SLOW SOLITON INTERACTION WITH DELTA IMPURITIES

JUSTIN HOLMER AND MACIEJ ZWORSKI

ABSTRACT. We study the Gross-Pitaevskii equation with a delta function potential, $q\delta_0$, where $|q|$ is small and analyze the solutions for which the initial condition is a soliton with initial velocity v_0. We show that up to time $(|q|+v_0^2)^{-\frac{1}{2}}\log(1/|q|)$ the bulk of the solution is a soliton evolving according the classical dynamics of a natural effective Hamiltonian, $(\xi^2 + q\,\mathrm{sech}^2(x))/2$.

1. INTRODUCTION

The Gross-Pitaevskii equation (NLS) with a delta function potential and soliton initial data,

$$
\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u - q\delta_0(x)u + u|u|^2 = 0 \\ u(x,0) = e^{iv_0x}\mathrm{sech}(x-a_0)\,, \end{cases} \tag{1.1}
$$

offers a surprising wealth of dynamical phenomena. In [12], (and numerically in [13]), the authors and J. Marzuola studied the high velocity, $v_0 \gg 1$, case and showed that the scattering matrix of the delta potential controls the dynamics. In this paper we describe the case of small q. The most interesting dynamics is visible for initial velocities satisfying $v_0^2 \leq |q|$. The low v_0 regime has been studied in the physics literature [3],[9],[2], and the behaviour in the intermediate range of q's and v_0's, that is between the fully quantum and semiclassical cases studied in [12] and in this paper respectively, is still unclear. We state the main result here with a slightly more precise version given in Theorem 2 in §7 below.

Theorem 1. *Suppose that in* (1.1) *we have* $|q| \ll 1$*. Then, on a time interval* $0 \leq t \leq \delta(v_0^2 + |q|)^{-1/2}\log(1/|q|)$,

$$
\|u(t,\bullet) - e^{i\bullet v(t)}e^{i\gamma(t)}\mathrm{sech}(\bullet - a(t))\|_{H^1(\mathbb{R})} \leq C|q|^{1-3\delta}\,, \tag{1.2}
$$

where a, v, and γ solve the following system of equations

$$
\frac{d}{dt}a = v\,, \quad \frac{d}{dt}v = -\frac{1}{2}q\partial_x(\mathrm{sech}^2)(a)\,, \quad \frac{d}{dt}\gamma = \frac{1}{2} + \frac{v^2}{2} - q\mathrm{sech}^2(a) - \frac{1}{2}q\partial_x(\mathrm{sech}^2)(a)\,, \tag{1.3}
$$

with initial data $(a_0, v_0, 0)$.

Remark. The arguments presented in the paper apply to much more general impurities. In fact, one can replace δ_0 by any $V \in H^{-1}(\mathbb{R})$. The effective potential $q\mathrm{sech}^2(a)$ is then replaced by $qV * \operatorname{sech}^2(a)$. Since our motivation comes from [12] we present only the special case.

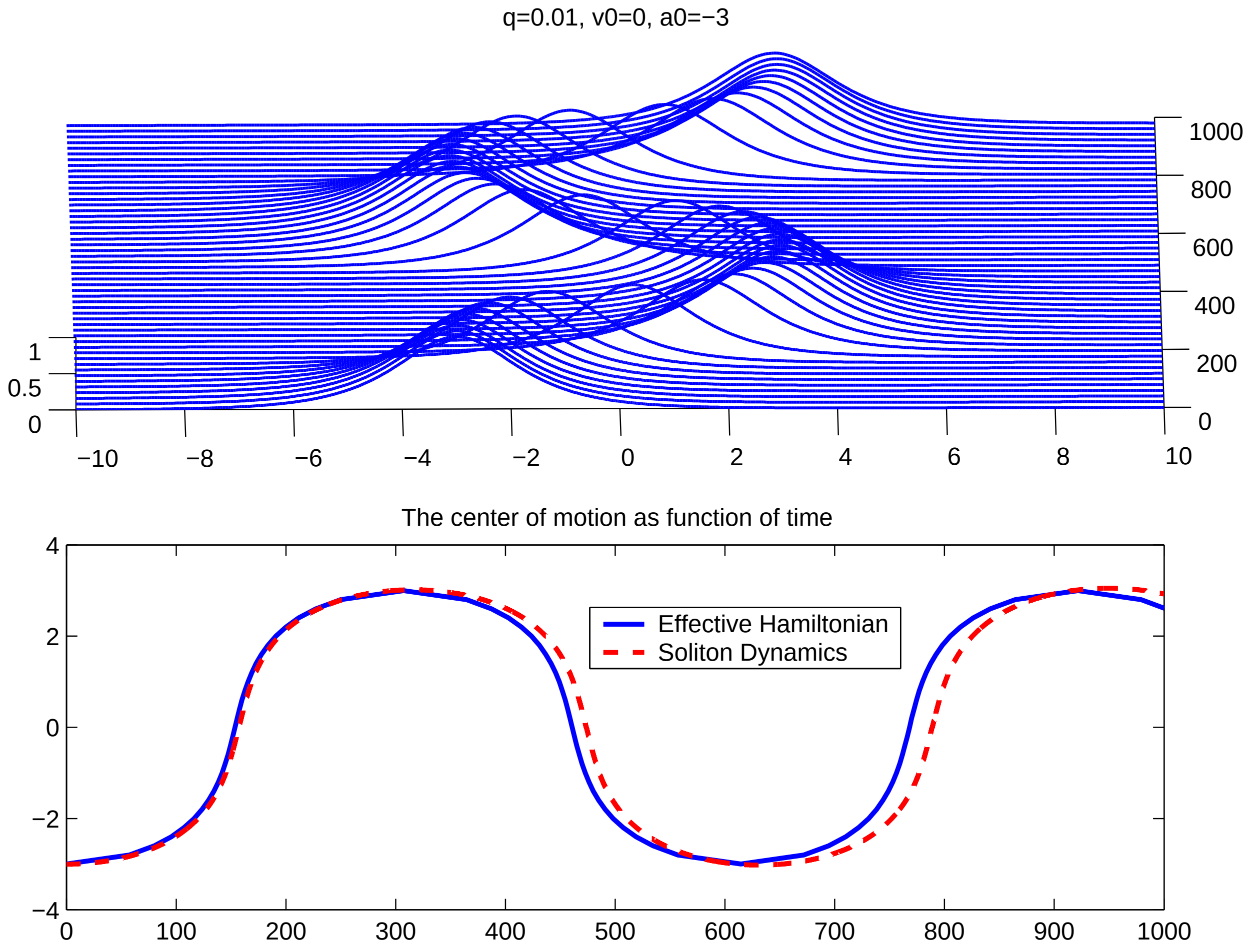

FIGURE 1. The top figure shows the evolution of $|u(x,t)|$ for $v_0 = 0$, $a_0 = -3$, $q = -0.01$ for $0 \leq t \leq 1000$. In the bottom figure the dashed curve is the computed center of motion, and the continuous curve, the plot of $a(t)$ given by (1.3). More figures illustrating other cases, some with an even more dramatic agreement can be found at `http://math.berkeley.edu/~zworski/HZ1.pdf`

Compared to numerical results, the theorem gives a remarkably good description of the dynamics of a slow soliton interacting with a small delta function potential. For example consider $v_0 = 0$, $a_0 < 0$ fixed, and $|q| \to 0$, illustrated in Fig.1. When $q < 0$, the bulk of the solution is oscillatory about the origin, with the center moving from $a_0 < 0$ to $-a_0 > 0$.

Since

$$\frac{1}{2}v^2 + \frac{1}{2}q\eta^2(a) = \frac{1}{2}q\eta^2(a_0)\,, \quad \eta(x) \stackrel{\text{def}}{=} \operatorname{sech}(x)\,,$$

the time to complete one cycle of oscillation is

$$\int_{a_0}^{-a_0} \frac{2\,dx}{|q|^{1/2}\sqrt{\eta^2(x) - \eta^2(a_0)}}$$

which is of size comparable to $|q|^{-1/2}$. Since the theorem provides an accurate description up to time $\sim |q|^{-1/2}\log(1/|q|)$, it covers many cycles for small enough $|q|$. When $q > 0$ the soliton is repulsed by the δ potential and slowly slides to negative infinity with the terminal velocity $q^{1/2}$ – see Fig.3 below.

The proof of our theorem follows the long tradition of the study of stability of solitons which started with the work of M.I. Weinstein [16]. The interaction of solitons with external potentials was studied in the stationary semiclassical setting by Floer and A. Weinstein [5] and Oh [14], and the first dynamical result belongs to Bronski and Jerrard [1]. The semiclassical regime is equivalent to considering slowly varying potentials,

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u - W(hx)u + u|u|^2 = 0\,, & 0 < h \ll 1 \\ u(x,0) = e^{iv_0 x}\operatorname{sech}(x - a_0)\,, & \|W^{(k)}\|_\infty \le C\,, \quad k \le 2\,, \end{cases} \tag{1.4}$$

and that case has been studied in various settings and degrees of generality in [6], [7], [8] (see these papers for additional references). The approach of these works was our starting point. The results of [6] in the special case of (1.4) give

$$\|u(t,\bullet) - e^{i\bullet v(t)}e^{i\gamma(t)}\operatorname{sech}(\bullet - a(t))\|_{H^1(\mathbb{R})} \le Ch\,, \quad 0 \le t \le C\log(1/h)/h\,, \tag{1.5}$$

where

$$\begin{gathered} \frac{d}{dt}a = v + \mathcal{O}(h^2)\,, \quad \frac{d}{dt}v = -hW'(ha) + \mathcal{O}(h^2)\,, \\ \frac{d}{dt}\gamma = \frac{1}{2} + \frac{v^2}{2} - W(ha) + \mathcal{O}(h^2)\,, \end{gathered} \tag{1.6}$$

with initial data $(a_0, v_0, 0)$.† We note that unlike in (1.3) the ordinary differential system (1.6) is not exact – see Fig.2 and the discussion below.

At first the equations (1.1) and (1.4) appear to be very different: a delta function potential is very far from being slowly varying. The similarity of (1.3) and (1.6) is however a result of the same underlying structure. As we recall in §2 the Gross-Pitaevski equations, (1.1) or (1.4), are the equation for Hamiltonian flow of

$$H_V(u) \stackrel{\text{def}}{=} \frac{1}{4}\int (|\partial_x u|^2 - |u|^4)dx + \frac{1}{2}\int V|u|^2\,, \quad V = q\delta_0\,, \quad V = W(h\bullet)\,, \tag{1.7}$$

†Strictly speaking the result in [6] describes the dynamics for $0 \le t \le c_0/h$ only. That corresponds to small time dynamics of the potential W. Iterating the full strength of the result of [6] seems to give the expected extension to Ehrenfest time $\log(1/h)/h$ [11].

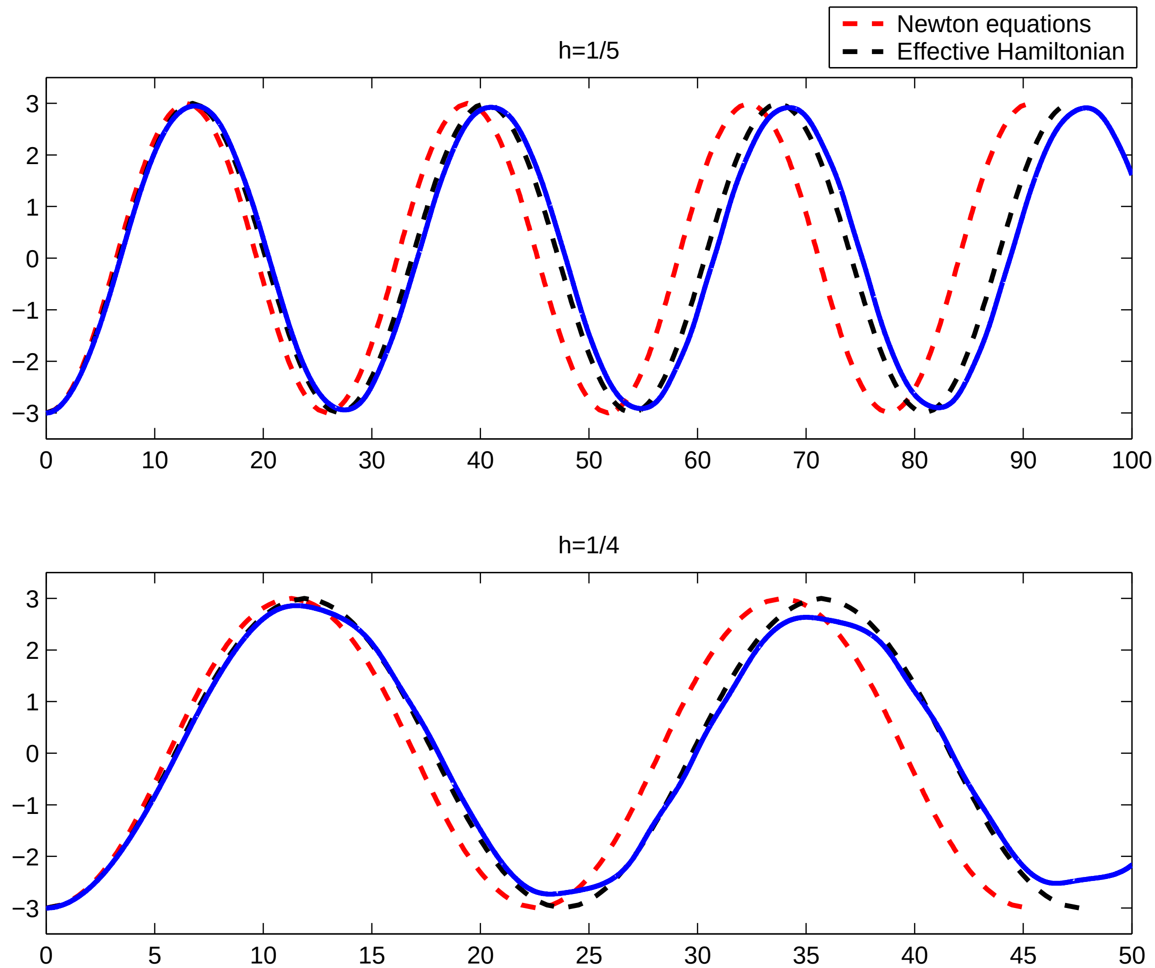

FIGURE 2. Comparison of the dynamics of the center of motion of the soliton for the Gross-Pitaevskii equation with a slowly varying potential,

$$iu_t = -\frac{1}{2}u_{xx} - |u|^2 u - \operatorname{sech}^2(hx)u\,, \quad h = 1/5\,, \quad h = 1/4\,,$$

and initial condition in (1.1) with $v_0 = 0$, $a_0 = -3$. The dashed red curve shows the solution to Newton's equations used in [1] and [6], the blue curve shows the center of the approximate soliton u, and the black dashed curve is given by the equations of motion of the effective Hamiltonian

$$\frac{1}{2}\left(v^2 - \operatorname{sech}^2(h\bullet) * \operatorname{sech}^2(a)\right)\,.$$

The improvement of the approximation given by the effective Hamiltonian is remarkable even in the case of $h = 1/4$ in which we already see radiative dissipation in the first cycle.

with respect to the symplectic form on $H^1(\mathbb{R},\mathbb{C})$ (considered as a real Hilbert space):

$$\omega(u,v) = \operatorname{Im}\int u\bar{v}\,, \quad u,v \in H^1(\mathbb{R},\mathbb{C})\,. \tag{1.8}$$

When $V \equiv 0$, $\eta = \operatorname{sech}$ is a critical value (minimizer) of H_0 with prescribed L^2 norm:

$$d\mathcal{E}_\eta = 0\,, \quad \mathcal{E}(u) \stackrel{\text{def}}{=} H_0(u) + \frac{1}{4}\|u\|^2_{L^2}\,. \tag{1.9}$$

The flow of H_0 is tangent to the manifold of solitons,

$$M = \{e^{i\gamma}e^{iv(x-a)}\mu\operatorname{sech}(\mu(x-a))\,, \quad a,v,\gamma\in\mathbb{R}\,, \quad \mu\in\mathbb{R}_+\}\,,$$

which of course corresponds to the fact that the solution of (1.1) with $q = 0$ and $u_0(x,0) = e^{i\gamma+iv_0(x-a_0)}\mu\operatorname{sech}(\mu(x-a_0))$, is

$$u(x,t) = e^{i\gamma+iv_0(x-a_0)+i(\mu^2-v^2)t/2}\mu\operatorname{sech}(\mu(x-a_0-v_0t))\,. \tag{1.10}$$

The symplectic form (1.8) restricted to M is

$$\omega\restriction_M = \mu dv\wedge da + vd\mu\wedge da + d\gamma\wedge d\mu\,, \tag{1.11}$$

see §2.4. The evolution of the parameters (a,v,γ,μ) in the solution $u(x,t)$ follows the Hamilton flow of

$$H_0\restriction_M = \frac{\mu v^2}{2} - \frac{\mu^3}{6}\,,$$

with respect to the symplectic form $\omega\restriction_M$.

The systems of equations (1.3) and (1.6) are obtained using the following basic idea: if a Hamilton flow of H, with initial condition on a symplectic submanifold, M, stays close to M, then the flow is close to the Hamilton flow of $H\restriction_M$.

In our case M is the manifold of solitons and H is given by (1.7)

$$H_V\restriction_M (a,v,\gamma,\mu) = \frac{\mu v^2}{2} - \frac{\mu^3}{6} + \frac{1}{2}\mu^2(V*\operatorname{sech}^2)(\mu a)\,, \tag{1.12}$$

and in particular

$$H_{q\delta_0}\restriction_M = H_0\restriction_M + \frac{1}{2}\mu^2\operatorname{sech}^2(\mu a)\,, \quad H_{W(h\bullet)}\restriction_M = H_0\restriction_M + \frac{1}{2}\mu^2\,W(h\bullet)*\operatorname{sech}^2(\mu\bullet)\,.$$

The equations (1.3) are simply the equations of the flow of $H_{q\delta_0}\restriction_M$ – see §2.5. The equations of the flow of $H_{W(h\bullet)}\restriction_M$ are easily seen to imply (1.6) but some h corrections are built into the classical motion. It would be interesting to see if this provides improvement of the analysis of [6]. Since our interests lie in the study of various aspects of the delta impurity we satisfy ourselves with a numerical experiment which shows that the improvement is indeed dramatic – see Fig.2.

In either case, all of this hinges on the proximity of the flow to M and to show that we use the Lyapunov function, $L(w)$, introduced in [16] – see §5. Typically, and as is done in [6], $L(w)$ is bounded from below so that it controls the norm of w (roughly speaking the expression estimated in (1.2) and (1.4)), while $(d/dt)L(w)$ is estimated from above. In this paper due to the irregularity of the potential that approach for upper bounds does not seem to be applicable but we can estimate $L(w)$ directly, controlling the propagation of a, v, γ, and μ more precisely.

The paper is organized as follows. In §2 we recall the Hamiltonian structure of the nonlinear flow of (1.1) and describe the manifold of solitons. Its identification with the Lie group $G = H_3 \ltimes \mathbb{R}_+$, where H_3 is the Heisenberg group, provides useful notational shortcuts. In §3 we describe the reparametrized evolution. The starting point there is an application of the implicit function theorem and a decomposition of the solution into symplectically orthogonal components. That method has a long tradition in soliton stability and we learned it from [6]. In §4 we give a self-contained and constructive presentation of well known spectral estimates. Weinstein's Lyapunov function is adapted to our problem in §5. It is estimated using classical energy. The ODE estimates needed for the iteration of our stability argument are given in §6 and a stronger version of Theorem 1 is proved in §7.

Finally, we make comments on the numerics. The computations of solutions of (1.1) and (1.4) were done using the `FORTRAN` code described in [13, §3] and written as part of that project by J. Marzuola. Other computations and all the graphics were done using `MATLAB`.

Acknowledgments. We would like to thank Jeremy Marzuola for allowing us the use of his code for NLS computations, and to Patrick Kessler and Jon Wilkening for generous help with various computing issues. The work of the first author was supported in part by an NSF postdoctoral fellowship, and that of the second second author by an NSF grant DMS-0200732.

2. The Hamiltonian structure and the manifold of solitons

In this section we recall the well known facts about the Hamiltonian structure of the nonlinear Schrödinger equation. The manifold of solitons is given as an orbit of a semidirect product of the Heisenberg group and $\mathbb{R}_+$.

2.1. **Symplectic structure.** Let V be a complex Hilbert space with the inner product $\langle \bullet, \bullet \rangle_V$. For W, a totally real subspace of V ($W \cap iW = \{0\}$), we have $V = W + iW \simeq W^2$, and we can consider W and V as real Hilbert spaces.

As a real Hilbert space V is equipped with the natural inner product or metric

$$g(X, Y) = \mathrm{Re}\langle X, Y\rangle_V \,,$$

and the natural symplectic form

$$\omega(X, Y) = \mathrm{Im}\langle X, Y\rangle_V = g(X, iY) \,.$$

In other words g, ω, and J, multiplication by $1/i$ form a compatible triple:

$$\omega(X, Y) = g(JX, Y) \,, \quad g(X, Y) = \omega(X, iY) \,. \tag{2.1}$$

In terms of W^2, we have

$$g(X,Y) = \left\langle \begin{bmatrix} \operatorname{Re} X \\ \operatorname{Im} X \end{bmatrix}, \begin{bmatrix} \operatorname{Re} Y \\ \operatorname{Im} Y \end{bmatrix} \right\rangle_{W^2} = \omega\left(\begin{bmatrix} \operatorname{Re} X \\ \operatorname{Im} X \end{bmatrix}, J \begin{bmatrix} \operatorname{Re} Y \\ \operatorname{Im} Y \end{bmatrix} \right)$$

and

$$\omega(X,Y) = g\left(J \begin{bmatrix} \operatorname{Re} X \\ \operatorname{Im} X \end{bmatrix}, \begin{bmatrix} \operatorname{Re} Y \\ \operatorname{Im} Y \end{bmatrix} \right)$$

where J is the matrix representing multiplication by $-i$:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

For example, when we consider $V = \mathbb{C}^n$ and $W = \mathbb{R}^n$, then ω is just the standard symplectic form.

In our work, we take $V = H^1(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C})$, and on V we use the L^2 inner product. The symplectic form ω is thus

$$\omega(u,v) = \operatorname{Im} \int u\bar{v}\,, \tag{2.2}$$

and the metric g is

$$g(u,v) = \operatorname{Re} \int u\bar{v}$$

Now we consider Hamiltonians and associated Hamiltonian flows. Let $H : V \to \mathbb{R}$ be a function, our Hamiltonian. The associated Hamiltonian vector field is a map $\Xi_H : V \to TV$, which means that for a particular point $u \in V$, we have $(\Xi_H)_u \in T_uV$. The vector field Ξ_H is defined by the relation

$$\omega(v, (\Xi_H)_u) = d_uH(v)\,, \tag{2.3}$$

where $v \in T_uV$, and $d_uH : T_uV \to \mathbb{R}$ is defined by

$$d_uH(v) = \frac{d}{ds}\Big|_{s=0} H(u+sv)\,.$$

In the notation of (2.1) if we use g to define functionals, $dH_u(v) = g(v, \nabla H_u)$, then $(\Xi_H)_u = J\nabla H_u$.

If we take $V = H^1(\mathbb{R}, \mathbb{C})$ with the symplectic form (2.2), and

$$H(u) = \int \frac{1}{4}|\partial_x u|^2 - \frac{1}{4}|u|^4$$

then we can compute

$$\begin{aligned} d_uH(v) &= \operatorname{Re} \int ((1/2)\partial_x u \partial_x \bar{v} - |u|^2 u\bar{v}) \\ &= \operatorname{Re} \int (-(1/2)\partial_x^2 u - |u|^2 u)\bar{v}\,. \end{aligned}$$

Thus, in view of (2.1) and (2.3),

$$(\Xi_H)_u = \frac{1}{i}\left(-\frac{1}{2}\partial_x^2 u - |u|^2 u\right)$$

The flow associated to this vector field (Hamiltonian flow) is

$$\dot{u} = (\Xi_H)_u = \frac{1}{i}\left(-\frac{1}{2}\partial_x^2 u - |u|^2 u\right) . \tag{2.4}$$

For future reference we state two general lemmas of symplectic geometry:

Lemma 2.1. *Suppose that $g : V \to V$ is a diffeomorphism such that $g^*\omega = \mu(g)\omega$, where $\mu(g) \in C^\infty(V;\mathbb{R})$. Then for $f \in C^\infty(V,\mathbb{R})$,*

$$(g^{-1})_* \Xi_f(g(\rho)) = \frac{1}{\mu(g)} \Xi_{g^* f}(\rho) \,, \quad \rho \in V \,. \tag{2.5}$$

Proof. This is a straightforward generalization of Jacobi's theorem which is the case of $\mu(g) \equiv 1$. To compute $(g^{-1})_*\Xi_f(g(\rho))$, we note

$$\begin{aligned}
\omega_\rho((g^{-1})_* X,\, (g^{-1})_*\Xi_f(g(\rho))) = ((g^{-1})^*\omega)_{g(\rho)}(X,\, \Xi_f(g(\rho))) &= \frac{1}{\mu(g)}\omega_{g(\rho)}(X, \Xi_f(g(\rho))) \\
&= \frac{1}{\mu(g)}[df(g(\rho))](X) = \frac{1}{\mu(g)}[g^* df(\rho)]((g^{-1})_* X) \\
&= \frac{1}{\mu(g)}\omega_\rho((g^{-1})_* X, \Xi_{g^* f}(\rho))
\end{aligned}$$

and the lemma follows. ☐

Suppose that $f \in C^\infty(V;\mathbb{R})$ and that $df(\rho_0) = 0$. Then the Hessian of f at ρ_0, $f''(\rho_0) : T_{\rho_0}V \mapsto T^*_{\rho_0}V$, is well defined. The Hamiltonian map $F : T_{\rho_0}V \to T_{\rho_0}V$ is given by the relation

$$\left[f''(\rho_0)X\right](Y) = \omega_{\rho_0}(Y, FX) \,. \tag{2.6}$$

In this notation we have

Lemma 2.2. *Suppose that $N \subset V$ is a finite dimensional symplectic submanifold of V, and $f \in C^\infty(V,\mathbb{R})$ satisfies*

$$\Xi_f(\rho) \in T_\rho N \subset T_\rho V \,, \quad \rho \in N \,.$$

If at $\rho_0 \in N$, $df(\rho_0) = 0$, then the Hamiltonian map defined by (2.6) satisfies

$$F(T_{\rho_0}N) \subset T_{\rho_0}N \,.$$

Proof. Since N is assumed to be finite dimensional we only need to prove the lemma for a finite dimensional V (any particular $Y \in (T_\rho V)^\perp$ can be a value of a vector field in a finite dimensional submanifold of V containing N). We can then assume that $\rho_0 = (0,0)$, and that in local coordinates near $(0,0)$, $N = \{(x,\xi) \mid x'' = \xi'' = 0\}$, $x = (x', x'')$, $\xi = (\xi', \xi'')$, $\bullet' = (\bullet_1, \cdots, \bullet_k)$, where $2k = \dim N$ (see for instance [15, Theorem 21.2.4]). The conditions of f mean that

$$d_{x''} f(x', \xi', 0, 0) = d_{\xi''} f(x', \xi', 0, 0) = 0\,, \quad df(0,0) = 0\,,$$

where we wrote $(x,\xi) = (x', \xi', x'', \xi'')$. Hence, the Hessian at $(0,0)$ is given by

$$f''(0,0) = \begin{bmatrix} f''_{x',\xi'}(0,0) & 0 \\ 0 & f''_{x'',\xi''}(0,0) \end{bmatrix} .$$

This means that

$$\langle f''(\rho_0) X, Y \rangle = 0 \ \ \forall\, X \in T_\rho N\,,\ Y \in (T_\rho N)^\perp\,.$$

where $\bullet^\perp$ denotes the symplectic orthogonal. Since the Hamiltonian map, F, is defined by $\langle f''(\rho_0) X, Y \rangle = \omega(Y, FY)$ this proves the lemma. ☐

2.2. **Associated symmetries and Noether's theorem.** For completeness we comment on the Hamiltonian version of Noether's theorem which states that the following three statements are equivalent

$$\Xi_H E \stackrel{\text{def}}{=} \omega(\Xi_H, \Xi_E) = 0\,,$$

E is preserved by the Hamiltonian flow of H,

H is preserved by the Hamiltonian flow of E.

For example, consider the mass $M = \int |u|^2$. The associated Hamiltonian vector field is $(\Xi_M)_u = iu$. We compute

$$\omega(\Xi_M, \Xi_H) = -\operatorname{Im} \int iu\, \overline{i(\partial_x^2 u + |u|^2 u)} = 0$$

The flow associated to Ξ_M is $u \mapsto e^{is}u$, which is the phase invariance of H and thus solutions to $\partial_t u = i(\partial_x^2 u + |u|^2 u)$.

Similarly, the time translation, $u(x,t) \mapsto u(x, t+s)$ gives the conservation of energy, $H(u)$, the space translation, $u(x,t) \mapsto u(x+y, t)$, gives the conservation of momentum, $\operatorname{Im} \int u_x \bar{u}$.

2.3. **Manifold of solitons as an orbit of a group.** For $g = (a, v, \gamma, \mu) \in \mathbb{R}^3 \times \mathbb{R}_+$ we define the following map

$$H^1 \ni u \longmapsto g \cdot u \in H^1\,, \quad (g \cdot u)(x) \stackrel{\text{def}}{=} e^{i\gamma} e^{iv(x-a)} \mu u(\mu(x-a))\,. \tag{2.7}$$

This action gives a group structure on $\mathbb{R}^3 \times \mathbb{R}_+$ and it is easy to check that this transformation group is a semidirect product of the Heisenberg group H_3 and $\mathbb{R}_+$:

$$G = H_3 \ltimes \mathbb{R}_+ \,, \quad \mu \cdot (a, v, \gamma) = (\frac{a}{\mu}, \mu v, \gamma) \,.$$

We recall that the Heisenberg group can be identified with the group of matrices of the form

$$\begin{bmatrix} 1 & v & \gamma \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \,, \quad a, v, \gamma \in \mathbb{R} \,,$$

and that the semidirect product of H and $\mathbb{R}_+$ is defined by

$$(h, \mu) \cdot (h', \mu') = (h \cdot (\mu \cdot h'), \mu\mu') \,, \quad h, h' \in H \,.$$

Explicitly, the group law on G is given by

$$(a, v, \gamma, \mu) \cdot (a', v', \gamma', \mu') = (a'', v'', \gamma'', \mu'') \,,$$

where

$$v'' = v + v'\mu \,, \quad a'' = a + \frac{a'}{\mu} \,, \quad \gamma'' = \gamma + \gamma' + \frac{va'}{\mu} \,, \quad \mu'' = \mu\mu'$$

Remark. As was pointed to us by Bjorn Poonen, the group acts faithfully on the 4-dimensional space spanned by 1,v,a,γ viewed as functions on the group. This can be used to see that the group is faithfully represented by the group of matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ v & \mu & 0 & 0 \\ a & 0 & 1/\mu & 0 \\ \gamma & 0 & v/\mu & 1 \end{bmatrix} \,, \quad v, a, \gamma \in \mathbb{R} \,, \ \mu \in \mathbb{R}_+ \,,$$

but we will not use this below.

The action of G is not symplectic but it is *conformally symplectic* in the sense that

$$g^*\omega = \mu(g)\omega \,, \quad g = (h(g), \mu(g)) \,, \quad \mu(g) \in \mathbb{R}_+ \,, \tag{2.8}$$

as is easily seen from (2.2).

The Lie algebra of G, denoted by $\mathfrak{g}$, is generated by e_1, e_2, e_3, e_4,

$$\exp(te_1) = (t, 0, 0, 1) \,, \quad \exp(te_2) = (0, t, 0, 1) \,,$$
$$\exp(te_3) = (0, 0, t, 1) \,, \quad \exp(te_4) = (0, 0, 0, e^t) \,,$$

and the bracket acts as follows:

$$[e_1, e_4] = e_1, \quad [e_2, e_4] = -e_2, \quad [e_1, e_2] = -e_3, \quad [e_3, \bullet] = 0 \,, \tag{2.9}$$

so e_3 is in the center. The infinitesimal representation obtained from (2.7) is given by

$$e_1 = -\partial_x \,, \quad e_2 = ix \,, \quad e_3 = i \,, \quad e_4 = \partial_x \cdot x \,. \tag{2.10}$$

It acts, for instance on $\mathcal{S}(\mathbb{R}) \subset H^1$, and by $X \in \mathfrak{g}$ we will denote a linear combination of the operators e_j.

We have the following standard

Lemma 2.3. *Suppose $\mathbb{R} \ni t \mapsto g(t)$ is a C^1 function and that $u \in \mathcal{S}(\mathbb{R})$. Then, in the notation of* (2.7),

$$\frac{d}{dt} g(t) \cdot u = g(t) \cdot (X(t) u) \,,$$

where $X(t) \in \mathfrak{g}$ is given by

$$X(t) = \dot{a}(t)\mu(t) e_1 + \frac{\dot{v}(t)}{\mu(t)} e_2 + (\dot{\gamma}(t) - \dot{a}(t) v(t)) e_3 + \frac{\dot{\mu}(t)}{\mu(t)} e_4 \,, \tag{2.11}$$

where $g(t) = (a(t), v(t), \gamma(t), \mu(t))$.

Proof. We differentiate

$$g(t) \cdot u = \exp(i\gamma(t)) \exp(-a(t)\partial_x) \exp(iv(t)x) \exp(\theta(\partial_x \cdot x)) u \,, \quad \exp\theta(t) = \mu(t) \,,$$

and note that

$$\begin{aligned} \partial_x \exp(ivx) &= \exp(ivx)(\partial_x + iv) \,, \\ \partial_x \exp(\theta(\partial_x \cdot x)) &= \exp(\theta(\partial_x \cdot x)) e^{\theta} \partial_x \,, \\ ix \exp(\theta(\partial_x \cdot x)) &= \exp(\theta(\partial_x \cdot x))(e^{-\theta} ix) \,, \end{aligned}$$

either by direct computation or using (2.9). The formula (2.11) follows. □

The manifold of solitons is an orbit of this group, $G \cdot \eta$, to which Ξ_H, defined in (2.3), is tangent. In view of (2.4) that means that

$$i \left(\frac{1}{2} \partial_x^2 \eta + |\eta|^2 \eta \right) = X \cdot \eta \,,$$

for some $X \in \mathfrak{g}$. The simplest choice is given by taking $X = \lambda i$, $\lambda \in \mathbb{R}$, so that η solves a nonlinear elliptic equation

$$-\frac{1}{2}\eta'' - \eta^3 + \lambda \eta = 0 \,.$$

This has a solution in H^1 if $\lambda = \mu^2/2 > 0$ and it then is $\eta(x) = \mu \mathrm{sech}(\mu x)$. We will fix $\mu = 1$ so that

$$\eta(x) = \mathrm{sech} x \,.$$

Using Lemma 2.1 we can check that $G \cdot \eta$ is the *only* orbit of G to which Ξ_H is tangent.

We define the submanifold of solitons, $M \subset H_1$, as the orbit of η under G,

$$M = G \cdot \eta \subset H_1$$

and thus we have the identifications

$$M = G \cdot \eta \simeq G/\mathbb{Z} \,, \quad T_\eta M = \mathfrak{g} \cdot \eta \simeq \mathfrak{g} \,. \tag{2.12}$$

The quotient corresponds to the $\mathbb{Z}$-action

$$(a, v, \gamma, \mu) \mapsto (a, v, \gamma + 2\pi k, \mu)\,, \quad k \in \mathbb{Z}$$

2.4. **Symplectic structure on the manifold of solitons.** We first compute the symplectic form $\omega\restriction_M$ on $T_\eta M$ using the identification (2.12):

$$(\omega\restriction_M)_\eta(e_i, e_j) = \operatorname{Im} \int (e_i \cdot \eta)(x)(\overline{e_j \cdot \eta})(x)\,.$$

Since

$$\int \eta^2(x)dx = 2\,, \quad \int \eta(x)\partial_x\eta(x) = 0\,, \quad \int \partial_x\eta(x)x\eta(x)dx = -1\,,$$

we obtain from (2.10) that

$$(\omega\restriction_M)_\eta(e_2, e_1) = 1\,, \quad (\omega\restriction_M)_\eta(e_3, e_4) = 1\,, \tag{2.13}$$

and all the other $(\omega\restriction_M)_\eta(e_i, e_j)$'s vanish. In other words,

$$(\omega\restriction_M)_\eta = (dv \wedge da + d\gamma \wedge d\mu)_{(0,0,0,1)} = (d(vda + \gamma d\mu))_{(0,0,0,1)}\,.$$

To find an expression for $\omega\restriction_M$ we use (2.8) and the following elementary

Lemma 2.4. *If σ is a one form on $\mathbb{R}^3 \times \mathbb{R}_+$ such that*

$$\sigma_{(0,0,0,1)} = (vda + \gamma d\mu)_{(0,0,0,1)}\,, \quad g^*\sigma = \mu(g)\sigma\,, \quad g \in G\,,$$

then

$$\sigma = \mu vda + \gamma d\mu\,.$$

□

We conclude that using the identification (2.12)

$$\omega\restriction_M = \mu dv \wedge da + vd\mu \wedge da + d\gamma \wedge d\mu \tag{2.14}$$

Now let f be a function defined on M, $f = f(a, v, \gamma, \mu)$. The associated Hamiltonian vectorfield, Ξ_f, is defined by

$$\omega(\cdot, \Xi_f) = df = f_a da + f_v dv + f_\mu d\mu + f_\gamma d\gamma\,.$$

Using (2.14) we obtain

$$\Xi_f = \frac{f_v}{\mu}\partial_a + \left(-\frac{f_a}{\mu} - \frac{vf_\gamma}{\mu}\right)\partial_v + f_\gamma\partial_\mu + \left(v\frac{f_v}{\mu} - f_\mu\right)\partial_\gamma\,. \tag{2.15}$$

The Hamilton flow is obtained by solving

$$\dot{v} = -\frac{f_a}{\mu} - \frac{vf_\gamma}{\mu}\,, \quad \dot{a} = \frac{f_v}{\mu}\,, \quad \dot{\mu} = f_\gamma\,, \quad \dot{\gamma} = v\frac{f_v}{\mu} - f_\mu\,.$$

The restriction of

$$H(u) = \frac{1}{4}\int |\partial_x u|^2 - \frac{1}{4}\int |u|^4$$

to M is given by computing by

$$f(a, v, \gamma, \mu) = H(g \cdot \eta) = \frac{\mu v^2}{2} - \frac{\mu^3}{6}\,, \quad g = (a, v, \gamma, \mu)\,. \tag{2.16}$$

The flow of (2.15) for this f describes the evolution of a soliton.

2.5. **The Gross-Pitaevski Hamiltonian restricted to the manifold of solitons.** We now consider the Gross-Pitaevski Hamiltonian for the delta function potential

$$H_q(u) \stackrel{\text{def}}{=} \frac{1}{4}\int (|\partial_x u|^2 - |u|^4) dx + \frac{1}{2} q |u(0)|^2\,, \tag{2.17}$$

and its restriction to $M = G \cdot \eta$:

$$H_q\restriction_M = f(a, v, \gamma, \mu) = \frac{\mu v^2}{2} - \frac{\mu^3}{6} + \frac{1}{2} q \mu^2 \operatorname{sech}^2(\mu a)\,. \tag{2.18}$$

This is obtained from (2.16) and from calculating

$$\frac{1}{2} q |(g \cdot \eta)|(0) = \frac{1}{2} q \mu^2 \eta^2(-\mu a) = \frac{1}{2} q \mu^2 \operatorname{sech}^2(\mu a)\,.$$

The flow of $(H_q)\restriction_M$ can be read off from (2.15):

$$\begin{aligned}
\dot{v} &= -\frac{f_a}{\mu} - \frac{v f_\gamma}{\mu} = \mu^2 q \operatorname{sech}^2(\mu a) \tanh(\mu a) \\
\dot{a} &= \frac{f_v}{\mu} = v \\
\dot{\mu} &= f_\gamma = 0 \\
\dot{\gamma} &= v \frac{f_v}{\mu} - f_\mu = \frac{1}{2} v^2 + \frac{1}{2}\mu^2 - q\mu \operatorname{sech}^2(\mu a) - \frac{1}{2} q \mu^2 a \operatorname{sech}^2(\mu a) \tanh(\mu a)
\end{aligned} \tag{2.19}$$

This are the same equations as (1.3). The evolution of a and v is simply the Hamiltonian evolution of $(v^2 + q\mu^2 \operatorname{sech}^2(\mu a))/2$, $\mu = \text{const}$. The more mysterious evolution of the phase γ is now explained by (2.18).

Since μ is constant by the third equation, solving this system reduces to solving the first two equations. The turning position, a_{turn}, is given by

$$|a_{\text{turn}}| = \operatorname{sech}^{-1}\left(\frac{v}{\sqrt{q}}\right)$$

and Fig.3 gives a comparison between a_{turn} and the numerically computed turning point of the center of the soliton.

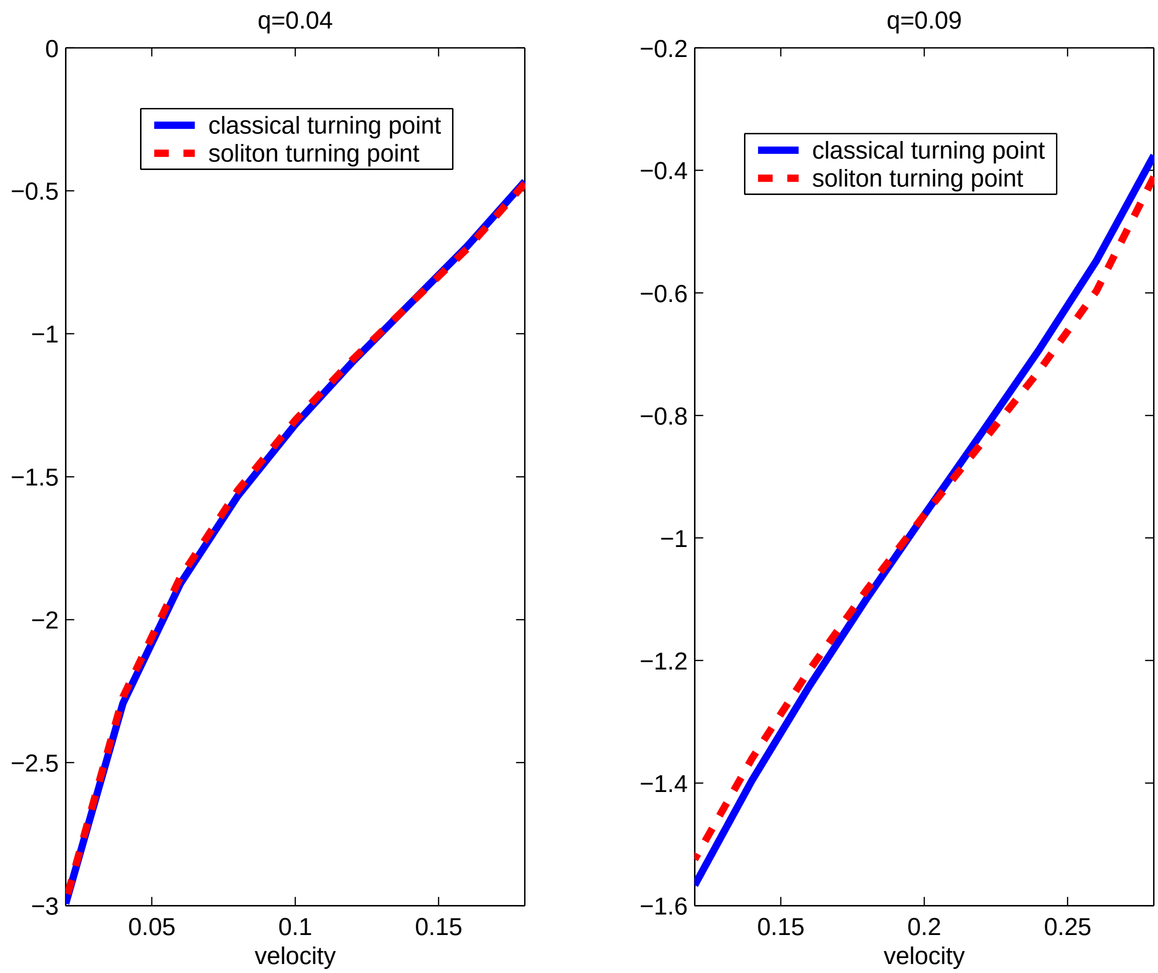

FIGURE 3. Two plots with $q = 0.04$ and $q = 0.09$, respectively, and $a_0 = -10$. The blue line is the theoretical prediction of the turning point of the soliton, $|a_{\rm turn}| = \operatorname{sech}^{-1}\left(v/\sqrt{q}\right)$, and the red dashed line is the actual soliton turning point. For smaller values of q the agreement is outstanding.

3. Reparametrized evolution

To see the effective dynamics described in §2.5 we write the solution of (1.1) as

$$u(t) = g(t) \cdot (\eta + w(t)) \,, \quad w(t) \in H^1(\mathbb{R}, \mathbb{C}) \,,$$

where $w(t)$ satisfies

$$\omega(w(t), X\eta) = 0 \,, \quad \forall X \in \mathfrak{g} \,.$$

To see that this decomposition is possible, initially for small times, we apply the following consequence of the implicit function theorem and the nondegeneracy of $\omega\restriction_M$ (see [6, Proposition 5.1] for a more general statement):

Lemma 3.1. *For $\Sigma \Subset G/\mathbb{Z}$ (where the topology on $G/\mathbb{Z}$ is given by the identification with $\mathbb{R}\times\mathbb{R}\times S^1\times\mathbb{R}_+$) let*

$$U_{\Sigma,\delta} = \{\, u \in H_1 \;:\; \inf_{g\in\Sigma} \|u - g\cdot\eta\|_{H^1} < \delta \} \,.$$

If $\delta \leq \delta_0 = \delta_0(\Sigma)$ then for any $u \in U_{\Sigma,\delta}$, there exists a unique $g(u) \in \Sigma$ such that

$$\omega(g(u)^{-1}\cdot u - \eta, X\cdot\eta) = 0 \quad \forall X \in \mathfrak{g}\,.$$

Moreover, the map $u \mapsto g(u)$ is in $C^1(U_{\Sigma,\delta},\Sigma)$.

Proof. We define the transformation

$$F \;:\; H^1(\mathbb{R},\mathbb{C})\times G \;\longrightarrow\; \mathfrak{g}^*\,, \quad [F(u,h)](X) \stackrel{\text{def}}{=} \omega(h\cdot u - \eta, X\cdot\eta)\,.$$

We want to solve $F(u,h) = 0$ for $h = h(u)$ and by the implicit fuction theorem that follows for u near $G\cdot\eta$ if for any $g_0 \in G$ the linear transformation

$$d_h F(g_0\cdot\eta, g_0) \;:\; T_{g_0}G \longrightarrow \mathfrak{g}^*\,,$$

is invertible. Clearly we only need to check it for $g_0 = e$, that is that $d_hF(\eta,e) \;:\; \mathfrak{g}\to\mathfrak{g}^*$, is invertible. But as an element of $\mathfrak{g}^*\otimes\mathfrak{g}^*$, $d_hF(\eta,e) = (\omega\restriction_M)_\eta$, which is nondegenerate. ☐

For §§2.1 and 2.5 we recall that the equation for u (1.1) can be written as

$$\partial_t u = \Xi_{H_q}(u)\,, \quad H_q(u) \stackrel{\text{def}}{=} \frac{1}{4}\int (|\partial_x u|^2 - |u|^4)dx + \frac{1}{2}q|u(0)|^2\,. \tag{3.1}$$

Using Lemma 3.1 we define

$$w(t) = g(t)^{-1}u(t) - \eta\,, \quad g(t) \stackrel{\text{def}}{=} g(u(t))\,, \tag{3.2}$$

and we want to to derive an equation for $w(t)$.

By the chain rule and Lemma 2.3

$$\partial_t u(t) = \partial_t(g(t)\cdot(\eta + w(t))) = g(t)\cdot(Y(t)(\eta+w(t)) + \partial_t w(t))\,,$$

$$Y(t) \stackrel{\text{def}}{=} \dot a(t)\mu(t)e_1 + \frac{\dot v(t)}{\mu(t)}e_2 + (\dot\gamma(t) - \dot a(t)v(t))e_3 + \frac{\dot\mu(t)}{\mu(t)}e_4\,,$$

$g(t) = (a(t), v(t), \gamma(t), \mu(t))$. Combined with (3.1) this gives

$$\partial_t w(t) = -Y(t)\eta - Y(t)w + g(t)^{-1}\Xi_{H_q}((g(t)\cdot(\eta + w(t))))\,. \tag{3.3}$$

To make this more explicit we apply Lemma 2.1 to see that

$$g(t)^{-1}\Xi_{H_q} g(t) = \frac{1}{\mu(t)}\Xi_{g(t)^*H_q}$$

(since the action of $g(t)$ is linear on H^1, $g(t)^{-1}$ and $(g(t)^{-1})_*$ are identified). We compute

$$\begin{aligned} g^* H_q(\tilde{u}) &= \frac{1}{4}\int (\mu|\partial_x(e^{ixv}\tilde{u}(\mu x))|^2 - \mu^4|\tilde{u}(\mu x)|^4)dx + \frac{1}{2}q\mu^2|\tilde{u}(-\mu a)|^2 \\ &= \frac{1}{4}\int (\mu^3|\partial_x\tilde{u}(x)|^2 - 2v\mu^2\operatorname{Im}\partial_x\tilde{u}(x)\overline{\tilde{u}(x)} + \operatorname{Re} v^2\mu|\tilde{u}(x)|^2 - \mu^3|\tilde{u}(x)|^4)dx \\ &\qquad + \frac{1}{2}\mu^2 q|\tilde{u}(-\mu a)|^2 \,, \end{aligned} \tag{3.4}$$

so that

$$\frac{1}{\mu(g)}\Xi_{g^*H_q}(\tilde{u}) = \frac{1}{i}\left(-\frac{\mu^2}{2}\tilde{u}_{xx} + v\mu\tilde{u}_x - \mu^2|\tilde{u}|^2\tilde{u} + \frac{v^2}{2}\tilde{u} + \mu q\delta(\bullet + \mu a)\tilde{u}\right) .$$

For us $\tilde{u}(t) = \eta + w(t)$ and we expand the nonlinear term

$$|\eta + w|^2(\eta + w) = \eta^3 + \underbrace{2\eta^2 w + \eta^2\bar{w}}_{\text{linear}} + \underbrace{2|w|^2\eta + \eta w^2}_{\text{quadratic}} + \underbrace{|w|^2 w}_{\text{cubic}}$$

Inserting this in (3.3) gives

Lemma 3.2. *If $w(t)$ is given by* (3.2) *then*

$$\partial_t w = X(t)w + X(t)\eta - i\mu^2\mathcal{L}w + i\mu^2\mathcal{N}w - iq\mu\delta_0(x + \mu a)\eta - iq\mu\delta_0(x + \mu a)w \,, \tag{3.5}$$

where $X(t) \in \mathfrak{g}$ is given by

$$X(t) \stackrel{\text{def}}{=} (-\mu\dot{a} + v\mu)\, e_1 - \frac{\dot{v}}{\mu}e_2 + \left(-\dot{\gamma} + v\dot{a} - \frac{v^2}{2} + \frac{\mu^2}{2}\right) e_3 - \frac{\dot{\mu}}{\mu}e_4 \,, \tag{3.6}$$

and

$$\mathcal{L}w = -\frac{1}{2}\partial_x^2 w - 2\eta^2 w - \eta^2\bar{w} + \frac{1}{2}w \,, \quad \mathcal{N}w = 2|w|^2\eta + \eta w^2 + |w|^2 w \,.$$

□

We now want to estimate the coefficients of $X(t)$ in (3.5) using the symplectic orthogonality of $Y\eta$, $Y \in \mathfrak{g}$ and w. For that we define

$$P \;:\; \mathcal{S}'(\mathbb{R}, \mathbb{C}) \longrightarrow \mathfrak{g}$$

as the unique linear map satisfying

$$\omega(u - P(u)\eta, Y\eta) = 0 \;\; \forall\, Y \in \mathfrak{g} \,.$$

We will need the following

Lemma 3.3. *Let $\| \bullet \|$ be a norm on $\mathfrak{g}$ obtained by using the standard $\mathbb{R}^4$ norm in the basis given by* (2.10)*. Then for $w \in H^1$, and $Y \in \mathfrak{g}$,*

$$
\begin{aligned}
&\|P(Yw)\| \leq C\|Y\|\|w\|_{L^2} \,, \\
&\|P(i\mathcal{N}u)\| \leq C\|w\|_{L^2}^2 \left(1 + \|w\|_{H^1}^{\frac{1}{2}}\|w\|_{L^2}^{\frac{1}{2}}\right) \,, \\
&\|P((i\delta_0(x-x_0)w)\| \leq C\|w\|_{H^1}^{\frac{1}{2}}\|w\|_{L^2}^{\frac{1}{2}} \,,
\end{aligned}
$$

with the constant independent of x_0.

Proof. We start with an explicit expression for P which follows from (2.13):

$$
P = \sum_{j=1}^{4} e_j P_j \,, \quad P_j \;:\; \mathcal{S}' \longrightarrow \mathbb{R} \,,
$$

$$
\begin{aligned}
P_1(u) &= -\omega(u, e_2\eta) = \mathrm{Re} \int u(x)x\eta(x)dx \,, \\
P_2(u) &= \omega(u, e_1\eta) = -\,\mathrm{Im} \int u(x)\partial_x\eta(x)dx \,, \\
P_3(u) &= \omega(u, e_4\eta) = \mathrm{Im} \int u(x)\partial_x(x\eta(x))dx \,, \\
P_4(u) &= -\omega(u, e_3\eta) = \mathrm{Re} \int u(x)\eta(x)dx \,.
\end{aligned} \tag{3.7}
$$

We now recall that $\|u\|_{L^\infty(\mathbb{R})}^2 \leq C\|u\|_{L^2(\mathbb{R})}\|u\|_{H^1(\mathbb{R})}$ and the estimates follow. □

We defined the following modified curve in $\mathfrak{g}$:

$$
X_1(t) \stackrel{\text{def}}{=} X(t) - q\mu P(i\delta_0(\bullet + a\mu)\eta) \,, \tag{3.8}
$$

which is estimated in

Proposition 3.4. *Suppose that $w(t)$ is given in Lemma 3.2 and that $X_1(t)$ is given by* (3.8)*. Then*

$$
\|X_1(t)\| \leq Cq\|w\|_{H^1} + C(\|w\|_{L^2}^2 + \|w\|_{H^1}^3) \,. \tag{3.9}
$$

Proof. Since $Pw_t = \partial_t Pw = 0$, (3.5) gives

$$
\begin{aligned}
X_1(t) = &- P(X_1(t)w) - q\mu P(P(i\delta_0(\bullet + a\mu)\eta)w) \\
&+ \mu^2 P(i\mathcal{L}w) - \mu^2 P(i\mathcal{N}w) + q\mu P(i\delta_0(x + \mu a)w) \,.
\end{aligned}
$$

The linear operator $\mathcal{L}$ is the Hessian of $\mathcal{E}$, given in (1.9), at the critical point η. The fact that $\Xi_{\mathcal{E}}$ is tangent to M and Lemma 2.2 (or a direct computation) show that

$$
P(i\mathcal{L}w) = 0 \,,
$$

and hence that term can be dropped from the right hand side. We can then use Lemma 3.3 to obtain

$$\|X_1(t)\| \le C\|w\|_{L^2}\|X_1(t)\| + Cq\mu(\|w\|_{L^2} + \|w\|_{H^1}^{\frac{1}{2}}\|w\|_{L^2}^{\frac{1}{2}}) + C\|w\|_{L^2}^2\left(1 + \|w\|_{H^1}^{\frac{1}{2}}\|w\|_{L^2}^{\frac{1}{2}}\right) .$$

The estimate (3.9) follows from the assumed smallness of $\|w\|_{L^2}$. $\square$

Finally we interpret the coefficients of $X_1(t)$. First we use (3.7) to see that

$$P(i\delta_0(\bullet + a\mu)\eta) = \frac{1}{2}\partial_x(\eta^2)(a\mu)e_2 + \left(\eta^2(a\mu) + \frac{1}{2}a\mu\partial_x(\eta^2)(a\mu)\right)e_3 .$$

Then we combine this with (3.6) and (3.8) to obtain

$$\begin{aligned} X_1(t) &= (-\mu\dot{a} + v\mu)\, e_1 + \left(-\frac{1}{2}q\mu\partial_x(\eta^2)(a\mu) - \frac{\dot{v}}{\mu}\right) e_2 \\ &\quad + \left(-\mu q\eta^2(a\mu) - \frac{1}{2}qa\mu^2\partial_x(\eta^2)(a\mu) - \dot{\gamma} + v\dot{a} - \frac{v^2}{2} + \frac{\mu^2}{2}\right) e_3 - \frac{\dot{\mu}}{\mu}e_4 . \end{aligned} \tag{3.10}$$

We now see that

$$X_1(t) = 0 \iff \text{equations (2.19) hold.}$$

4. Spectral estimates

In this section we will recall the now standard estimates on the operator $\mathcal{L}$ which arises as Hessian of $\mathcal{E}$ at η:

$$\mathcal{L}w = -\frac{1}{2}\partial_x^2 w - 2\eta^2 w - \eta^2\bar{w} + \frac{1}{2}w ,$$

or

$$\mathcal{L}w = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}\begin{bmatrix} \operatorname{Re} w \\ \operatorname{Im} w \end{bmatrix} , \quad L_\pm = -\frac{1}{2}\partial_x^2 - (2 \pm 1)\eta^2 + \frac{1}{2} .$$

In our special case we can be more precise than in the general case (see [16], and also [6, Appendix D]). The self-adjoint operators $L_\pm$ belong the class of Schrödinger operators with *Pöschl-Teller* potentials and their spectra can be explicitly computed using hypergeometric functions – see for instance [10, Appendix]. This gives

$$\sigma(L_-) = \{0\} \cup [1/2, \infty) , \quad \sigma(L_+) = \{0, -3/2\} \cup [1/2, \infty) .$$

The eigenfuctions can computed by the same method but a straightforward verification is sufficient to see that

$$L_-\eta = 0 , \quad L_+(\partial_x\eta) = 0 , \quad L_+(\eta^2) = -\frac{3}{2}\eta^2 .$$

We now have

Proposition 4.1. *Suppose that for every $X \in \mathfrak{g}$*

$$\omega(w, X \cdot \eta) = 0\,, \quad w \in H^1(\mathbb{R}, \mathbb{C})\,.$$

Then, with $\langle w, v\rangle \stackrel{\text{def}}{=} \operatorname{Re}\int w\bar{v}$ *on* $H^1(\mathbb{R},\mathbb{C})$ *(considered as a real Hilbert space),*

$$\langle \mathcal{L} w, w\rangle \geq \rho_0 \|w\|^2_{L^2}\,, \quad \rho_0 = \frac{9}{2(12+\pi^2)} \simeq 0.2058\,. \tag{4.1}$$

We need the following elementary

Lemma 4.2. *Let V be a real vector space with an inner product $\langle \bullet, \bullet\rangle$, and let L be a symmetric operator on V. Suppose that for $v_0, v_1 \in V$, $\|v_j\| = 1$, we have*

$$\begin{gathered} Lv_0 = -c_0 v_0\,, \quad c_0 \geq 0\,, \quad \langle v_0, v_1\rangle^2 = c_2\,, \\ \langle w, v_0\rangle = 0 \implies \langle Lw, w\rangle \geq c_1 \|w\|^2\,, \quad c_1 \geq 0\,. \end{gathered} \tag{4.2}$$

Then

$$\langle v, v_1\rangle = 0 \implies \langle Lv, v\rangle \geq c_3 \|v\|^2\,, \quad c_3 \stackrel{\text{def}}{=} c_1 c_2 - c_0(1-c_2)\,. \tag{4.3}$$

Proof. For reader's convenience we present the straightforward argument in which we can assume that $0 < c_2 < 1$. For $v \in V$ we write $v = \alpha v_0 + w$, $\langle v_0, w\rangle = 0$. The condition $\langle v, v_1\rangle = 0$ gives

$$\alpha^2 = \frac{1}{c_2}\langle w, v_1\rangle^2 = \frac{1}{c_2}\langle w, v_1 - c_2^{\frac{1}{2}} v_0\rangle^2 \leq \frac{1-c_2}{c_2}\|w\|^2\,. \tag{4.4}$$

Hence

$$\begin{aligned} \langle Lv, v\rangle &\geq c_1\|w\|^2 - c_0\alpha^2 \\ &\geq c_1\delta\|w\|^2 + \left(c_1(1-\delta)\frac{c_2}{1-c_2} - c_0\right)\alpha^2 \\ &= (c_1c_2 - c_0(1-c_2))\|v\|^2\,, \end{aligned}$$

if we choose $c_1\delta = (c_1c_2 - c_0(1-c_2))$. □

Proof of Proposition 4.1: The assumption means that

$$\operatorname{Im}\int w \left\{ \begin{array}{c} i\eta \\ \partial_x \eta \\ ix\eta \\ \partial_x(x\eta) \end{array} \right\} dx = 0\,.$$

Working with real and imaginary parts the proof reduces to lower bounds on $L_\pm$:

$$\begin{gathered} \langle v, \eta\rangle = \langle v, x\eta\rangle = 0 \implies \langle L_+ v, v\rangle \geq \rho_0\|v\|^2_{L^2}\,, \\ \langle v, \partial_x\eta\rangle = \langle v, (x\partial_x + 1)\eta\rangle = 0 \implies \langle L_- v, v\rangle \geq \rho_0 \|v\|^2_{L^2}\,, \end{gathered} \tag{4.5}$$

where now $v \in H^1(\mathbb{R};\mathbb{R})$. Noting that

$$\langle \eta, \partial_x \eta \rangle = \langle x\eta, \eta^2 \rangle = \langle \partial_x \eta, \eta^2 \rangle = 0$$

we can apply Lemma 4.2 in the following three cases:

$$\begin{gathered}
V = (\partial_x \eta)^\perp \cap H^2(\mathbb{R},\mathbb{R}) \,, \quad v_0 = \frac{\sqrt{3}}{2}\eta^2 \,, \quad v_1 = \frac{1}{\sqrt{2}}\eta \,, \quad L = L_+ \\
c_0^1 = \frac{3}{2} \,, \quad c_1^1 = \frac{1}{2} \,, \quad c_2^1 = \frac{3\pi^2}{32} \,, \\
V = (\eta^2)^\perp \cap H^2(\mathbb{R},\mathbb{R}) \,, \quad v_0 = \frac{\sqrt{3}}{\sqrt{2}}\partial_x \eta \,, \quad v_1 = \frac{\sqrt{6}}{\pi} x\eta \,, \quad L = L_+ \\
c_0^2 = 0 \,, \quad c_1^2 = \frac{1}{2} \,, \quad c_2^2 = \frac{9}{\pi^2} \,, \\
V = H^2(\mathbb{R},\mathbb{R}) \,, \quad v_0 = \frac{1}{\sqrt{2}}\eta \,, \quad v_1 = \frac{2\sqrt{2}}{\sqrt{12+\pi^2}}\partial_x(x\eta) \,, \quad L = L_- \,, \\
c_0^3 = 0 \,, \quad c_1^3 = \frac{1}{2} \,, \quad c_2^3 = \frac{9}{12+\pi^2} \,.
\end{gathered}$$

Here we used

$$\begin{gathered}
\int_{\mathbb{R}} \operatorname{sech}^2(x)dx = 2 \,, \quad \int_{\mathbb{R}} \operatorname{sech}^4(x)dx = \frac{4}{3} \,, \quad \int_{\mathbb{R}} \operatorname{sech}^3(x)dx = \frac{\pi}{2} \,, \quad \int_{\mathbb{R}} x^2 \operatorname{sech}^2(x)dx = \frac{\pi^2}{6} \,, \\
\int_{\mathbb{R}} \tanh^2(x)\operatorname{sech}^2(x)dx = \frac{2}{3} \,, \quad \int_{\mathbb{R}} (\partial_x(x\operatorname{sech}(x)))^2 dx = \frac{1}{18}(12+\pi^2) \,.
\end{gathered}$$

It follows that we can take

$$\rho_0 = \min_{j=1,2,3} (c_1^j c_2^j - c_0^j(1-c_2^j)) = \min\left(\frac{3\pi^2}{16} - \frac{3}{2}, \frac{9}{2\pi^2}, \frac{9}{2(12+\pi^2)}\right) = \frac{9}{2(12+\pi^2)} \,,$$

completing the proof. □

Proposition 4.1 gives a slightly stronger statement:

$$\begin{aligned}
\langle \mathcal{L}w, w \rangle &\geq (1-\delta)\langle \mathcal{L}w, w\rangle + \delta\rho_0 \|w\|_{L^2}^2 \\
&\geq (1-\delta)\left(\frac{1}{2}\|\partial_x w\|^2 - \frac{5}{2}\|w\|^2\right) + \delta\rho_0\|w\|_{L^2}^2 \\
&\geq \frac{2\rho_0}{5+2\rho_0}\|\partial_x w\|^2 \simeq 0.0760\|\partial_x w\|^2 \,, \quad \delta = \frac{5}{5+2\rho_0} \,.
\end{aligned}$$

In addition,

$$\langle \mathcal{L}w, w\rangle \geq \frac{2\rho_0}{7+2\rho_0}\|w\|_{H^1}^2 \simeq 0.0555\|w\|_{H^1}^2 \,. \tag{4.6}$$

Remark. The smallness of these constants gives a possible explanation of the size of q's for which the asymptotic result agrees with numerical simulations. The implicit constants in §5 are closely related to the constants above.

5. Estimates on the Lyapunov function

Suppose $u = u(x,t)$ solves (1.1) with[‡] $|q| \ll 1$ and initial data

$$u_0(x) = e^{ixv_0}\eta(x - a_0), \qquad |v_0| \ll 1 \tag{5.1}$$

Let $T > 0$ be the maximal time such that on $[0,T]$, the smallness condition $\delta \leq \delta_0$ in Lemma 3.1 is met. From Lemma 3.1, obtain the C^1 parameters $\mu = \mu(t)$, $\gamma = \gamma(t)$, $v = v(t)$, $a = a(t)$ satisfying the symplectic orthogonality conditions stated there. Let $\tilde{u} = \tilde{u}(x,t)$ be defined by

$$u(x,t) = g(t) \cdot \tilde{u}(x,t) \stackrel{\text{def}}{=} e^{i\gamma} e^{ixv} \mu \tilde{u}(\mu(x-a), t)\,, \tag{5.2}$$

and let

$$w(x,t) = \tilde{u}(x,t) - \eta(x)\,.$$

The Lyapunov function of [16] and [6] is given by

$$L(w) \stackrel{\text{def}}{=} \mathcal{E}(\eta + w) - \mathcal{E}(\eta)\,. \tag{5.3}$$

The lower bound on $L(w)$ follows from the spectral estimates of §4, and in particular from (4.6). For the upper bound we will use the conservation of $H_q(u)$ and its relation to $\mathcal{E}(\eta + w)$.

For future reference we state the following crucial consequence of the orthogonality conditions on w, and in particular of the condition that $\operatorname{Im}\int i\eta\bar{w} = \operatorname{Re}\int \bar{w}\eta = 0$:

Lemma 5.1. *Suppose that for every $X \in \mathfrak{g}$*

$$\omega(w, X\cdot\eta) = 0\,, \quad w \in H^1(\mathbb{R}, \mathbb{C})\,.$$

Then

$$\|w\|_{L^2}^2 = \frac{2}{\mu}(1-\mu)\,, \tag{5.4}$$

Proof. We first compute

$$\|\eta + w\|_{L^2}^2 = \|g^{-1}u\|_{L^2}^2 = \frac{1}{\mu(g)}\|u\|_{L^2}^2 = \frac{2}{\mu(g)}\,,$$

where we used the conservation of the L^2 norm. As noted before the statement of the lemma $\operatorname{Re}\langle w, \eta\rangle = 0$ and hence

$$\|\eta + w\|_{L^2}^2 = 2 + \|w\|_{L^2}^2\,,$$

[‡]The symbol $\ll 1$ means smaller than an *absolute* positive constant, i.e. one independent of all parameters in this problem.

from which the conclusion follows. $\square$

As a consequence, we can dispense with μ in the estimates of Proposition 3.4, and we reformulate it as

Proposition 5.2. *Suppose* $1-\mu \ll 1$ *and* $|q| \leq 1$. *Then*

$$
\begin{aligned}
|v-\dot{a}| + |\dot{v} + q\partial_x \eta^2(a)/2| + |-q\eta^2(a) - qa\partial_x\eta^2(a)/2 - \dot{\gamma} + v^2/2 + 1/2| \\
\leq C(|q|\|w\|_{H^1}^2 + \|w\|_{H^1}^2 + \|w\|_{H^1}^3)\,.
\end{aligned}
$$

Proof. We use (5.4) in (3.9). For example,

$$
\begin{aligned}
\Big|\frac{1}{2}q\partial_x\eta^2(a) + \dot{v}\Big| &\leq \mu\Big|\frac{1}{2}\frac{q}{\mu}\partial_x\eta^2(a) + \frac{\dot{v}}{\mu}\Big| \\
&\leq \mu\Big|\frac{1}{2}q\mu\partial_x\eta^2(a\mu) + \frac{\dot{v}}{\mu}\Big| + c|q||1-\mu| \\
&\leq 2\Big|\frac{1}{2}q\mu\partial_x\eta^2(a\mu) + \frac{\dot{v}}{\mu}\Big| + c|q|\|w\|_{L^2}^2
\end{aligned}
$$

We also use the estimate for $|v-\dot{a}|$ to replace $v\dot{a}$ by v^2 in the equation for $\dot{\gamma}$. $\square$

We adopt the following notational convention: denote the initial (time $t=0$) configuration of the system by 0-subscripts – $u_0 = u(0)$, $w_0 = w(0)$, and $a_0 = a(0)$, $v_0 = v(0)$, $\mu_0 = \mu(0)$, $\gamma_0 = \gamma(0)$. Similarly, denote the configuration of the system at some fixed time t_i by i-subscripts. Finally, the configuration at any arbitrary time t we denote without subscripts – $w = w(t)$, $u = u(t)$ and $a = a(t)$, $v = v(t)$, $\mu = \mu(t)$, $\gamma = \gamma(t)$.

With this notation we now state

Lemma 5.3. *Suppose* $\mu_0 = 1$ *and* $w_0 = 0$ *(equivalently, suppose* (5.1) *holds), and suppose that* $T > 0$ *is the maximal time for which the smallness condition in Lemma 3.1 holds. Suppose that for an interval of time* $[t_i, t_{i+1}] \subset [0,T]$, *the following conditions hold*

$$
\begin{aligned}
0 \leq 1-\mu \ll 1, \quad \max_{t_i \leq s \leq t_{i+1}} |v(s)| \ll 1, \quad \|w_i\|_{H^1} \leq 1\,, \\
|q||t_{i+1}-t_i| \ll 1, \quad |t_{i+1}-t_i| \max_{t_i \leq s \leq t_{i+1}} |v(s)| \ll 1\,.
\end{aligned}
\tag{5.5}
$$

Then there is an absolute constant $c_* > 1$ *such that*

$$
\sup_{t_i \leq s \leq t_{i+1}} \|w(s)\|_{H^1}^2 \leq c_*\|w_i\|_{H^1}^2 + c_*|q|^2\,.
$$

We remark that the inequality, $0 \leq 1-\mu$, in (5.5) is not an assumption but follows from Lemma 5.1.

The main result of this section is the following consequence of this:

Proposition 5.4. *Suppose $\mu_0 = 1$ and $w_0 = 0$, and suppose that $T > 0$ is the maximal time for which the smallness condition in Lemma 3.1 holds. Let*

$$n \leq \frac{\delta \log(1/|q|)}{\log c_*} - 1$$

and suppose there is a partition of the time axis

$$0 = t_0 < t_1 < \cdots < t_n \leq T$$

such that on each subinterval $[t_i, t_{i+1}]$, (5.5) in Lemma 5.3 holds. Then,

$$\sup_{0 \leq s \leq t_n} \|w(s)\|_{H^1}^2 \leq |q|^{2-\delta}$$

Proof of Lemma 5.3. We start by noting that in the argument that follows, we will not use any information about w or the parameters μ, γ, a, and v for times $0 < t < t_i$; only that $\mu_0 = 1$ and $w_0 = 0$.

We will conveniently reexpress $L(w)$ given by (5.3) using the conserved Hamiltonian and mass. Since $u = g \cdot \tilde{u}$, we recall (3.4) to obtain:

$$\begin{aligned} H_q(u) = g^* H_q(\tilde{u}) = \ & \frac{1}{4}\mu v^2 \int |\tilde{u}|^2 + \frac{1}{2}\mu^2 v \operatorname{Im} \int \partial_x \tilde{u}\, \bar{\tilde{u}} + \frac{1}{4}\mu^3 \int |\partial_x \tilde{u}|^2 \\ & - \frac{1}{4}\mu^3 \int |\tilde{u}|^4 + \frac{1}{2} q \mu^2 |\tilde{u}(-\mu a, t)|^2 \end{aligned} \tag{5.6}$$

The expression for the mass, $M(u) = \int |u|^2$, becomes $M(u) = \mu \int |\tilde{u}|^2$. Using this and (5.6), we obtain

$$\mathcal{E}(\tilde{u}) = \frac{1}{\mu^3} H_q(u) + \frac{1}{4\mu} M(u) - \frac{v^2}{4\mu^3} M(u) - \frac{v}{2\mu} \operatorname{Im} \int \bar{\tilde{u}} \partial_x \tilde{u} - \frac{q}{2\mu} |\tilde{u}(-\mu a)|^2$$

Now substitute $\tilde{u} = \eta + w$ and use the orthogonality condition $\operatorname{Im} \int w \partial_x \eta = 0$ to obtain

$$\begin{aligned} \mathcal{E}(\eta + w) = \ & \frac{1}{\mu^3} H_q(u) + \frac{1}{4\mu} M(u) - \Big(\frac{v^2}{4\mu^3} M(u) + \frac{q}{2\mu} \eta(-\mu a)^2 \Big) \\ & - \frac{v}{2\mu} \operatorname{Im} \int \bar{w} \partial_x w - \frac{q}{\mu} \eta(-\mu a) \operatorname{Re} w(-\mu a) - \frac{q}{2\mu} |w(-\mu a)|^2 \end{aligned} \tag{5.7}$$

Note that the classical energy term (with the μ terms dropped)

$$E(u) \stackrel{\text{def}}{=} \frac{1}{4} v^2 M(u) + \frac{1}{2} q \eta(a)^2 \,,$$

has appeared in this expression. Evaluate (5.7) at $t = t_i$ to obtain

$$\begin{aligned} \mathcal{E}(\eta + w_i) = \ & \frac{1}{\mu_i^3} H_q(u) + \frac{1}{4\mu_i} M(u) - \Big(\frac{v_i^2}{4\mu_i^3} M(u) + \frac{q}{2\mu_i} \eta(-\mu_i a_i)^2 \Big) \\ & - \frac{v_i}{2\mu_i} \operatorname{Im} \int \bar{w}_i \partial_x w_i - \frac{q}{\mu_i} \eta(-\mu_i a_i) \operatorname{Re} w(-\mu_i a_i) - \frac{q}{2\mu_i} |w(-\mu_i a_i)|^2 \end{aligned} \tag{5.8}$$

By taking the difference of the right hand sides of (5.7) and (5.8), we obtain

$$
\begin{aligned}
&\mathcal{E}(\eta+w)-\mathcal{E}(\eta)\\
&\quad= \Big(\frac{1}{\mu^3}-\frac{1}{\mu_i^3}\Big)H_q(u)+\frac{1}{4}\Big(\frac{1}{\mu}-\frac{1}{\mu_i}\Big)M(u)\\
&\qquad-\Big(\frac{v^2}{4\mu^3}M(u)+\frac{q}{2\mu}\eta(-\mu a)^2\Big)+\Big(\frac{v_i^2}{4\mu_i^3}M(u)+\frac{q}{2\mu_i}\eta(-\mu_i a_i)^2\Big)\\
&\qquad-\frac{v}{2\mu}\operatorname{Im}\int\bar{w}\partial_x w-\frac{q}{\mu}\eta(-\mu a)\operatorname{Re}w(-\mu a)-\frac{q}{2\mu}|w(-\mu a)|^2\\
&\qquad+\frac{v_i}{2\mu_i}\operatorname{Im}\int\bar{w}_i\partial_x\bar{w}_i+\frac{q}{\mu_i}\eta(-\mu_i a_i)\operatorname{Re}w_i(-\mu_i a_i)+\frac{q}{2\mu_i}|w_i(-\mu_i a_i)|^2\\
&\qquad+(\mathcal{E}(\eta+w_i)-\mathcal{E}(\eta))\\
&\quad=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}
\end{aligned}
\tag{5.9}
$$

where each line has been labeled by a Roman numeral. From the spectral estimate Proposition 4.1 (see (4.6)), we have

$$
c_1\|w\|_{H^1}^2-\|w\|_{H^1}^3-\frac{1}{4}\|w\|_{H^1}^4\le\mathcal{E}(\eta+w)-\mathcal{E}(\eta) \tag{5.10}
$$

We next estimate the right-hand side of (5.9), line by line. For $t_i\le t\le t_{i+1}$, let

$$
\epsilon(t)^2=\sup_{t_i\le s\le t}\|w(s)\|_{H^1}^2
$$

Estimate of the 1st line of (5.9). By the assumption $w_0=0$ and $\mu_0=1$, we have

$$
M(u)=M(\eta)=2 \tag{5.11}
$$

and

$$
H_q(u)=-\frac{1}{6}+\frac{1}{2}v_0^2+\frac{q}{2}\eta^2(a_0) \tag{5.12}
$$

By substituting (5.11) and (5.12) into Term I, we obtain $\mathrm{I}=\mathrm{I}_a+\mathrm{I}_b$, where

$$
\mathrm{I}_a=-\frac{1}{6}\Big(\frac{1}{\mu^3}-\frac{1}{\mu_i^3}\Big)+\frac{1}{2}\Big(\frac{1}{\mu}-\frac{1}{\mu_i}\Big)
$$

and

$$
\mathrm{I}_b=\Big(\frac{1}{\mu^3}-\frac{1}{\mu_i^3}\Big)\Big(\frac{1}{2}v_0^2+\frac{|q|}{2}\eta^2(a_0)\Big)
$$

Inserting (5.4) in Term I_a, gives

$$\begin{aligned}\mathrm{I}_a &= \frac{1}{6}\Big(\frac{1}{\mu}-\frac{1}{\mu_i}\Big)\Big(3-\frac{1}{\mu^2}-\frac{1}{\mu\mu_i}-\frac{1}{\mu_i^2}\Big)\\ &= \frac{1}{6}\Big(\frac{1}{\mu}-\frac{1}{\mu_i}\Big)\Big[\Big(1-\frac{1}{\mu^2}\Big)+\Big(1-\frac{1}{\mu\mu_i}\Big)+\Big(1-\frac{1}{\mu_i^2}\Big)\Big]\\ &= \frac{1}{6}\Big(-\frac{1}{2}\|w\|_{L^2}^2+\frac{1}{2}\|w_i\|_{L^2}^2\Big)\Big[\Big(\frac{1}{2}(1+\frac{1}{\mu})\|w\|_{L^2}^2\Big)\\ &\qquad +\Big(\frac{1}{2}\|w\|_{L^2}^2+\frac{1}{2\mu}\|w_i\|_{L^2}^2\Big)+\Big(\frac{1}{2}(1+\frac{1}{\mu_i})\|w_i\|_{L^2}^2\Big)\Big]\end{aligned}$$

and thus

$$|\mathrm{I}_a| \le \frac{1}{6}(\|w\|_{L^2}^2+\|w_i\|_{L^2}^2)^2$$

For Term I_b, we have

$$\mathrm{I}_b = \Big(\frac{1}{\mu}-\frac{1}{\mu_i}\Big)\Big(\frac{1}{\mu^2}+\frac{1}{\mu\mu_i}+\frac{1}{\mu_i^2}\Big)\Big(\frac{1}{2}v_0^2+\frac{|q|}{2}\eta^2(a_0)\Big)$$

and thus

$$|\mathrm{I}_b| \le \frac{3}{4}(v_0^2+|q|)(\|w\|^2+\|w_i\|^2)$$

Collecting these estimates, we obtain

$$|\mathrm{I}| \le \epsilon^4 + 2(v_0^2+|q|)\epsilon^2 \tag{5.13}$$

Remark: This direct calculation is in fact the consequence of $d\mathcal{E}_\eta = 0$. We are using

$$\mathcal{E}(\mu\cdot\eta)-\mathcal{E}(\eta) = \mathcal{O}((1-\mu)^2)\,,$$

which follows from

$$\partial_\mu \mathcal{E}(\mu\cdot\eta)\restriction_{\mu=1} = 0\,.$$

Estimate of the 2nd line of (5.9) *(classical energies).* We compute

$$\begin{aligned}\partial_t\Big(\frac{v^2}{2}+\frac{q}{2}\eta^2(a)\Big) &= v\dot v + \frac{1}{2}q\partial_x\eta^2(a)\dot a\\ &= \Big(\dot v+\frac{1}{2}q\partial_x\eta^2(a)\Big)v + \frac{1}{2}q\partial_x\eta^2(a)(\dot a - v)\end{aligned}$$

and thus by Proposition 5.2,

$$\Big|\partial_t\Big(\frac{v^2}{2}+\frac{q}{2}\eta^2(a)\Big)\Big| \le c(\|w\|_{H^1}^2+|q|\|w\|_{H^1}+\|w\|_{H^1}^3)(|v|+|q|)$$

By the fundamental theorem of calculus,

$$\Big|\Big(\frac{v^2}{2}+\frac{q}{2}\eta^2(a)\Big)-\Big(\frac{v_i^2}{2}+\frac{q}{2}\eta^2(a_i)\Big)\Big| \le c(\epsilon^2+|q|\epsilon+\epsilon^3)\big(|t-t_i|\max_{t_i\le s\le t}|v(s)|+|q||t-t_i|\big)$$

As in the proof of Proposition 5.2, we can install μ's in this expression using (5.4) to obtain

$$|\mathrm{II}| \le c(\epsilon^2 + |q|\epsilon)\big(|t - t_i| \max_{t_i \le s \le t} |v(s)| + |q||t - t_i| + v^2 + |q|\big) \tag{5.14}$$

Estimate of the 3rd and 4th lines of (5.9). By the Cauchy-Schwarz inequality and the Sobolev embedding theorem,

$$|\mathrm{III}| \le |v|\|w\|_{H^1}^2 + |q|\|w\|_{H^1} + |q|\|w\|_{H^1}^2$$

Similarly,

$$|\mathrm{IV}| \le |v|\|w_i\|_{H^1}^2 + |q|\|w_i\|_{H^1} + |q|\|w_i\|_{H^1}^2$$

and thus

$$|\mathrm{III}| + |\mathrm{IV}| \le 2(|v| + |q|)\epsilon^2 + 2|q|\epsilon \tag{5.15}$$

Estimate of the 5th line of (5.9). By definition of $\mathcal{E}$, we have

$$\mathcal{E}(\eta + w_i) = \frac{1}{4}\int |\partial_x \eta + \partial_x w_i|^2 - \frac{1}{4}\int |\eta + w_i|^4 + \frac{1}{4}\int |\eta + w_i|^2 \tag{5.16}$$

Substitute into (5.16) the three expansions:

$$\begin{aligned}
&|\partial_x \eta + \partial_x w_i|^2 = |\partial_x \eta|^2 + 2\,\mathrm{Re}\,\partial_x \eta\, \partial_x w_i + |\partial_x w_i|^2 \\
&|\eta + w_i|^4 = \eta^4 + 4\,\mathrm{Re}\,\eta^3 w_i + 2\eta^2(2(\mathrm{Re}\, w_i)^2 + |w_i|^2) + 4\eta(\mathrm{Re}\, w_i)|w_i|^2 + |w_i|^4 \\
&|\eta + w_i|^2 = \eta^2 + 2\eta\,\mathrm{Re}\, w_i + |w_i|^2
\end{aligned}$$

and observe that the linear terms cancel since η solves $-\frac{1}{2}\eta + \frac{1}{2}\eta'' + \eta^3 = 0$. Thus, we obtain the estimate

$$|\mathrm{V}| \le 8\|w_i\|_{H^1}^2 + 4\|w_i\|_{H^1}^3 + \|w_i\|_{H^1}^4 \le 10\|w_i\|_{H^1}^2 \tag{5.17}$$

This completes the line-by-line estimation of the right-hand side of (5.9). By combining (5.10), and the estimates (5.13),(5.14),(5.15),(5.17) for the right-hand side of (5.9), we obtain

$$\begin{aligned}
c_1\epsilon^2 \le\ & \epsilon^3 + \frac{1}{4}\epsilon^4 + c(\epsilon^2 + |q|\epsilon + \epsilon^3)(|t - t_i| \max_{t_i \le s \le t} |v(s)| + |q||t - t_i| + v^2 + |q|) \\
& + [\epsilon^4 + 2(v_0^2 + |q|)\epsilon^2] + [2(|v| + |q|)\epsilon^2 + 2|q|\epsilon] + 10\|w_i\|_{H_1}^2
\end{aligned}$$

By hypothesis, every ϵ^2 term on the right side has a small coefficient, and thus can be absorbed on the left side. Therefore, we obtain

$$\epsilon^2 \le c(|q|\epsilon + \|w_i\|_{H^1}^2)$$

By applying the Peter-Paul inequality $|q|\epsilon \le \frac{1}{2}c|q|^2 + \frac{\epsilon^2}{2c}$, we obtain the desired estimate. $\square$

Proof of Proposition 5.4. Now let

$$\epsilon^2(t) = \sup_{0\le s\le t} \|w(s)\|_{H^1}^2$$

On the first interval $[0,t_1]$, we apply Lemma 5.3 with $i=0$, and since $w_0=0$, we obtain

$$\epsilon(t_1)^2 \le c_*|q|^2$$

On the second interval $[t_1,t_2]$, we apply Lemma 5.3 with $i=1$, and since $\|w_1\|_{H^1}^2 \le c_*|q|^2$, we obtain

$$\epsilon(t_2)^2 \le (c_*+c_*^2)|q|^2$$

We continue, and after the n applications, we obtain

$$\epsilon(t_n)^2 \le c_*\left(\sum_{j=0}^{n-1} c_*^j\right)|q|^2 = c_*\left(\frac{c_*^n-1}{c_*-1}\right)|q|^2 \le c_*^{n+1}|q|^2$$

Since we want $c_*^{n+1}q^2 \le |q|^{2-\delta}$, we require

$$n+1 \le \frac{\delta\log(1/|q|)}{\log c_*}$$

□

6. ODE analysis

The assumptions of Lemma 5.3 involve estimates on $v(s)$. To control these we use Proposition 5.2 and ODE estimates which we present in this section.

Lemma 6.1. *Suppose q is a constant, $|q|\ll 1$, and $a=a(t)$, $v=v(t)$, $\epsilon_1=\epsilon_1(t)$, $\epsilon_2=\epsilon_2(t)$ are C^1 real-valued functions. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a C^2 mapping such that $|f|$ and $|f'|$ are uniformly bounded. Suppose that on $[0,T]$,*

$$\begin{cases} \dot a = v+\epsilon_1 \\ \dot v = qf(a)+\epsilon_2 \end{cases}, \qquad \begin{matrix} a(0)=a_0 \\ v(0)=v_0 \end{matrix} \tag{6.1}$$

Let $\bar a=\bar a(t)$ and $\bar v=\bar v(t)$ be the C^1 real-valued functions satisfying the exact equations

$$\begin{cases} \dot{\bar a} = \bar v \\ \dot{\bar v} = qf(\bar a) \end{cases}, \qquad \begin{matrix} \bar a(0)=a_0 \\ \bar v(0)=v_0 \end{matrix}$$

with the same initial data. Suppose that on $[0,T]$, we have $|\epsilon_j|\le|q|^{2-\delta}$ for $j=1,2$. Then provided $T\le\delta|q|^{-1/2}\log(1/|q|)$, we have on $[0,T]$ the estimates

$$|a-\bar a|\le|q|^{1-2\delta}\log(1/|q|), \qquad |v-\bar v|\le|q|^{\frac32-2\delta}\log(1/|q|)$$

Before proceeding to the proof, we recall some basic tools.

Gronwall estimate. Suppose $b = b(t)$ and $w = w(t)$ are C^1 real-valued functions, q is a constant, and (b, w) satisfy the differential inequality:

$$\begin{cases} |\dot{b}| \leq |w| & b(0) = b_0 \\ |\dot{w}| \leq |q||b| \ , & w(0) = w_0 \end{cases} \tag{6.2}$$

Let $x(t) = |q|^{1/2} b(|q|^{-1/2} t)$, $y(t) = w(|q|^{-1/2} t)$. Then

$$\begin{cases} |\dot{x}| \leq |y| & x(0) = x_0 = |q|^{1/2} b_0 \\ |\dot{y}| \leq |x| \ , & y(0) = y_0 = w_0 \end{cases}$$

Let $z(t) = x^2 + y^2$. Then $|\dot{z}| = |2x\dot{x} + 2y\dot{y}| \leq 2|x||y| + 2|x||y| \leq 2(x^2 + y^2) = 2z$, and hence $z(t) \leq z(0)e^{2t}$. Thus

$$|x(t)| \leq \sqrt{2} \max(|x_0|, |y_0|) \exp(t)$$
$$|y(t)| \leq \sqrt{2} \max(|x_0|, |y_0|) \exp(t)$$

Converting from (x, y) back to (b, w), we obtain the Gronwall estimate

$$\begin{aligned} |b(t)| &\leq \sqrt{2} \max(|q|^{1/2}|b_0|, |w_0|) \frac{\exp(|q|^{1/2} t)}{|q|^{1/2}} \\ |w(t)| &\leq \sqrt{2} \max(|q|^{1/2}|b_0|, |w_0|) \exp(|q|^{1/2} t) \end{aligned} \tag{6.3}$$

Duhamel's formula. For a two-vector function $X(t) : \mathbb{R} \to \mathbb{R}^2$, a two-vector $X_0 \in \mathbb{R}^2$, and a 2×2 matrix function $A(t) : \mathbb{R} \to (2 \times 2 \text{ matrices})$, let $X(t) = S(t, t')X_0$ denote the solution to the ODE system $\dot{X}(t) = A(t)X(t)$ with $X(t') = X_0$. In other words, $\frac{d}{dt} S(t, t')X_0 = A(t)S(t, t')X_0$ and $S(t', t')X_0 = X_0$. Then, for a given two-vector function $f(t) : \mathbb{R} \to \mathbb{R}^2$, the solution to the inhomogeneous ODE system

$$\dot{X}(t) = A(t)X(t) + F(t) \tag{6.4}$$

with initial condition $X(0) = 0$ is given by Duhamel's formula

$$X(t) = \int_0^t S(t, t')F(t')dt' \tag{6.5}$$

Proof of Lemma 6.1. Let $\tilde{a} = a - \bar{a}$ and $\tilde{v} = v - \bar{v}$; these perturbative functions satisfy

$$\begin{cases} \dot{\tilde{a}} = \tilde{v} + \epsilon_1 & \tilde{a}(0) = 0 \\ \dot{\tilde{v}} = qg\tilde{a} + \epsilon_2 \ , & \tilde{v}(0) = 0 \end{cases}$$

where $g = g(t)$ is given by

$$g = \begin{cases} \dfrac{f(a) - f(\bar{a})}{a - \bar{a}} & \text{if } \bar{a} \neq a \\ f'(a) & \text{if } a = \bar{a} \end{cases}$$

which is C^1 (in particular, uniformly bounded). Set

$$A(t) = \begin{bmatrix} 0 & 1 \\ qg(t) & 0 \end{bmatrix}, \quad F(t) = \begin{bmatrix} \epsilon_1(t) \\ \epsilon_2(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} \tilde{a}(t) \\ \tilde{v}(t) \end{bmatrix}$$

in (6.4), and appeal to Duhamel's formula (6.5) to obtain

$$\begin{bmatrix} \tilde{a}(t) \\ \tilde{v}(t) \end{bmatrix} = \int_0^t S(t,t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix} dt' \tag{6.6}$$

Apply the Gronwall estimate (6.3) with

$$\begin{bmatrix} b(t) \\ w(t) \end{bmatrix} = S(t+t',t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix}, \quad \begin{bmatrix} b_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix}$$

to conclude that

$$\left| S(t,t') \begin{bmatrix} \epsilon_1(t') \\ \epsilon_2(t') \end{bmatrix} \right| \leq \sqrt{2} \begin{bmatrix} |q|^{-1/2} \exp(|q|^{1/2}(t-t')) \\ \exp(|q|^{1/2}(t-t')) \end{bmatrix} \max(|q|^{1/2}|\epsilon_1(t')|, |\epsilon_2(t')|)$$

Feed this into (6.6) to obtain that on $[0,T]$

$$\begin{aligned} |\tilde{a}(t)| &\leq \sqrt{2}\, T \frac{\exp(|q|^{1/2}T)}{|q|^{1/2}} \sup_{0\leq s\leq T} \max(|q|^{1/2}|\epsilon_1(s)|, |\epsilon_2(s)|) \\ |\tilde{v}(t)| &\leq \sqrt{2}\, T \exp(|q|^{1/2}T) \sup_{0\leq s\leq T} \max(|q|^{1/2}|\epsilon_1(s)|, |\epsilon_2(s)|) \end{aligned}$$

Taking $T \leq \delta |q|^{-1/2} \log(1/|q|)$, we obtain the claimed bounds. □

7. Main theorem and proof

Here we put all the components together and give a stronger version of Theorem 1. The basic procedure is the iteration of Lemmas 5.3 and 6.1 which can roughly be described as follows: if the conditions (5.5) hold, and the initial condition satisfies $\|w_i\|_{H^1} \leq |q|^{1-\delta}$, say, then on the interval $[t_i, t_{i+1}]$, $\|w(t)\|_{H^1} \leq 2\|q\|^{1-\delta}$. That means that the evolution of the parameters $g(t) \in G$ is close to the evolution using the effective Hamiltonian, in the way that makes Lemma 6.1 applicable. But that gives us a lower bound on t_{i+1} for which (5.5) holds on $[t_i, t_{i+1}]$, closing the bootstrap loop.

More precisely, we have

Theorem 2. *Suppose $|q| \ll 1$ and $|v_0| \ll 1$. Let u solve*

$$i\partial_t u + \partial_x^2 u - q\delta_0(x)u + |u|^2 u = 0$$

with initial data $u_0(x)$ satisfying

$$\|u_0 - e^{i\bullet v_0}\eta(\bullet - a_0)\|_{H^1} \leq C|q|\,.$$

Then, for times $0 \leq t \leq \delta(v_0^2 + |q|)^{-1/2} \log(1/|q|)$, *the smallness condition in Lemma 3.1 is met, and thus there are* C^1 *parameters* μ, v, γ, a *satisfying the symplectic orthogonality conditions stated there. Furthermore, we have*

$$\|u - \mu e^{ixv} e^{i\gamma} \eta(\mu(x-a))\|_{H^1} \leq c|q|^{1-\frac{1}{2}\delta}$$

Moreover, if $\bar{a}$, $\bar{v}$, $\bar{\gamma}$ *solve the ODE system*

$$\dot{\bar{a}} = \bar{v}\,, \quad \dot{\bar{v}} = -\frac{1}{2} q \partial_x \eta^2(\bar{a})\,, \quad \dot{\bar{\gamma}} = \frac{1}{2}\bar{v}^2 + \frac{1}{2} - q\eta^2(\bar{a}) + \frac{1}{2} q \bar{a} \partial_x \eta^2(\bar{a})\,. \tag{7.1}$$

with initial data $(a_0, v_0, 0)$, *then*

$$|a - \bar{a}| \leq c|q|^{1-3\delta}\,, \quad |\gamma - \bar{\gamma}| + |v - \bar{v}| \leq c|q|^{\frac{3}{2}-3\delta}\,, \quad |\mu - 1| \leq c|q|^{2-\delta}\,.$$

Proof. The equations (7.1) imply the conservation of energy

$$\frac{1}{2}\bar{v}^2 + \frac{1}{2} q\eta^2(\bar{a}) = \frac{1}{2} v_0^2 + \frac{1}{2} q \eta^2(a_0)$$

from which we obtain the bound

$$|\bar{v}| \leq \sqrt{v_0^2 + 2|q|}\,. \tag{7.2}$$

Let

$$\epsilon(t)^2 = \sup_{0 \leq s \leq t} \|w(s)\|_{H^1}^2\,.$$

By Proposition 5.2,

$$|\dot{a} - v| + |\dot{v} + \frac{1}{2} q \partial_x \eta^2(a)| \leq c_0 (q\|w\|_{H^1} + \|w\|_{H^1}^2 + \|w\|_{H^1}^3)\,. \tag{7.3}$$

Let t_1 with $T \geq t_1 > 0$ be the maximal time for which the assumptions of Lemma 5.3 (5.5) hold with $i = 0$. Then by Proposition 5.4 with $n = 1$, we have $\epsilon^2(t_1) \leq |q|^{2-\delta}$. The estimate (7.3) implies (6.1) in Lemma 6.1 for $t \in [0, t_1]$, with $f(a) = -\partial_x \eta(a)/2$. By Lemma 6.1 and (7.2), we have

$$\max_{0 \leq s \leq t_1} |v(s)| \leq 2\sqrt{v_0^2 + 2|q|}\,.$$

Reviewing (5.5), we now see that

$$T \geq t_1 \geq c_4 (v_0^2 + 2|q|)^{-1/2}\,,$$

where c_4 depends only on the implicit absolute constant in (5.5).

Now let t_2 with $T \geq t_2 > t_1$ be the maximum time such that (5.5) holds with $i = 1$. Then by Proposition 5.4 with $n = 2$, we have

$$\epsilon^2(t_2) \leq |q|^{2-\delta}\,.$$

By (7.3), we have that (6.1) in Lemma 6.1 holds on $[0, t_2]$. By Lemma 6.1 and (7.2), we have

$$\max_{0 \leq s \leq t_2} |v(s)| \leq 2\sqrt{v_0^2 + 2|q|}\,.$$

Reviewing (5.5), we now see that

$$|t_2 - t_1| \geq c_4 (v_0^2 + 2|q|)^{-1/2} ,$$

with the same c_4 as in the previous paragraph.

Continue until the nth step is reached, where

$$n = \frac{\delta \log(1/|q|)}{\log c_*} - 1 ,$$

which is the most allowed in Proposition 5.4. But now we know that

$$T \geq t_n \geq c\delta(v_0^2 + 2|q|)^{-1/2} \log(1/|q|) ,$$

and that on $[0, t_n]$,

$$|a - \bar{a}| \leq |q|^{1-2\delta} \log(1/|q|), \quad |v - \bar{v}| \leq |q|^{\frac{3}{2}-2\delta} \log(1/|q|)$$

We also have from Proposition 5.2,

$$\left|\dot{\gamma} - \left(\frac{1}{2}v^2 + \frac{1}{2} - q\eta^2(a) + \frac{1}{2}qa\partial_x\eta(a)\right)\right| \leq 2\|w\|_{H^1}^2 + |q|\|w\|_{H^1}$$

Subtracting the equations for $\dot{\gamma}$ and $\dot{\bar{\gamma}}$ and using that $\|w\| \leq |q|^{2-\delta}$, we obtain

$$\begin{aligned} |\dot{\gamma} - \dot{\bar{\gamma}}| &\leq |v^2 - \bar{v}^2| + |q||\eta^2(a) - \eta^2(\bar{a})| + |q||a - \bar{a}|\eta^2(\bar{a}) + |q||\bar{a}||\partial_x^2\eta(a) - \partial_x^2\eta(\bar{a})| \\ &\leq \left(|q|^{1/2}|q|^{\frac{3}{2}-2\delta} + |q||q|^{1-2\delta} + |q||q|^{1-2\delta}\right) \log(1/|q|) + |q|^{3-4\delta} \log^2(1/|q|) \\ &\leq |q|^{2-2\delta} \log(1/|q|) \end{aligned}$$

Since we restrict to times $t \leq \delta|q|^{-1/2} \log(1/|q|)$, we integrate to obtain $|\gamma - \bar{\gamma}| \leq |q|^{\frac{3}{2}-3\delta}$. $\square$

Remark. There remains the case of initial velocities, v_0, which are not small. When $|q| \to 0$ and $v_0 > 0$ is fixed, the dynamics is not interesting and the solution can be approximated by the solution with $q = 0$, that is by the propagating soliton (1.10). The proof of that follows from the arguments of [12, §3.1] and the details can be found in [4].

References

[1] J. C. Bronski and R. L. Jerrard, *Soliton dynamics in a potential,* Math. Res. Lett. **7**(2000), 329-342.

[2] C. Lee and J. Brand, *Enhanced quantum reflection of matter-wave solitons,* Europhys. Lett. **73**(2006), 321–327.

[3] X.D. Cao and B.A. Malomed, *Soliton-defect collisions in the nonlinear Schrödinger equation,* Physics Letters A **206**(1995), 177–182.

[4] Kiril Datchev, *A perturbative approximation for the nonlinear Schroedinger equation,* `http://math.berkeley.edu/~datchev/app.pdf`

[5] A. Floer and A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. **69**(1986), 397–408.

[6] J. Fröhlich, S. Gustafson, B.L.G. Jonsson, and I.M. Sigal, *Solitary wave dynamics in an external potential,* Comm. Math. Physics, **250**(2004), 613–642.

[7] J. Fröhlich, S. Gustafson, B.L.G. Jonsson, and I.M. Sigal, *Long time motion of NLS solitary waves in a confining potential,* Ann. H. Poincaré, **7**(2006), 621-660.

[8] J. Fröhlich, T.-P. Tsai, and H.-T. Yau, *On the point-particle (Newtonian) limit of the non-linear Hartree equation,* Comm. Math. Phys. **225**(2002), 223–274.

[9] R.H. Goodman, P.J. Holmes, and M.I. Weinstein, *Strong NLS soliton-defect interactions,* Physica D **192**(2004), 215–248.

[10] L. Guillopé and M. Zworski, *Upper bounds on the number of resonances on noncompact Riemann surfaces.* J. Func. Anal. **129**(1995), 364-389.

[11] S. Gustafson, *private communication*, December 2005.

[12] J. Holmer, J. Marzuola, and M. Zworski, *Fast soliton scattering by delta impurities,* to appear in Comm. Math. Phys.

[13] J. Holmer, J. Marzuola, and M. Zworski, *Soliton splitting by delta impurities,* preprint 2006.

[14] Y.G. Oh, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple-well potentials,* Comm. Math. Phys. **131**(1990), 223-253.

[15] L. Hörmander, *The Analysis of Linear Partial Differential Operators, vol.III,IV,* Springer Verlag, 1985.

[16] M.I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations,* Comm. Pure. Appl. Math. **29**(1986), 51-68.

E-mail address: holmer@math.berkeley.edu

E-mail address: zworski@math.berkeley.edu

Mathematics Department, University of California, Evans Hall, Berkeley, CA 94720, USA