

Solution for Problem 1

(1) The limit exists.

Since $0 \leq \left| \frac{xy \cos y}{\sqrt{x^2+y^2}} \right| \leq |xy/\sqrt{x^2}| = |y| \rightarrow 0$, as $y \rightarrow 0$. By squeeze theorem, the limit exists and equals 0.

Remark: We can first try finding limits along a few typical paths, like $x=0, y=0, x=y$, it is not hard to see all the limits are 0, so we doubt that the limit exists, then we use squeeze theorem proving our guess.

(2) The limit does not exist.

First let's approach $(0,0)$ along the x -axis, then $y=0$ gives $f(x,0) = \frac{x^2}{2x^2} = 1/2$ for all $x \neq 0$, so the limit along the line $y=0$ is $1/2$. We then approach along the y -axis by putting $x=0$, it gives $f(0,y) = \frac{\sin^2(y)}{3y^2}$, and the limit as $y \rightarrow 0$ is $1/3$, (we can show this by L'Hospital's Rule) so limit along the line $x=0$ is $1/3 \neq 1/2$, the limit does not exist.

Solution for Problem 2

The level surface is $F(x,y,z) = x^2 - 2x + 4y^2 - z^2 = 0$, so the normal vector of the tangent plane to the level surface is the gradient vector of F , $\nabla F = \langle 2x-2, 8y, -2z \rangle$. If two planes are parallel means their normal vectors are parallel, and the normal vector for the given plane is $\langle 2, 2, 1 \rangle$, so we get the following equation $\langle 2x-2, 8y, -2z \rangle = \lambda \langle 2, 2, 1 \rangle$, also keep in mind that the point we are looking for lies on the level surface $F(x,y,z) = 0$. We then can solve $\lambda = 1, -1$, so we get a pair of points which satisfy the requirement $(2, 1/4, -1/2)$, and $(0, -1/4, 1/2)$.

Solution for Problem 3:

Let A denote the area, so $A = xy \sin(\theta)/2$. From the information provided, $x'(t) = 2, y'(t) = -3, \theta' = 2$, so A is indirectly a function of t , we take the derivative of A with respect to t by applying the Chain Rule.

$$dA/dt = \frac{\partial A}{\partial x} \frac{dx}{dt} + \frac{\partial A}{\partial y} \frac{dy}{dt} + \frac{\partial A}{\partial \theta} \frac{d\theta}{dt} = 2y \sin \theta / 2 + (-3x \sin \theta) / 2 + 2xy \cos \theta / 2 \quad (1)$$

put $x=4, y=2, \theta = \pi/6$ into (1), we get $dA/dt = 1 - 3 + 4\sqrt{3} = 4\sqrt{3} - 2$

Solution for Problem 4

If we set the northeast to be x -axis, and the northwest to be the y -axis, (since these two directions are orthogonal) then the south direction will be corresponding with the unit vector $u = \langle -\sqrt{2}/2, -\sqrt{2}/2 \rangle$, let L

denotes the altitude from the ground. Since the rate of change along x-axis is $\sqrt{2}$, namely $\partial L/\partial x = \sqrt{2}$, similarly $\partial L/\partial y = -2\sqrt{2}$, so $\partial L/\partial u = \sqrt{2}(-\sqrt{2}/2) + (-2\sqrt{2})(-\sqrt{2}/2) = 1$, the rate of change when traveling south is 1.

Solution for Problem 5

It is not hard to see $f(x, y)$ is continuous, and the set is closed and bounded, so the absolute maximum and minimum exist. First find out all the critical points, since all the partial derivatives of $f(x, y)$ exists, if (x, y) is the critical points, the only case is $\partial f/\partial x = e^{-x^2-y^2}(-2x)(x^2 + 2y^2) + 2xe^{-x^2-y^2} = 0$ and $\partial f/\partial y = e^{-x^2-y^2}(-2y)(x^2 + 2y^2) + 4ye^{-x^2-y^2} = 0$. Since $e^{-x^2-y^2} \neq 0$, we only need to solve the following two equations $(-2x)(x^2 + 2y^2) + 2x = 0$, and $(-2y)(x^2 + 2y^2) + 4y = 0$. By solving the two equations, we get the following critical points: $(0, 0), (0, \pm 1), (\pm 1, 0)$. Evaluate at the critical points, we get $f(0, 0) = 0, f(0, \pm 1) = 2e^{-1}, f(\pm 1, 0) = e^{-1}$

Next, we need to find out the maximum and minimum on the boundary $x^2 + y^2 = 4$, notice that on the boundary $e^{-x^2-y^2} = e^{-4}$ which is a constance, we only need to worry the term $x^2 + 2y^2$ by Lagrange multiplier method. $\nabla(x^2 + y^2) = \langle 2x, 2y \rangle, \nabla(x^2 + 2y^2) = \langle 2x, 4y \rangle$, so we need to solve the equation $\langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle$ together with the constraint $x^2 + y^2 = 4$, it is not hard to solve that f has possible extreme values at the point $(0, \pm 2), (\pm 2, 0)$, evaluate at these points, we get $f(\pm 2, 0) = 4e^{-4}, f(0, \pm 2) = 8e^{-4}$. Compare within all the candidates, we get the maximum value is $f(0, \pm 1) = 2e^{-1}$, (notice that $e > 2$ so $2e^{-1} > 8e^{-4}$) and the minimum is $f(0, 0) = 0$.

Solution for Problem 6

$F(x, y) = \sin(x - y) - \frac{2}{\pi}xe^y = 0$, the slope is dy/dx , by implicit differentiation, we know $dy/dx = -\frac{F_x}{F_y}$. $F_x = \cos(x - y) - \frac{2}{\pi}e^y$, evaluate at the point $(\pi/2, 0)$, we get $F_x(\pi/2, 0) = -\frac{2}{\pi}$. $F_y = -\cos(x - y) - \frac{2}{\pi}xe^y$, evaluate at the point $(\pi/2, 0)$, we get $F_y(\pi/2, 0) = 0 - 1 = -1$, $dy/dx = -\frac{2}{\pi}$.

Solution for Problem 7

Set up the problem first, suppose the glass cost 1 dollar per unit area we want to minimize the cost function $c(x, y, z) = 10xy + 2xz + 2yz$ given $xyz = V$, by lagrange multiplier, we need to solve the equation $\langle 10y + 2z, 10x + 2z, 2x + 2y \rangle = \lambda \langle yz, xz, xy \rangle$, by the same trick as in the book (P936, Example 1), we solve $x = y, z = 5x$, together with the constraint $xyz = V$, we get $x = y = (V/5)^{1/3}, z = 5(V/5)^{1/3}$

Solution for problem 8

$$V = \int_{-2}^2 \int_0^\pi (2 + e^x \sin y) dy dx = \int_{-2}^2 (2\pi + 2e^x) dx = 8\pi + 2(e^2 - e^{-2})$$