\mathbb{Z})z^{-1}}$. On the other hand, $\Lambda_{U_1(\mathbb{Z}),U_2(\mathbb{Z})}$ is commensurable to $H(\mathbb{Z})$. By Proposition 3.3.4, the orbit $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is closed. □

3.3.8. Since H is a normal subgroup of finite index in $\mathbf{H}(\mathbb{R})$, the following is another corollary of Theorem 3.3.2 and Proposition 3.3.4.

Lemma.

(1) $\mathbf{H}(\mathbb{R}).\Delta_{F_i}$ *is closed if and only if* $H.\Delta_{F_i}$ *is closed.*
(2) $\delta\mathbf{H}(\mathbb{R}).(\Delta_{F_1}, \Delta_{F_2})$ *is closed if and only if* $\delta H.(\Delta_{F_1}, \Delta_{F_2})$ *is closed.*

3.3.9. Proposition. *Suppose that* Γ_{F_1,F_2} *is discrete and that* $H.\Delta_{F_1}$ *and* $H.\Delta_{F_2}$ *are closed. If* $\overline{\delta(\mathbf{H}(\mathbb{R})).(\Delta_{F_1}, \Delta_{F_2})}$ *contains a closed orbit* $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ *and* Γ_{E_1,E_2} *is an irreducible lattice in* G*, then* $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$ *is closed.*

Proof. Let $\mathbf{H}(\mathbb{R}) = \bigcup_{i=1,..,n} g_i H$ and $\delta(H).(\Delta_{E_1}, \Delta_{E_2}) \subset \overline{\delta(\mathbf{H}(\mathbb{R})).(\Delta_{F_1}, \Delta_{F_2})}$ be a closed orbit. Then $\delta(H).(\Delta_{E_1}, \Delta_{E_2}) \subset \delta(H).(Ad_1(g_k)\Delta_{F_1}, Ad_2(g_k)\Delta_{F_2})$ for some $1 \leqq k \leqq n$. Let $\Delta'_{E_i} = Ad_i(g_k^{-1})\Delta_{E_i}$ for each $i = 1, 2$. Since H is a normal subgroup of $\mathbf{H}(\mathbb{R})$, we have that $\delta(H).(\Delta'_{E_1}, \Delta'_{E_2}) \subset \overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$. Since $g_k^{-1}(\Gamma_{E_1,E_2})g_k = \Gamma_{E'_1,E'_2}$, the subgroup $\Gamma_{E'_1,E'_2}$ is an irreducible lattice in G.

Therefore we may replace $\mathbf{H}(\mathbb{R})$ by H in the statement of the proposition. By Ratner's theorem, there exists a connected closed subgroup M of $SL(\mathcal{U}_1(\mathbb{R})) \times SL(\mathcal{U}_2(\mathbb{R}))$ containing $\delta(Ad(H))$ such that $\overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$ is $M.(\Delta_{F_1}, \Delta_{F_2})$. Since $H.\Delta_{F_1}$ and $H.\Delta_{F_2}$ are closed by the assumption, $M \subset H \times H$. Assume that $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$ is not closed. We claim that $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is open in $M.(\Delta_{E_1}, \Delta_{E_2})$. It is enough to show that for a sequence $(Ad_1(x_i)\Delta_{E_1}, Ad_2(y_i)\Delta_{E_2}), i \geq 1$ in $M.(\Delta_{E_1}, \Delta_{E_2})$ converging to $(\Delta_{E_1}, \Delta_{E_2})$, there exists $n \in \mathbb{N}$ such that $(Ad(x_i)\Delta_{E_1}, Ad(y_i)\Delta_{E_2}) \in \delta(H).(\Delta_{E_1}, \Delta_{E_2})$ for all $i > n$. Denote by Γ_i the subgroup generated by $x_i E_1 x_i^{-1}$ and $y_i E_2 y_i^{-1}$. Since $M.(\Delta_{E_1}, \Delta_{E_2})$ lies inside $M.(\Delta_{F_1}, \Delta_{F_2})$, each Γ_i is discrete by Proposition 3.2.55 and the sequence of Γ_i's converges to Γ_{E_1,E_2}. Therefore there exists a neighborhood V of the identity e such that $V \cap \Gamma_i = \{e\}$ (see the proof of Proposition 3.2.5). It is well known that an irreducible lattice in a semisimple Lie group with real rank at least 2 is finitely presentable and locally rigid (for example, see [11]). Let $h_1, h_2, ..., h_s$ be generators of Γ_{E_1,E_2} with relations $w_j(h_1, h_2, ..., h_s) = e$, $j = 1, 2, ..., t$. We may assume the generators lie in the set $E_1 \cup E_2$.

Denote by $h_{ik} \in x_i E_1 x_i^{-1} \cup y_i E_2 y_i^{-1}$ the element in Γ_i such that $x_i h_{ik} x_i^{-1} \to h_k$ if $h_k \in E_1$, and $y_i h_{ik} y_i^{-1} \to h_k$ if $h_k \in E_2$ as $i \to \infty$, for each $i \geqq 1$ and $k = 1, 2, ..., s$. Since the number of generators is finite, we can find a sufficiently large n such that $w_j(h_{i1}, h_{i2}, ..., h_{is}) \in V \cap \Gamma_i = \{e\}$ fgr each $j = 1, 2, ...t$ and $i > n$. Thus we can define a homomorphism $r_i : \Gamma_{E_1,E_2} \to \Gamma_i$ which carries h_k to h_{ik} for each $k = 1, 2, ..., s$ and for all $i > n$. Note that the sequence $\{r_i \mid i > n\}$ tends to the identity map id of Γ_{E_1,E_2} in the limit. It follows from the local rigidity of Γ_{E_1,E_2} that there is $n_0 > n$ such that r_i is an inner automorphism for each $i > n_0$. Now if $r_i = \text{int}\,(g_i)$ for $g_i \in G$, $g_i \in N(U_1) \cap N(U_2) = \mathbf{L}$, since $g_i E_1 g_i^{-1} \subset U_1(\mathbb{R})$ and $g_i E_2 g_i^{-1} \subset U_2(\mathbb{R})$.

Therefore $Ad(g_i) = Ad_1(x_i)$ and $Ad(g_i) = Ad_2(y_i)$, and hence $(x_i, y_i) \in (H \times H) \cap \delta(\mathbf{L}(\mathbb{R})) = \delta(H)$ for all $i > n_0$. It follows that $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is open. On the other hand $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is also closed by the assumption. Since both $\delta(H)$ and M are connected, $\delta(H) = M$. This contradiction establishes the closedness of the orbit $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$. □

3.3.10. Proposition. *There exists a lattice E_i in $U_i(\mathbb{R})$ such that $\Delta_{E_i} \in SL(\mathcal{U}_i(\mathbb{R})).\Delta_{F_i}$ and the orbit $H.\Delta_{E_i}$ is closed.*

Proof. By Proposition 1.4.2, there exists a $\mathbb{Q}$-form of $\mathbf{G}$ such that every parabolic $\mathbb{R}$-subgroup is defined over $\mathbb{Q}$. Therefore $N(U_1)$ and $N(U_2)$ are defined over $\mathbb{Q}$. We may assume that the determinant of $\Delta_{U_i(\mathbb{Z})}$ is the same as Δ_{F_i} by modifying the $\mathbb{Q}$-form by the inner automorphism by an element of $\mathbf{L}$. It is now enough to set $E_i = U_i(\mathbb{Z})$ to conclude the proof. □

3.3.11. Lemma. *Let H be an algebraic group, $H = L \ltimes N$, Λ a Zariski dense arithmetic subgroup of L and F a Zariski dense arithmetic subgroup of N which normalizes Λ. Then $\Lambda \ltimes F$ is a Zariski dense arithmetic subgroup of H.*

Proof. Consider the $\mathbb{Q}$-forms of L and N such that Λ and F are commensurable to $L(\mathbb{Z})$ and $N(\mathbb{Z})$, respectively . Since Λ normalizes F, $L(\mathbb{Q})$ normalizes $N(\mathbb{Q})$ and hence they define a $\mathbb{Q}$-form of H. The lemma now follows from the well-known fact that $L(\mathbb{Z})N(\mathbb{Z})$ is of finite index in $H(\mathbb{Z})$. □

3.3.12. Proposition. *Suppose that G is absolutely simple and that U_1 is either commutative or Heisenberg. Then the orbit $H.\Delta_{F_i}$ is closed if and only if there exists a $\mathbb{Q}$-form of G with respect to which U_i and $N(U_1) \cap N(U_2)$ are defined over $\mathbb{Q}$ and F_i is commensurable to $U_i(\mathbb{Z})$.*

Proof. Suppose that the orbit $H.\Delta_{F_i}$ is closed. By Proposition 3.3.4, there exists a $\mathbb{Q}$-form of H such that Λ_{F_i} is commensurable to $H(\mathbb{Z})$. Since F_i is a Zariksi dense arithmetic subgroup of U_i and Λ_{F_i} normalizes F_i, we obtain a $\mathbb{Q}$-form on $\mathbf{H}U_i$ Since $\mathbf{H}U_i = [N(U_i), N(U_i)]$, it follows from Proposition 1.5.3 that there exists a $\mathbb{Q}$-form of G which we are looking for, proving the claim since in the proof we see $N(U_1) \cap N(U_2)$ is defined over $\mathbb{Q}$.

Let G have a $\mathbb{Q}$-form of G with respect to which U_i and H are defined over $\mathbb{Q}$. If F_i is commensurable to $U_i(\mathbb{Z})$, then Λ_{F_i} is commensurable to $H(\mathbb{Z})$ and hence a lattice in H. It follows from Theorem 3.3.2 that $H.\Delta_{F_i}$ is closed. □

3.3.13. Lemma. *Let G be a semisimple $\mathbb{Q}$-group such that U_1 and $N(U_1) \cap N(U_2)$ are defined over $\mathbb{Q}$. Then U_2 is also defined over $\mathbb{Q}$.*

Proof. The map $P' \to P' \cap P$ is a bijection between the set of opposite parabolic subgroups to a parabolic subgroup P and the Levi subgroups of P ([5, 4.8]). It follows that an opposite parabolic subgroup P' to P is defined over $\mathbb{Q}$ if P and $P \cap P'$ are defined over $\mathbb{Q}$. Therefore $N(U_2)$ is defined over $\mathbb{Q}$; hence so is U_2. □

4. THE PROOF OF THE MAIN THEOREM

For the entire section 4, we assume that $\mathbf{G}$ is an adjoint absolutely simple algebraic $\mathbb{R}$-group with real rank at least 2 and U_1, U_2 is a pair of opposite horospherical $\mathbb{R}$-subgroups. We recall the notation $G = \mathbf{G}(\mathbb{R})^0$, $\mathbf{L} = N(U_1) \cap N(U_2)$, $\mathbf{H} = [\mathbf{L}, \mathbf{L}]$ and $H = \mathbf{H}(\mathbb{R})^0$ from section 3.2.1 and the terminology Property (A) from section 1.1.16.

4.1. Commutative horospherical subgroup cases

We are now ready to give the proof of the main theorem for commutative horospherical subgroup cases. A main feature in this case is the fact that $Ad_i(H)$ is a maximal connected closed subgroup of $SL(\mathcal{U}_i(\mathbb{R}))$; hence by Ratner's theorem $H.\Delta_{F_i}$ is either closed or dense in Ω_i. Moreover, unless G is of type A_n, $\delta(H)$ is a maximal connected closed subgroup of $Ad_1(H) \times Ad_2(H)$; hence $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$ is either closed or dense in $H.\Delta_{F_1} \times H.\Delta_{F_2}$. This feature facilitates finding a closed $\delta(H)$-orbit lying in $\overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$, which is enough to prove the main theorem by Proposition 3.3.6.

4.1.1. Theorem. *Let U_1, U_2 be a pair of commutative horospherical $\mathbb{R}$-subgroups of $\mathbf{G}$. Assume that the $\mathbb{R}$-form of $\mathbf{G}$ is not of type ${}^1E_{6,2}^{28}$ and that H has no compact factors. Then the triple $(\mathbf{G}, U_1, U_2)$ has property (A).*

Proof. Let F_1, F_2 be arbitrary lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively . To prove that Γ_{F_1,F_2} is an arithmetic subgroup, it is enough to find a closed orbit $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ lying in $\overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$ by Propositions 3.3.6 and 3.3.9.

We first claim that there exists $(\Delta_{E_1}, \Delta_{E_2}) \in \overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$ such that $H.\Delta_{E_1}$ and $H.\Delta_{E_2}$ are closed. Since $Ad_i(H)$ is a maximal connected closed subgroup of $SL(\mathcal{U}_i(\mathbb{R}))$ (see Proposition 1.3.6) it follows from Ratner's theorem that $\overline{H.\Delta_{F_1}}$ is either $H.\Delta_{F_1}$ or $SL(\mathcal{U}_i(\mathbb{R})).\Delta_{F_1}$. It then follows from Proposition 3.3.10 that $\overline{H.\Delta_{F_1}}$ contains a closed orbit $H.\Delta_{E_1}$ for some lattice E_1 in $U_1(\mathbb{R})$. By Proposition 3.2.8, there exists a lattice E_2 in $U_2(\mathbb{R})$ such that $(\Delta_{E_1}, \Delta_{E_2}) \in \overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$. If $H.\Delta_{E_2}$ is not closed, then let E_2' be a lattice in $U_2(\mathbb{R})$ such that $\Delta_{E_2'} \in \overline{H.\Delta_{E_2}}$ and $H.\Delta_{E_2'}$. Again we get a lattice E_1' in $U_1(\mathbb{R})$ such that $\Delta_{E_1'} \in \overline{H.\Delta_{E_1}}$ such that $(\Delta_{E_1'}, \Delta_{E_2'}) \in \overline{\delta(H).(\Delta_{E_1}, \Delta_{E_2})}$. Since $H.\Delta_{E_1}$ is closed, so is $H.\Delta_{E_1'}$, proving the claim.

Case (1): U_1 is reflexive. Since $H.\Delta_{E_1}$ is closed, by Proposition 3.3.12 there exists a $\mathbb{Q}$-form of G with respect to which U_1 and H defined over $\mathbb{Q}$ and E_1 is commensurable to $U_1(\mathbb{Z})$. Since U_2 is also defined over $\mathbb{Q}$ by Lemma 3.3.13, there exists $w \in G(\mathbb{Q})$ such that $wU_1(\mathbb{Q})w^{-1} = U_2(\mathbb{Q})$ by Lemma 1.2.2. By Lemma 2.2.5, we may assume that $E_2 = x(wE_1w^{-1})x^{-1}$ for some $x \in (\mathbf{L})(\mathbb{R})$. Write $x = yz$ for $y \in H(\mathbb{R})$ and $z \in Z(L)(\mathbb{R})$. Suppose that $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is not closed and denote by M a connected closed subgroup of $Ad_1(H) \times Ad_2(H)$ such that $\delta(H) \subset M$ and $\overline{\delta(H).(\Delta_{E_1}, \Delta_{E_2})} = M(\Delta_{E_1}, \Delta_{E_2})$ (Theorem 3.3.1).

Subcase (i): $M = Ad_1(H) \times Ad_2(H)$.

In that case, it is clear that $\overline{\delta(\mathbf{H}(\mathbb{R})).(\Delta_{E_1}, \Delta_{E_2})}$ is $\mathbf{H}(\mathbb{R}).\Delta_{E_1} \times \mathbf{H}(\mathbb{R}).\Delta_{E_2}$. Therefore we have found a closed orbit $\delta(H).(\Delta_{U_1(\mathbb{Z})}, \Delta_{zU_2(\mathbb{Z})z^{-1}})$ (by Corollary 3.3.7) lying inside $\overline{\delta(\mathbf{H}(\mathbb{R})).(\Delta_{E_1}, \Delta_{E_2})}$ up to commensurablity. it follows from Proposition 3.3.9 that $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is closed

Subcase (ii): M is a proper subgroup of $Ad_1(H) \times Ad_2(H)$.

If the absolute type of $\mathbf{G}$ is not A_n, then H is a simple group and hence $\delta(H)$ is a maximal closed connected subgroup of $Ad_1(H) \times Ad_2(H)$. Therefore the case when M is a proper subgroup of $Ad_1(H) \times Ad_2(H)$ can happen only when $\mathbf{G}$ has type A_n.

Let's now suppose that $\mathbf{G}$ has type A_n. From Fig 1.3.2, $\mathbf{H}$ is the product of two absolutely simple $\mathbb{R}$-groups, say, $\mathbf{N}_1 \times \mathbf{N}_2$. Also $Ad_1(\mathbf{H}) = \mathbf{N}_1 \otimes \mathbf{N}_2$ and $Ad_2(\mathbf{H}) = \mathbf{N}_2 \otimes \mathbf{N}_1$. There are only two proper connected closed subgroups of $Ad_1(H) \times Ad_2(H)$ which properly contain $\delta(H)$: $M_1 = \{(Ad_1(A,B), Ad_2(C,B)) \mid A, C \in \mathbf{N}_1(\mathbb{R})^0, B \in \mathbf{N}_2(\mathbb{R})^0\}$ and $M_2 = \{(Ad_1(A,B), Ad_2(A,D)) \mid A \in \mathbf{N}_1(\mathbb{R})^0, B, D \in \mathbf{N}_2(\mathbb{R})^0\}$. Assume that $M = M_1$. If $y \in H(\mathbb{R})$ is (a_1, a_2) where $a_i \in \mathbf{N}_i(\mathbb{R})$ for each $i = 1, 2$, then $(\Delta_{U_1(\mathbb{Z})}, Ad_2(a_2)Ad_2(z)\Delta_{U_2(\mathbb{Z})}) \in M_1.(\Delta_{E_1}, \Delta_{E_2})$. Recall that $E_1 = U_1(\mathbb{Z})$ and denote by E_2'' the lattice $(a_2z)U_2(\mathbb{Z})(a_2z)^{-1}$. We claim that the M_2-orbit of $(\Delta_{E_1}, \Delta_{E_2''})$ is closed. Observe that the subgroup $\{(Ad_1(A,B), Ad_2(A,D)) \mid Ad_1(A,B)\Delta_{E_1} = \Delta_{E_1}, Ad_2(A,D)\Delta_{E_2''} = \Delta_{E_2''}\}$ is equal to $\{(Ad_1(A,B), Ad_2(A, a_2Da_2^{-1})) \mid (A,B) \in$

$H(\mathbb{Z}), (A, D) \in H(\mathbb{Z})\}$. Since this group, which is the intersection of the stabilizer of $(\Delta_{E_1}, \Delta_{E_2''})$ and M_2, is a lattice in M_2, it follows that M_2-orbit of $(\Delta_{E_1}, \Delta_{E_2''})$ is closed. Since $M_1 \cap M_2 = \delta(H)$ and $M_1(\Delta_{E_1}, \Delta_{E_2''})$ and $M_2(\Delta_{E_1}, \Delta_{E_2''})$ are closed, $\delta(H).(\Delta_{E_1}, \Delta_{E_2''})$ is closed. It follows that $\Delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is closed since $\delta(H).(\Delta_{E_1}, \Delta_{E_2''}) \subset M.(\Delta_{E_1}, \Delta_{E_2})$. The case when $M = M_2$ can be dealt with in the same way.

This establishes the proof of the theorem when U_1 is not reflexive.

Case (2): U_1 is not reflexive.

First we observe that the only $\mathbb{Q}$-forms having a non-reflexive conjugacy class of a commutative horospherical $\mathbb{Q}$-subgroup are the following: ${}^1A_n$ with $\mathbb{Q}$-rank ≥ 2, D_{2m+1} with $\mathbb{Q}$-rank $2m+1$, ${}^1E_{6,2}^{28}$ and ${}^1E_{6,6}^{0}$ [21]. Since $H.\Delta_{E_1}$ and $H.\Delta_{E_2}$ are closed, there are $\mathbb{Q}$-forms $\bar{G}$ and $\tilde{G}$ of G such that E_1 and E_2 are commensurable to $U_1 \cap \bar{G}(\mathbb{Z})$ and $U_2 \cap \tilde{G}(\mathbb{Z})$ respectively. Suppose that the $\mathbb{Q}$-rank of $\bar{G}$ is less than or equal to that of $\tilde{G}$.

Subcase (i): $\bar{G}$ is split over $\mathbb{Q}$.

Then $\tilde{G}$ is also split over $\mathbb{Q}$. By the uniqueness of $\mathbb{Q}$-split forms of G, there exists a $\mathbb{Q}$-isomorphism between $\bar{G}$ and $\tilde{G}$. Since U_1 and U_2 are defined over $\mathbb{Q}$ with respect to both of the $\mathbb{Q}$-forms, we may assume that $\tilde{G}(\mathbb{Q}) = x\bar{G}(\mathbb{Q})x^{-1}$ for some $x \in N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$. It follows that E_2 is commensurable to $x(U_2 \cap \bar{G}(\mathbb{Z}))x^{-1}$. Let $x = yz$ for $z \in Z(N(U_1) \cap N(U_2))(\mathbb{R}))$ and $y \in \mathbf{H}(\mathbb{R})$. Then $(\Delta_{E_1}, \Delta_{yE_2y^{-1}}) \in H.\Delta_{E_1} \times H.\Delta_{E_2}$ while E_1 and yE_2y^{-1} are commensurable to $U_1 \cap \bar{G}(\mathbb{Z})$ and $z(U_2 \cap \bar{G}(\mathbb{Z}))z^{-1}$ respectively. By the same argument as in the previous case, we can show that $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is closed.

Subcase (ii): $\bar{G}$ is not split over $\mathbb{Q}$; hence the $\mathbb{Q}$-form of $\bar{G}$ has type either ${}^1A_n$ or ${}^1E_{6,2}^{28}$.

There exist a maximal torus $\bar{T}$ containing a maximal $\mathbb{Q}$-split torus $\bar{S}$ of $\bar{G}$ and a basis Δ of $\Phi(\bar{T}, G)$ such that $U_1 = V_{\Delta - \alpha}$ and $U_2 = V^-_{\Delta - \alpha}$ for some commutative root $\alpha \in \Delta$. Since U_1 is a $\mathbb{Q}$-subgroup of $\bar{G}$, by Proposition 1.4.1, α is fixed by the $*$-action (in the Tits index of $\bar{G}$) of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and $\alpha \notin \Delta_0$, Δ_0 the subset of Δ whose elements vanish on $\bar{S}$. Set $U_1' = V_{\Delta - \{\alpha, i(\alpha)\}}$ and $U_2' = V^-_{\Delta - \{\alpha, i(\alpha)\}}$ where i is the opposition involution on Δ (see 2.3.4). Since i commutes with the $*$-action of the Galois group, $i(\alpha)$ is also fixed by the $*$-action and $i(\alpha) \notin \Delta_0$. It follows that U_1' and U_2' are $\mathbb{Q}$-subgroups of $\bar{G}$.

Since U_1 is a $\mathbb{Q}$-subgroup of $\tilde{G}$, it is enough to note the fact that i is invariant under an isomorphism between two Dynkin diagrams to see that U_1' and U_2' are also $\mathbb{Q}$-subgroups of $\tilde{G}$. Therefore $E_1' = \bar{G}(\mathbb{Z}) \cap U_1'(\mathbb{R})$ and $E_2' = \tilde{G}(\mathbb{Z}) \cap U_2'(\mathbb{R})$ are lattices in $U_1'(\mathbb{R})$ and $U_2'(\mathbb{R})$ respectively . Note that for each $k = 1, 2$, $E_k' \cap U_k(\mathbb{R}) = E_k$ up to commensurability.

Set $i(U_1) = V_{\Delta - i(\alpha)}$ and $i(U_2) = V^-_{\Delta - i(\alpha)}$. Then $U_k \cap i(U_k) = Z(U'_k)$ for each $k = 1, 2$ (see the following lemma 4.1.2). In particular we have $Z(U'_k) \subset U_k$. Then $Z(U'_k(\mathbb{R})) \cap E_k$ is a lattice in $Z(U'_k(\mathbb{R}))$ for each $k = 1, 2$ since $Z(U'_k)$ is a $\mathbb{Q}$-subgroup with respect to the both $\mathbb{Q}$-forms $\bar{G}$ and $\tilde{G}$. Denote by $\mathbf{G_0}$ the algebraic $\mathbb{R}$-subgroup of $\mathbf{G}$ generated by $Z(U'_1)$ and $Z(U'_2)$, $G_0 = \mathbf{G_0}(\mathbb{R})^0$ and by Γ_0 the subgroup generated by $Z(U'_1(\mathbb{R})) \cap E_1$ and $Z(U'_2(\mathbb{R})) \cap E_2$. Note that Γ_0 is discrete since $\Gamma_0 \subset \Gamma_{E_1, E_2}$.

By Proposition 2.3.4, $\mathbf{G}_0$ is an absolutely simple algebraic $\mathbb{R}$-group and $Z(U'_1)$ and $Z(U'_2)$ are opposite commutative horospherical $\mathbb{R}$-subgroups of $\mathbf{G}_0$. Since i obviously preserves the set $\{\alpha, i(\alpha)\}$, U'_1 is reflexive and it follows that $Z(U'_1)$ is also reflexive.

We now claim that the commutator subgroup, say S', of $N(Z(U'_1)) \cap N(Z(U'_2)) \cap \mathbf{G}_0$ has no $\mathbb{R}$-anisotropic factors. Recall from Proposition 2.5.5 that S' is a semisimple normal algebraic $\mathbb{R}$-subgroup of $N(U'_1) \cap N(U'_2)$. If the absolute type of $\mathbf{G}$ is E_6, the $\mathbb{R}$-form of G is split by the assumption that the $\mathbb{R}$-form of $\mathbf{G}$ is not ${}^1E^{28}_{6,2}$,and hence so are the $\mathbb{R}$-forms of $\mathbf{G}_0$ and S', proving the claim in that case. For reference the type of $\mathbf{G}_0$ in this case is D_5. Now if the absolute type of $\mathbf{G}$ is A_n and hence $\mathbf{H}$ is $H_1 \times H_2$ where respectively H_1 and H_2 are of the type A_k and A_{n-k-1}, $k > 1$ (from the assumption that $\bar{G}$ is not $\mathbb{Q}$-split), respectively . In that case G_0 is of type A_m for some $m < n$ and the type of S' is $H_j \times H_j$ where $j = 1$ if $k < n - k - 1$; otherwise $j = 2$. The claim follows from the assumption that $\mathbf{H}$ has no $\mathbb{R}$-anisotropic factors.

It also follows that the $\mathbb{R}$-rank of $\mathbf{G}$ is at least 2 since the $\mathbb{R}$-rank of S' is at least 2.

Therefore there exists a $\mathbb{Q}$-form of $\mathbf{G}_0$ with respect to which Γ_0 is commensurable to $\mathbf{G}_0(\mathbb{Z})$ by the previous result (case (1)) on the case of reflexive horospherical subgroup cases. Note that $\Gamma_0 \subset \Gamma_{E'_1, E'_2}$ and $\Gamma_{E'_1, E'_2} \cap S'(\mathbb{R}) = S' \cap \mathbf{G}_0(\mathbb{Z})$ is a Zariski dense arithmetic subgroup of S' since $S'(\mathbb{R})$ has no compact factors. It follows from Proposition 2.4.2 that there exists a $\mathbb{Q}$-form of $\mathbf{G}$ such that $\Gamma_{E'_1, E'_2} \subset \mathbf{G}(\mathbb{Q})$. Since Γ_{E_1, E_2} is a discrete subgroup contained in $\mathbf{G}(\mathbb{Q})$, by Proposition 2.4.5 it is an arithmetic subgroup, or equivalently, $\delta(H)(\Delta_{E_1}, \Delta_{E_2})$ is closed. □

4.1.2. We now give a proof of the lemma used in the proof of the above theorem.

Lemma. *$U_k \cap i(U_k) = Z(U'_k)$ for each $k = 1, 2$.*

Proof. It suffices to prove that $(\Phi^+ - [\Delta - \alpha]) \cap (\Phi^+ - [\Delta - i(\alpha)]) = \{\beta \in \Phi^+ - [\Delta - \{\alpha, i(\alpha)\}] \mid \beta + \gamma \notin \Phi^+$ for any $\gamma \in \Phi^+ - [\Delta - \{\alpha, i(\alpha)\}]\}$. That the coefficient of β is non-zero with respect to each α and $i(\alpha)$ implies that the coefficient is 1 in each case by Lemma 1.3.1 and by the assumption that α is a commutative root hence so is $i(\alpha)$. Since the highest root has the coefficient 1 with respect to each α and $i(\alpha)$, $\beta + \gamma$ cannot be a root for any $\gamma \in {}_{\mathbb{R}}\Phi^+ - [\Delta - \{\alpha, i(\alpha)\}]$, proving the inclusion $\subset$. Suppose that $U_\beta \subset Z(U'_1)$ and the coefficient of β with respect to α is zero. There exist simple roots

$\alpha_{i_1}, ..., \alpha_{i_m}$ such that $\beta+\alpha_{i_1}+...+\alpha_{i_j} \in {}_{\mathbb{R}}\Phi^+$ for each $j = 1, 2, ..., m$ and $\beta+\alpha_{i_1}+...+\alpha_{i_m}$ is the highest root. Then for some j, $\alpha_{i_j} = \alpha$. Since $\alpha \in \Phi^+ - [\Delta - \{\alpha, i(\alpha)\}]$ and $U_{\beta+\alpha_{i_1}+...\alpha_{i_{j-1}}} \subset Z(U_1')$, we obtain a contradiction to the assumption $U_\beta \subset Z(U_1')$. Similary we can show that the coefficient of β with respect to $i(\alpha)$ is non-zero. This proves the other inclusion relation. □

4.1.3. Remark. The case when H does have compact factors occurs only when the $\mathbb{R}$-type of G is ${}^1A_n$ and U_1 is conjugate to the minimal parabolic subgroup V_Θ where Θ is either $\Delta - \alpha_1$ or $\Delta - \alpha_n$.

4.1.4. Remark. We remark that in [14], we proved the closedness of the orbits $H.\Delta_{F_1}$ and $H.\Delta_{F_2}$ in a direct way, in the case of $G(\mathbb{R}) = SL_n(\mathbb{R})$ and this method works for the $\mathbb{R}$-split groups of classical types

4.2. Heisenberg horospherical subgroup cases

4.2.1. In this section, let U_1 and U_2 be a pair of opposite Heisenberg horospherical $\mathbb{R}$-subgroups. We recall the representations $Ad_i : \mathbf{L} \to GL(\mathcal{U}_i)$ from 1.2.12 and $Ad_i' : \mathbf{H} \to SL(\mathcal{V}_i)$ from 1.3.7. Denote by π the projection of $\mathcal{U}_i$ onto $\mathcal{V}_i$.

The following lemma easily follows from Lemma 2.1.10.

Lemma. *Let F_i be a lattice in $U_i(\mathbb{R})$ and $\Delta'_{F_i} = \pi(\Delta_{F_i})$. Then $\Delta_{F_i} \cap Z(\mathcal{U}_i)$ and Δ'_{F_i} are lattices in $Z(\mathcal{U}_i)$ and $\mathcal{V}_i$ respectively.*

The group H acts through Ad_i and Ad_i' on the space of lattices in $\mathcal{U}_i$ and $\mathcal{V}_i$, respectively. The notation $H.J_i$ and $H.J_i'$ denotes the orbits $Ad_i(H)(J_i)$ and $Ad_i'(H)(J_i')$, respectively, under this action.

4.2.2. To prove the main theorem in this case, we investigate the closures of the orbits $H.\Delta_{F_i}$ and $\delta(H).(\Delta_{F_1}, \Delta_{F_2})$ as we did in section 4.1.1. Unlike the case when U_1 and U_2 are commutative, we have various closed subgroups between $Ad_i(H)$ and $SL(\mathcal{U}_i(\mathbb{R}))$. The key point of narrowing the possibilities for the closures of those orbits will be that (1) we may assume that the closures are the homogeneous spaces under semisimple subgroups of $SL(\mathcal{U}_i(\mathbb{R})$, after replacing Δ_{F_1} and Δ_{F_2} by suitable lattices inside the closures and (2) $Ad_i'(H)$ is a maximal connected closed (resp. semisimple) subgroup of $Sp(\mathcal{V}_i(\mathbb{R}))$ if G is not of type A_n (resp. if G is of type A_n).

The following two propositions will contribute to the process (1).

Proposition. [20, Proposition 3.2] *Let $\mathbf{G} \subset SL_n(\mathbb{C})$ be an algebraic $\mathbb{Q}$-group, $G = \mathbf{G}^0_{\mathbb{R}}$, $\Gamma = G \cap SL_n(\mathbb{Z})$ and H a subgroup generated by algebraic unipotent one-parameter subgroups of G contained in H. Suppose that $\overline{H.\Gamma} = M.\Gamma$ for a connected Lie subgroup*

M of G. Let $\mathbf{M}$ be the smallest algebraic $\mathbb{Q}$-subgroup containing M. Then the radical of $\mathbf{M}$ is a unipotent algebraic $\mathbb{Q}$-subgroup and $M = \mathbf{M}^0_{\mathbb{R}}$. Furthermore if $\mathbf{M} = \mathbf{SU}$, $\mathbf{U} = R_u(\mathbf{M})$ is an $\mathbb{R}$-Levi- decomposition of $\mathbf{M}$, then $M = \mathbf{S}(\mathbb{R})^0\mathbf{U}(\mathbb{R})^0$.

In particular we note that the Levi component of $\mathbf{M}$ is semisimple in the above proposition since the radical of $\mathbf{M}$ is unipotent.

4.2.3. Proposition. *With the same notation as the above proposition, there exists an element $u \in \mathbf{U}(\mathbb{R})$ such that the orbit $S(\mathbb{R})^0.(u\Gamma)$ is closed.*

Proof. Since $\mathbf{M}$ is defined over $\mathbb{Q}$, there exists $u \in \mathbf{U}(\mathbb{R})$ such that $u\mathbf{S}u^{-1}$ is defined over $\mathbb{Q}$. It follows that $u^{-1}\mathbf{S}(\mathbb{R})^0 u.\Gamma$ is closed. Therefore its left-translation by u, $\mathbf{S}(\mathbb{R})^0.(u\Gamma)$, is closed. □

4.2.4. Since $[\mathcal{V}_i, \mathcal{V}_i] = Z(\mathcal{U}_i)$, we have a skew-symmetric bilinear product $<,>: \mathcal{V}_i(\mathbb{R}) \times \mathcal{V}_i(\mathbb{R}) \to Z(\mathcal{U}_i(\mathbb{R}))$ defined by $< v, w >= [v, w]$. We set $Sp(\mathcal{V}_i) = \{g \in SL(\mathcal{V}_i) \mid < gv, gw >=< v, w > \text{ for all } v, w \in \mathcal{V}_i\}$.

Lemma. *Let F_i be a lattice in $U_i(\mathbb{R})$ and $\Phi(F_i) = \{g \in SL(\mathcal{V}_i) \mid < g(v), g(w) > \in \Delta_{F_i} \cap Z(\mathcal{U}_i)$ for all $v, w \in \Delta'_{F_i}\}$. Then the connected component of identity in $\Phi(F_i)$ is $Sp(\mathcal{V}_i)$*

Proof. In a Lie group, the connected components coincide with the path-wise connected components. Denote by Φ° the path-wise connected component of the identity in $\Phi(F_i)$ and let $g \in \Phi^\circ$. Then there exists a continuous family $g_t \in \Phi^\circ, t \in [0, 1]$, such that $g_0 = e$ and $g_1 = g$. For any $v, w \in \Delta'_{F_i}$, $< g_t(v), g_t(w) > \in \Delta_{F_i} \cap Z(\mathcal{U}_i)$. Since $\Delta_{F_i} \cap Z(\mathcal{U}_i)$ is discrete, it follows that $< g_t(v), g_t(w) >=< v, w >$ for all $t \in [0, 1]$. In particular, it implies that $< g(v), g(w) >=< v, w >$ for all $v, w \in Z(\mathcal{U}_i)$ since Δ'_{F_i} is a lattice in $\mathcal{V}_i$. Hence $g \in SP(\mathcal{V}_i)$. □

4.2.5. Denote by P_i the parabolic subgroup of $SL(\mathcal{U}_i)$ which stabilizes the line $Z(\mathcal{U}_i)$. Then $[P_i, P_i]$ is isomorphic to $SL(\mathcal{V}_i)W_i$ where $W_i = R_u(P_i)$.

Lemma.

(1) $\overline{H.\Delta'_{F_i}} \subset Sp(\mathcal{V}_i(\mathbb{R})).\Delta'_{F_i}$.
(2) $\overline{H.\Delta_{F_i}} \subset Sp(\mathcal{V}_i(\mathbb{R}))W_i(\mathbb{R}).\Delta_{F_i}$.

Proof. By Ratner's theorem, there exists a connected closed subgroup $L'_i \subset SL(\mathcal{V}_i(\mathbb{R}))$ containing $Ad'_i(H)$ such that $\overline{H.\Delta'_{F_i}} = L'_i.\Delta'_{F_i}$. Since $Ad'_i(H) \subset Sp(V_i(\mathbb{R}))$, it is not difficult to see that $L'_i \subset \Phi(F_i)$ and hence (1) follows from Lemma 4.2.4. Since $\Delta_{F_i} \cap Z(\mathcal{U}_i)$ is a lattice in $Z(\mathcal{U}_i)$, it follows that P_i is defined over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-form of $SL(\mathcal{U}_i)$ given by Δ_{F_i}. Then (2) follows from (1) and Proposition 4.2.2. □

4.2.6. If $\mathbf{G}$ is not of type A_n, the orbit $H.\Delta'_{F_i}$ is either closed or dense in $Sp(\mathcal{V}_i(\mathbb{R})).\Delta'_{F_i}$ by Lemma 4.2.5. The following lemma, which is an immdediate corollary of Proposition 1.3.7 and Lemma 4.2.5, gives the possibilities for the closure of the orbit $H.\Delta_{F_1}$ for the groups of type A_n. For the notation P and P^*, see Proposition 1.3.7.

Lemma. *Suppose that the type of* $\mathbf{G}$ *is* A_n *and that* $\overline{H.\Delta_{F_i}} = M.\Delta_{F_i}$. *Then* $M = M_0$ *or* M_0W_i *where* M_0 *is one of the followings:*

$$Ad'_i(H),\ Ad'_i(H)R_u(P)(\mathbb{R}),\ Ad'_i(H)R_u(P^*)(\mathbb{R}),\ Sp(\mathcal{V}_i(\mathbb{R})).$$

4.2.7. Proposition. *For any lattice* E_i *in* $U_i(\mathbb{R})$*, the orbit* $Sp(\mathcal{V}_i(\mathbb{R})).\Delta'_{E_i}$ *is closed.*

Proof. Similarly to proof of Lemma 4.2.4, we can show that if $Sp(\mathcal{V}_i(\mathbb{R})).\Delta'_{E_1} = M'.\Delta'_{E_1}$ for some connected closed subgroup M' of $SL(\mathcal{V}_i(\mathbb{R}))$, $M' \subset Sp(\mathcal{V}_i(\mathbb{R}))$. Therefoe $M' = Sp(\mathcal{V}_i(\mathbb{R})).\Delta'_{E_1}$, proving that the orbit is closed. □

4.2.8. Proposition. *If* $Sp(\mathcal{V}_i(\mathbb{R})).\Delta_{F_i}$ *is closed, it contains a closed orbit* $H.\Delta_{E_i}$.

Proof. By Proposition 1.4.2, there exists a $\mathbb{Q}$-form of G such that U_1 and U_2 are defined over $\mathbb{Q}$. Set $E_i = U_i(\mathbb{Z})$. Since $\Lambda_{E_i} = H(\mathbb{Z})$ is a lattice in H, $H.\Delta_{E_i}$ is closed. We will show that Δ_{E_i} is contained in $Sp(\mathcal{V}_i(\mathbb{R})).\Delta_{F_i}$. It is enough to show that Δ'_{E_i} is contained in $Sp(\mathcal{V}_i(\mathbb{R})).\Delta'_{F_i}$. Fixing an isomorphism of $\mathcal{V}_i(\mathbb{R})$ to $\mathbb{R}^n$, let J_i be the standard lattice $\mathbb{Z}^n$. Since $Sp(\mathcal{V}_i(\mathbb{R})).\Delta'_{F_i}$ is closed, it follows that $\Delta_{F'_i}$ is commenuarbele to $gSP_n(\mathbb{Z})g^{-1}$ for some $g \in Sp(\mathcal{V}_i(\mathbb{R}))$. It implies that Δ'_{F_i} is commensurable to $g\mathbb{Z}^n$. For the same reason, Δ'_{E_i} is commensurable to $h\mathbb{Z}^n$ for some $h \in Sp(\mathcal{V}_i(\mathbb{R}))$, proving the claim. □

4.2.9. Proposition. $\overline{H.\Delta_{F_i}}$ *contains a closed orbit* $H.\Delta_{E_i}$.

Proof. By Ratner's theorem, there exists a connected closed subgroup M such that $\overline{H.\Delta_{F_i}} = M.\Delta_{F_i}$ By Propositions 4.2.3, we may assume that M is a semisimple subgroup of $Sp(\mathcal{V}_i(\mathbb{R}))$. Then M is either $Ad_i(H)$ or $Sp(\mathcal{V}_i(\mathbb{R}))$. Therefore the claim follows from Proposition 4.2.8. □

4.2.10. Lemma. *If* $H.\Delta_{F_i}$ *is closed, then* $\Delta_{F_i} \cap \mathcal{V}_i(\mathbb{R})$ *is a lattice in* $\mathcal{V}_i(\mathbb{R})$*. Furthermore* Δ_{F_i} *is, up to commensurability, determined by* $\Delta_{F_i} \cap \mathcal{V}_i(\mathbb{R})$.

Proof. It follows from the $\mathbb{Q}$-rationality of Ad_i that $\mathcal{V}_i$ is a $\mathbb{Q}$-subspace with respect to the $\mathbb{Q}$-structure of $\mathcal{U}_i$ given by Δ_{F_i} (see the proof of Proposition 1.5.5). It implies that $\Delta_{F_i} \cap \mathcal{V}_i(\mathbb{R})$ is Zariski dense and hence is a lattice in $\mathcal{V}_i(\mathbb{R})$. The second claim follows from the fact that $[\Delta_{F_i} \cap \mathcal{V}_i(\mathbb{R}), \Delta_{F_i} \cap \mathcal{V}_i(\mathbb{R})]$ is commensurable to $Z(\mathcal{U}_i) \cap \Delta_{F_i}$. □

4.2.11. Theorem. *Let U_1, U_2 be a pair of opposite Heisenberg horospherical $\mathbb{R}$-subgroups. Suppose that H is non-trivial and has no compact factors. Then $(\mathbf{G}, U_1, U_2)$ has property (A).*

Proof. By the same argument as in Theorem 4.1.3, it is enough to find a closed orbit $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ inside $\overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$. By Proposition 4.2.9, $\overline{H.\Delta_{F_1}}$ contains a closed orbit $H.\Delta_{E_1}$. By the same argument as the proof of Theorem 4.1.3, we may assume that there exists a lattice E_2 such that $H.\Delta_{E_2}$ is closed and $(\Delta_{E_1}, \Delta_{E_2}) \in \overline{\delta(H).(\Delta_{F_1}, \Delta_{F_2})}$. Therefore there exists a $\mathbb{Q}$-form of G with respect to which U_1, U_2 and H are defined over $\mathbb{Q}$ and E_1 is commensurable to $U_1(\mathbb{Z})$.

Assume that the orbit $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is not closed. Denote by M a connected closed subgroup of $Ad_1(H) \times Ad_2(H)$ such that $\delta(H) \subset M$ and $\overline{\delta(H).(\Delta_{E_1}, \Delta_{E_2})} = M(\Delta_{E_1}, \Delta_{E_2})$.

Case (1). $M = Ad_1(H) \times Ad_2(H)$. Since a Heisenberg horospherical subgroup is reflexive (see remark ?), there exists $w \in G(\mathbb{Q})$ such that $wU_1(\mathbb{Q})w^{-1}$ is conjugate to $NU_2(\mathbb{Q})$. By Lemma 2.3.3, we may assume that for some $x \in N(U_1)(\mathbb{R}) \cap N(U_2)(\mathbb{R})$, $\Delta_{E_2} \cap \mathcal{V}_2$ is commensurable to $\Delta_{xU_2(\mathbb{Z})x^{-1}} \cap \mathcal{V}_2$. By Lemma 2.3.3, Δ'_{E_2} is commensurable to $\Delta'_{xU_2(\mathbb{Z})x^{-1}}$. Write x as yz where $y \in H(\mathbb{R})$ and $z \in Z(L)(\mathbb{R})$ according to Levi decomposition. Then $(\Delta_{U_1(\mathbb{Z})}, \Delta_{y^{-1}E_2 y}) \in \overline{\delta(\mathbf{H}(\mathbb{R})).(\Delta_{E_1}, \Delta_{E_2})}$. Since the H-orbits of $\Delta_{y^{-1}E_2 y}$ and $\Delta_{zU_2(\mathbb{Z})z^{-1}}$ are closed and their projections onto $\mathcal{V}_i(\mathbb{R})$ are commensurable, by Lemma 4.2.10, $(\Delta_{U_1(\mathbb{Z})}, \Delta_{zU_2(\mathbb{Z})z^{-1}}) \in \overline{\delta(\mathbf{H}(\mathbb{R})).(\Delta_{E_1}, \Delta_{E_2})}$ up to commensurablity, contradicting that $\delta(H).(\Delta_{E_1}, \Delta_{E_2})$ is not closed by Proposition 3.3.7.

Case (2). M is a proper subgroup of $Ad_1(H) \times Ad_2(H)$.

This case happens only when the type of G is B_n or D_n; otherwise H is a simple Lie group, and hence $\delta(H)$ is a maximal connected closed subgroup of $Ad_1(H) \times Ad_2(H)$. Since in this case H is a product of two simple groups, by the same argument as case (1-ii) in Theorem 4.1.1, we can show that $\overline{\delta(H).(\Delta_{E_1}, \Delta_{E_2})}$ contains a closed $\delta(H)$-orbit. This concludes the proof. □

4.2.12. Remark. 1. The subgroup H is trivial only when G is $\mathbb{R}$-split of type A_2.

2. The subgroup H has compact factors only when G is of the (absolute) type A_n, B_n or D_n with the real rank 2.

4.3. Non $\mathbb{R}$-Heisenberg horospherical subgroup cases

We deal with non-$\mathbb{R}$-Heisenberg subgroup cases by reducing into commutative horospherical subgroup cases. If $Z(U_1)$ is not the root group of a highest real root, this can be done by considering $Z(U_1)$ and $Z(U_2)$. Otherwise we show that we can replace

U_1, U_2 by another pair U_1', U_2' of oppsoite horospherical subgroups such that $Z(U_1')$ is not a root group of a highest real root (Theorem 4.3.4).

4.3.1. In section 4.3.1-2, we assume that G is almost $\mathbb{R}$-simple. We define the pair of horospherical $\mathbb{R}$-subgroups U_1', U_2' mentioned in the beginning of this section and prove their needed properties.

For each $i = 1, 2$, set $\mathcal{U}_i = \mathrm{Lie}(U_i)$, $\tilde{\mathcal{U}}_i = \{X \in \mathcal{U}_i \mid [\mathcal{U}_i, X] \subset Z(\mathcal{U})\}$ and $\mathcal{U}_i' = \{X \in \mathcal{U}_i \mid [X, \tilde{\mathcal{U}}] = 0\}$. Denote by $\tilde{U}_i$ and U_i' the connected subgroups of U_i such that Lie $(\tilde{U}_i) = \tilde{\mathcal{U}}_i$ and Lie $(U_i') = \mathcal{U}_i'$. Equivalently, $\tilde{U}_i = \{u \in U \mid gug^{-1}u^{-1} \in Z(U_i)$ for all $g \in U_i\}$ and U_i' the centralizer of $\tilde{U}_i$ in U_i.

Proposition.

(1) *The subgroups $\tilde{U}_i$ and U_i' are $\mathbb{R}$-subgroups of U_i.*
(2) *If U_i is defined over $\mathbb{Q}$, then $\tilde{U}_i$ and U_i' are $\mathbb{Q}$-subgroups.*
(3) *If F_i is a lattice in $U_i(\mathbb{R})$, $F_i \cap U_i'(\mathbb{R})$ is a lattice in $U_i'(\mathbb{R})$.*

Proof. It is well known that the normal subgroups in the ascending cetral series of a unipotent $\mathbb{R}$-group are defined over $\mathbb{Q}$. Hence $\tilde{U}_i$ is defined over $\mathbb{R}$. Since U_i' is the centrailzer of $\tilde{U}_i$ in U_i, it is defined over $\mathbb{R}$. This argument works in the same way for the field $\mathbb{Q}$. (3) is directly follows from (2) and Lemma 2.1.11. □

4.3.2. The notation in this section is the same as in section 2.5.1 with $k = \mathbb{R}$. Set $\Psi_1 = \{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid$ the coefficient of α with respect to ${}_{\mathbb{R}}\Delta - (\Theta \cup {}_{\mathbb{R}}\Delta_H)$ is $0\}$ and $\Psi_2 = ({}_{\mathbb{R}}\Phi^+ - [\Theta]) - \Psi_1$. Then Ψ_1 and Ψ_2 are closed sets of ${}_{\mathbb{R}}\Phi^+$. It is easy to see that $\Psi_2 = {}_{\mathbb{R}}\Phi^+ - [\Theta \cup {}_{\mathbb{R}}\Delta_H]$.

To simplify the notation, let ${}_{\mathbb{R}}\Delta = \Delta$ and ${}_{\mathbb{R}}\Delta_H = \Delta_H$ in the following proposition.

Proposition. *For $U_1 = {}_{\mathbb{R}}V_\Theta$ and $U_2 = {}_{\mathbb{R}}V_\Theta{}^-$, suppose that $Z(U_1) = U_{\alpha_h}$ and $[U_1, U_1] \neq Z(U_1)$. Then*

(1) $\tilde{U}_1 = U_\psi$ *where* $\psi = \{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid \alpha_h - \alpha \in \Psi_1\}$.
(2) $U_1' = {}_{\mathbb{R}}V_{\Theta \cup \Delta_H}$, $U_2' = {}_{\mathbb{R}}V_{\Theta \cup \Delta_H}{}^-$. *In particular, U_1' and U_2' are opposite horospherical $\mathbb{R}$-subgroups of G and $Z(U_1')$ is strictly bigger than U_{α_h}.*

Proof. The last claim directly follows from Lemma 2.5.3. Since $Z(U_1) = U_{\alpha_h}$, we have that $\Delta_H \subset \Theta^c$ and that $\Delta - (\Theta \cup \Delta_H)$ is non-empty by the hypothesis that $U_1 \neq [U_1, U_1]$. We observe that $\{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid U_\alpha \subset \tilde{U}_1\} = \{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid$ if $\alpha + \beta \in {}_{\mathbb{R}}\Phi^+$ for $\beta \in {}_{\mathbb{R}}\Phi^+ - [\Theta]$, then $\alpha + \beta = \alpha_h\}$, which we will denote by $\tilde{\Phi}$. Also $\{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid U_\alpha \subset U_1'\} = \{\alpha \in {}_{\mathbb{R}}\Phi^+ - [\Theta] \mid \alpha + \beta \notin {}_{\mathbb{R}}\Phi^+$ for all β with $U_\beta \subset \tilde{U}_1\}$, which we will denote by Φ'. It is enough to show that $\tilde{\Phi} = \psi$ for (1) and $\Phi' = \Psi_2$ for (2).

(1): to show that $\psi \subset \tilde{\Phi}$, let $\beta = \alpha_h - \alpha$ with $\alpha \in \Psi_1$. If $\alpha + \gamma \in {}_{\mathbb{R}}\Phi^+$ for some $\gamma \in {}_{\mathbb{R}}\Phi^+ - [\Theta]$, then the coefficient of γ with respect to each simple $\mathbb{R}$-root in $\Delta - (\Theta \cup \Delta_H)$ must be 0. Since $\gamma \in {}_{\mathbb{R}}\Phi^+ - [\Theta]$, the sum of the coefficients of γ with respect to Δ_H should be non-zero and hence 1. Therefore the sum of all coefficients of $\beta + \gamma$ with respect to Δ_H is 2, yielding $\beta + \gamma = \alpha_h$ by Proposition 1.3.1.

To show the converse, assume that $\alpha \in \tilde{\Phi}$. If $\alpha \notin \psi$, then $\mathrm{ht}(\alpha) \leqq \mathrm{ht}(\alpha_h) - 2$. We can find $\mathbb{R}$-simple roots $\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_k}$ (not necessarily different) such that $\alpha + \alpha_{i_1} + \cdots + \alpha_{i_j} \in {}_{\mathbb{R}}\Phi^+$ for each $j = 1, 2, ..., k$ and $\alpha + \alpha_{i_1} + \cdots + \alpha_{i_k} = \alpha_h$. Let j be the smallest number such that $\alpha_{i_j} \in \Delta - \Theta$. Then $j < k$ since $\alpha \notin \psi$. Set $\alpha' = \alpha + \alpha_{i_1} + \cdots + \alpha_{i_{j-1}}$. It follows from the property of ascending series of U that if $U_\alpha \subset \tilde{U}$ and $\alpha + \alpha_m \in {}_{\mathbb{R}}\Phi^+$ for $\alpha_m \in \Delta$, then $U_{\alpha + \alpha_m} \subset \tilde{U}$. Therefore by induction we have $U_{\alpha'} \subset \tilde{U}$. But $\alpha' + \alpha_{i_j} \in {}_{\mathbb{R}}\Phi^+$, $\alpha_{i_j} \in {}_{\mathbb{R}}\Phi^+ - [\Theta]$ while $\alpha' + \alpha_{i_j} \neq \alpha_h$. Therefore $\alpha' \notin \tilde{\Phi}$, contradicting $U_{\alpha'} \subset \tilde{U}$.

(2): to show that $\Psi_2 \subset \Phi'$, assume $\alpha \in \Psi_2 - \Phi'$. Then $\alpha + \beta \in {}_{\mathbb{R}}\Phi^+$ for some $\beta \in \tilde{\Phi}$. Therefore $\alpha + \beta = \alpha_h$. Since $\beta = \alpha_h - \alpha \in \psi$ by (1), $\alpha \in \Psi_1 = {}_{\mathbb{R}}\Phi^+ - \Psi_2$, contradicting the assumption.

To show the converse, suppose that $\alpha \in \Phi' - \Psi_2$. Since $\Delta - (\Theta \cup \Delta_H)$ is non-empty and $\alpha \in \Psi_1$, the sum of the coefficients of α with respect to Δ_H is 1 and and there exists a simple root in Δ with respect to which the coefficient of α is 0. It follows from Lemma 2.5.1 that $\alpha_h - \alpha \in {}_{\mathbb{R}}\Phi^+$. Hence $\alpha_h - \alpha \in \psi$; $U_{\alpha_h - \alpha} \subset \tilde{U}$. But $\alpha = \alpha + (\alpha_h - \alpha)$, contradicting the assumption that $\alpha \subset \tilde{\Phi}$ by (1). □

4.3.3. Proposition. *Let G be a connected adjoint $\mathbb{R}$-simple algebraic $\mathbb{R}$-group with real rank at least 2 and U_1, U_2 a pair of opposite horospherical $\mathbb{R}$-subgroups. Let the notation G_0, $\mathbf{H}$ and S be the same as Proposition 2.5.4 and $Z_i = Z(U_i)$ for each $i = 1, 2$. Suppose that S is non-trivial and has no $\mathbb{R}$-anisotropic factors. If the triple (G_0, Z_1, Z_2) has property (A), then so does (G, U_1, U_2).*

Proof. By Proposition 2.5.4, G_0 is a connected almost $\mathbb{R}$-simple $\mathbb{R}$-group with real rank at least 2 and S is a semisimple normal algebraic $\mathbb{R}$-group of $\mathbf{H}$. Now let F_1 and F_2 be lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively such that the subgroup Γ_{F_1,F_2} generated by them is discrete. Note that $Z_i(\mathbb{R}) \cap F_i$ is a lattice in $Z_i(\mathbb{R})$ for $i = 1, 2$, by Lemma 2.1.10. By the assumption that (G_0, Z_1, Z_2) has property (A), there exists a $\mathbb{Q}$-form of G_0 with respect to which Z_1 and Z_2 are defined over $\mathbb{Q}$ and the subgroup Γ_0 generated by $Z_1 \cap F_1$ and $Z_2 \cap F_2$ is commensurable to the subgroup $G_0(\mathbb{Z})$. Since $S(\mathbb{R})$ has no compact factors, $S \cap \Gamma_0(\mathbb{Z}) \subset \Gamma_{F_1,F_2}$ is a Zariski dense arithmetic subgroup of $S(\mathbb{R})$. Therefore by Corollary 2.4.5, Γ_{F_1,F_2} is an arithmetic subgroup of G. □

4.3.4. Theorem. *Let* $\mathbf{G}$ *be an adjoint absolutely simple algebraic* $\mathbb{R}$*-group with real rank at least* 2 *and* U_1, U_2 *a pair of opposite horospherical* $\mathbb{R}$*-subgroups which satisfies either of the following condition:*

(1) $[U_1, U_1] = Z_1$, Z_1 *is not the root group of a highest real root, the commutator subgroup of* $N(U_1) \cap N(U_2) \cap G_0$ *has no* $\mathbb{R}$*-anisotropic factors and the* $\mathbb{R}$*-form of* G_0 *is not of type* ${}^1E_{6,2}^{28}$*, where* G_0 *is the subgroup generated by* Z_1 *and* Z_2*.*

(2) $[U_1, U_1] \neq Z_1$, Z_1 *is the root group of a highest real root, the commutator subgroup of* $N(U_1') \cap N(U_2') \cap G_0'$ *has no* $\mathbb{R}$*-anisotropic factors and the* $\mathbb{R}$*-form of* G_0' *is not of type* ${}^1E_{6,2}^{28}$*, where* U_i' *is the centralizer of* $\tilde{U}_i = \{g \in U_i \mid gug^{-1}u^{-1} \in Z_i \text{ for all } u \in U_i\}$ *in* U_i *and* G_0' *is the subgroup generated by the center of* U_1' *and of* U_2'*.*

Then the triple $(\mathbf{G}, U_1, U_2)$ *has Property A.*

Proof. (1): Let U_1, U_2 satisfy the first condition. By Proposition 4.1.1 and Proposition 2.5.4, the triple $(G_0, Z(U_1), Z(U_2))$ has property (A). This proves the theorem by the previous proposition.

(2): Since U_1, U_2 is conjugate to the pair ${}_{\mathbb{R}}V_\Theta$ and ${}_{\mathbb{R}}V_\Theta{}^-$ for some $\Theta \subset {}_{\mathbb{R}}\Delta$, by Proposition 4.3.2, U_1' and U_2' are opposite horospherical $\mathbb{R}$-subgroups and $Z(U_1')$ is not the root group of a highest real root. By Part (1), we have that the triple $(G_0', Z(U_1'), Z(U_2')$ has property (A).

Now let F_1 and F_2 be lattices in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively and Γ_{F_1,F_2} is discrete. Then $F_i' = F_i \cap U_i'(\mathbb{R})$ is a lattice in $U_i'(\mathbb{R})$ by Lemma 4.3.1 and $\Gamma_{F_1',F_2'}$ is discrete since $\Gamma_{F_1',F_2'} \subset \Gamma_{F_1,F_2}$. Therefore the subgroup $\Gamma_{F_1',F_2'}$ is an arithmetic subgroup; hence so is Γ_{F_1,F_2}. □

4.3.5. Remark. We recall the remark in the introduction that in Theorem 4.3.4, the assumptions on $\mathbb{R}$-anisotropic factors are weaker than the one that the commutator subgroup of $N(U_1) \cap N(U_2)$ has no $\mathbb{R}$-anisotropic factors.

This can be shown as follows: for case (1) in the statement of Theorem 4.3.4, we have proved this in (3) of Proposition 2.5.4. For case (2), we observe that if ν is the the subset of the simple roots in the Dynkin diagram of $\mathbf{G}$ corresponding to $N(U_1) \cap N(U_2)$, then $\nu \cup (j^{-1}({}_{\mathbb{R}}\Delta_H) \cap \Delta)$ is the attached subset to $N(U_1') \cap N(U_2')$ by Proposition 4.3.3. Therefore each connected component of $\nu \cup (j^{-1}({}_{\mathbb{R}}\Delta_H) \cap \Delta)$ contains some connected component of ν. Now it is enough to combine (3) in Proposition 2.5.4 with Proposition 1.4.1 which implies that the commutator subgroup of $N(U_1) \cap N(U_2)$ has an $\mathbb{R}$-anisotropic factor if and only if some connected component of ν entirely belongs to Δ_0.

4.3.6. Remark. As long as G_0 (resp. G_0') is not of type A_n, the commutator subgroup of $N(U_1) \cap N(U_2) \cap G_0'$ (resp. $N(U_1) \cap N(U_2) \cap G_0'$) is almost absolutely simple. It follows that in those cases, we can drop the assumption on $\mathbb{R}$-anisotropic factors in (2) and (3) in the statement of the main theorem.

4.4. Arithmetic subgroups of the form Γ_{F_1,F_2}

We discuss, for a given pair U_1 and U_2 of opposite horospherical subgroups, how specifically we can determine the discrete subgroups of the form Γ_{F_1,F_2} from the main theorem.

4.4.1. Corollary. *Let G be an absolutely simple algebraic $\mathbb{Q}$-group with the Tits index $(\Delta, \Delta_0, * -$ action of $Gal(\mathbb{C}/\mathbb{Q}))$ and $U_1 = V_\Theta$, $U_2 = V_\Theta^-$ for some $\Theta \subset \Delta$. If Γ_{F_1,F_2} is a subgroup commensurable to $G(\mathbb{Z})$ for some lattices F_1 and F_2 in $U_1(\mathbb{R})$ and $U_2(\mathbb{R})$ respectively , then Θ contains Δ_0 and is invariant under the $*$-action of $Gal(\mathbb{C}/\mathbb{Q})$.*

Proof. It follows from the assumption that Γ_{F_1,F_2} is commensurable to $G(\mathbb{Z})$ that F_1 is commensurable to $U_1(\mathbb{Z}) = U_1 \cap G(\mathbb{Z})$. Since F_1 is a lattice in $U_1(\mathbb{R})$, it follows that $U_1(\mathbb{Z})$ is Zariski dense in U_1. By Proposition 1.2.7, U_1 is defined over $\mathbb{Q}$. Therefore the normalizer $N(U_1)$ is defined over $\mathbb{Q}$. Since $N(U_1) = P_\Theta$, the corollary follows from Proposition 1.4.1. □

4.4.2. Let U_1, U_2 be as in the example 2.2.4 and F_1 and F_2 lattices in $M_{m\times k}(\mathbb{R})$ and $M_{k\times m}(\mathbb{R})$ respectively.

Corollary. *Let $n \geq 3$ and $G = SL_n(\mathbb{R})$. If the subgroup Γ_{F_1,F_2} is discrete, there exist elements $g \in GL_m(\mathbb{R})$ and $h \in GL_k(\mathbb{R})$ such that gF_1h^{-1} and hF_2g^{-1} are, up to commensurability, one of the following pairs:*

1. the pair consisting of $M_{r\times s}(D_\mathbb{Z})$ and $M_{s\times r}(D_\mathbb{Z})$ where D is an $\mathbb{R}$-algebra defined over $\mathbb{Q}$ with $D_\mathbb{R} = M_d(\mathbb{R})$ such that $D_\mathbb{Q}$ is a central division algebra over $\mathbb{Q}$, $d = Deg_\mathbb{Q} D_\mathbb{Q}$, $rd = m$, $sd = k$, $D_\mathbb{Z}$ is a $\mathbb{Z}$-order of the algebra $D_\mathbb{Q}$ and $M_{r\times s}(D_\mathbb{Z})$ denotes the set of $r \times s$ matrices over the ring $D_\mathbb{Z}$;

2. the pair consisting of $\{(X_{ij}) \in M_r(D_J) \mid X_{ij} + \sigma(X_{ji}) = 0\}$ repeated twice, where K is a real quadratic extension field of $\mathbb{Q}$, J is the ring of integers of K and D is an $\mathbb{R}$-algebra defined over K with $D_\mathbb{R} = M_d(\mathbb{R})$ such that D_K is a central division algebra with an involution of the second kind σ, $d = Deg_K D_K$, $rd = m = k$ and D_J is a J-order of the algebra D_K compatible with σ.

Moreover by conjugation by the element $\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$, the subgroup Γ_{F_1,F_2} is commensurable to either the subgroup $(SL_{r+s}D)_\mathbb{Z}$ or the subgroup $SU(h_0)_\mathbb{Z} = \{Y \in (SL_{2r}D)_J \mid {}^tY^\sigma h_0 Y = h_0\}$ where $h_0 = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix}$, respectively.

Proof. By Theorem 4.1.1, there exists a $\mathbb{Q}$-form of G such that Γ_{F_1,F_2} is commensurable to $G(\mathbb{Z})$. Let $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_{n-1}\}$ be the set of simple roots such that $\alpha_i(\text{diag}(t_1, t_2, ..., t_n)) = t_i - t_{i+1}$ for each i. Then $U_1 = V_{\Delta-\alpha_m}$. By the previous corollary, $\alpha_m \in \Delta - \Delta_0$. If the $\mathbb{Q}$-form G is inner, $G(\mathbb{Q}) = SL_j(D_{\mathbb{Q}})$, up to conjugation, where D is described as above and $jd = n$. Since $\Delta - \Delta_0 = \{a_{jd} \in \Delta\}$, we have $m = rd$ for some r and hence $k = (j-r)d$. If the $\mathbb{Q}$-form G is outer and has a minimal horospherical subgroup defined over $\mathbb{Q}$, then n should be even and the standard minimal horospherical $\mathbb{Q}$-subgroup is $V_{\Delta-\alpha_{n/2}}$. Therefore $m = n/2$, followed by $k = n/2$. $\square$

References

[1] A. A. Borel, *Linear algebraic groups (second enlarged edition)*, Berlin Heidelberg New York, Springer-Verlag, 1991.

[2] A. A. Borel, *Density and maximality of arithmetic subgroups*, J. Reine Angew. Math. 224 (1966), 78–89.

[3] A. A. Borel and G. Harder, *Existence of discrete cocompact subgroups of reductive groups over local fields*, J. Reine Angew. Math. 298 (1978), 53–64.

[4] A. A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. Math. 75 (1962), 485–535.

[5] A. A. Borel and J. Tits, *Groupes réductifs*, Publ. Math. A.H.E.S.27 (1965), 55–151.

[6] J. W. Cassels, *An introduction to the geometry of numbers*, New York, Springer-Verlag, 1959.

[7] R. Howe, *Perspectives on invariant theory*, I.M.C.P, vol 8 (1995), 1–182.

[8] J. Humphreys, *Introduction to Lie algebras and Representation Theory*, New York, Springer-Verlag, 1972.

[9] J. Gilman, *Two generator discrete subgroups of $PSL_2(\mathbb{R})$*, Memoirs of AMS, vol 561 (1995).

[10] G. A. Margulis, *Nonuniform lattices in semisimple algebraic groups*, Lie groups and their representations, ed. I. M. Gelfand. (1975), Wiley, NY.

[11] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Berlin Heidelberg New York, Springer-Verlag, 1991.

[12] G. A. Margulis, *Arithmetic properties of discrete groups*, Russian Math. Surveys 29 (1974), 107–156.

[13] G. A. Margulis, *Arithmeticity of irreducible lattices in semisimple groups of rank greater than 1, appendix to Russian translation of M. Raghunathan, Discrete groups of Lie groups. Mir, Moscow 1977 (in Russian)*, Invent. Math. 76 (1984), 93–120.

[14] H. Oh, *Discrete subgroups of $SL_n(\mathbb{R})$ generated by lattices in horospherical subgroups*, C. R. Acad. Sci, Paris, Serie I, Vol 323 (1996).

[15] A. Onishchik and E. Vinberg, *Lie groups and Lie algebras, EMS 41*, Springer-Verlag, 1994.

[16] M. S. Raghunathan, *Discrete subgroups of Lie groups*, New York, Springer-Verlag, 1972.

[17] M. S. Raghunathan, *A note on generators for arithmetic subgroups of algebraic groups*, Pacific Jour. Math, vol 152 (1992), 365–373.

[18] M. S. Raghunathan, *Discrete groups and $\mathbb{Q}$-structures on semisimple Lie groups*, Proc. Int. Coll. on Discrete subgroups of Lie groups and appl. to moduli, TIFR, Bombay (1975), 225–321.

[19] M. Ratner, *Raghunathan topological conjecture and distributions of unipotent flows*, Duke Math. J. (1991), 235–280.

[20] N. Shah, *Uniformly distributed orbits of certain flows on homogeneous spaces*, Math. Ann. 289 (1991), 315–334.

[21] J. Tits, *Classification of algebraic semisimple groups*, Proc. Symp. Pure. Math 9, AMS (1966), 33–62.

[22] J. Tits, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, J. Reine. Angew Math 247 (1971), 196–220.

[23] J. Tits, *Systems Generatteurs de Groupes de Congruence*, C. R. Acad. Sci, Paris, Serie A, 693 (1976), 693.

[24] L. N. Vaserstein, *The structure of Classical Arithmetic Groups of rank greater than one, (English translation)*, Math. USSR; Sbornik, 20 (1973), 465-492.

[25] T. N. Venkataramana, *On systems of generators of arithmetic subgroups of higher rank groups*, Pacific Jour. Math, vol 166, no. 1 (1994), 193–212.

[26] G. Warner, *Harmonic analysis on semisimple Lie groups I*, New York, Springer-Verlag, 1972.

[27] R. Zimmer, *Ergodic Theory and Semisimple Groups*, Boston Basel Stuttgart, Birkhäuser, 1984.

10 Hillhouse Avenue, Department of Mathematics, Yale University, New Haven, CT 06520, U.S.A

E-mail address: heeoh@math.yale.edu