ON BLOW-UP SOLUTIONS TO THE 3D CUBIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract. For the 3d cubic nonlinear Schrödinger (NLS) equation, which has critical (scaling) norms L^3 and $\dot{H}^{1/2}$, we first prove a result establishing sufficient conditions for global existence and sufficient conditions for finite-time blow-up. For the rest of the paper, we focus on the study of finite-time radial blow-up solutions, and prove a result on the concentration of the L^3 norm at the origin. Two disparate possibilities emerge, one which coincides with solutions typically observed in numerical experiments that consist of a specific bump profile with maximum at the origin and focus toward the origin at rate $\sim (T-t)^{1/2}$, where $T>0$ is the blow-up time. For the other possibility, we propose the existence of "contracting sphere blow-up solutions", i.e. those that concentrate on a sphere of radius $\sim (T-t)^{1/3}$, but focus towards this sphere at a faster rate $\sim (T-t)^{2/3}$. These conjectured solutions are analyzed through heuristic arguments and shown (at this level of precision) to be consistent with all conservation laws of the equation.

1. Introduction

Consider the cubic focusing nonlinear Schrödinger (NLS) equation on $\mathbb{R}^3$:

$$i\partial_t u + \Delta u + |u|^2 u = 0, \tag{1.1}$$

where $u = u(x,t)$ is complex-valued and $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$. The initial-value problem posed with initial-data $u(x,0) = u_0(x)$ is locally well-posed in H^1. Such solutions, during their lifespan $[0,T)$ (where $T = +\infty$ or $T < +\infty$), satisfy mass conservation $M[u](t) = M[u_0]$, where

$$M[u](t) = \int |u(x,t)|^2 \, dx,$$

and energy conservation $E[u](t) = E[u_0]$, where

$$E[u](t) = \frac{1}{2}\int |\nabla u(x,t)|^2 - \frac{1}{4}\int |u(x,t)|^4 \, dx.$$

If $\|xu_0\|_{L^2} < \infty$, then u satisfies the virial identity

$$\partial_t^2 \int |x|^2 |u(x,t)|^2 \, dx = 24E[u] - 4\|\nabla u(t)\|_{L^2_x}^2.$$

to appear in *Appl. Math. Res. eXpress*
submitted Sept 28, 2006,
accepted Feb 12, 2007.

The equation has scaling: $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is a solution if $u(x,t)$ is a solution. The scale-invariant Lebesgue norm for this equation is L^3, and the scale-invariant Sobolev norm is $\dot{H}^{1/2}$.

Fundamental questions include:

1. Under what conditions on the initial data u_0 is the solution u globally defined ($T = +\infty$)? If it is globally defined, does it scatter (approach the solution to a linear Schrödinger equation as $t \to +\infty$) or resolve into a sum of decoupled solitons plus a dispersive component? The latter type of inquiry leads one to the "soliton resolution conjecture" (see Tao [24, 26]).
2. If the solution fails to be globally defined (we say "blows-up in finite time"), can one provide a description of the behavior of the solution as $t \to T$, where T is the "blow-up time"? This will be the focus of the present paper.

It follows from the H^1 local theory optimized by scaling (see Cazenave [3] or Tao [25] for exposition), that if blow-up in finite-time $T > 0$ occurs, then there is a lower-bound on the "blow-up rate":

$$\|\nabla u(t)\|_{L^2_x} \geq \frac{c}{(T-t)^{1/4}} \tag{1.2}$$

for some absolute constant[1] c. Thus, to prove global existence, it suffices to prove a global *a priori* bound on $\|\nabla u(t)\|_{L^2}$. From the Strichartz estimates, there is a constant $c_{ST} > 0$ such that if $\|u_0\|_{\dot{H}^{1/2}} < c_{ST}$, then the solution u is globally defined and scatters. The *optimal*[2] constant c_{ST}, the "scattering threshold", has not to our knowledge been identified, under any regularity or decay assumptions on the solution.

Note that the quantities $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ and $M[u_0]E[u_0]$ are also scale-invariant. Another result of the above type follows from the conservation laws and the sharp Gagliardo-Nirenberg inequality of M. Weinstein [27]. Let $Q(x)$ be the minimal mass ground-state solution to the nonlinear elliptic equation

$$-\tfrac{1}{2}Q + \tfrac{3}{2}\Delta Q + Q^3 = 0$$

on $\mathbb{R}^3$, and set $u_Q(x,t) = e^{\frac{1}{2}it}Q(\sqrt{\tfrac{3}{2}}x)$. Then u_Q is a soliton solution to (1.1), and we have:

Theorem 1.1. *Suppose u is the (possibly nonradial) H^1 solution to* (1.1) *corresponding to initial data u_0 satisfying*

$$M[u_0]E[u_0] < M[u_Q]E[u_Q]\,. \tag{1.3}$$

[1]By "absolute constant" in this article we mean a constant that does not depend on any properties of the solution u under study (e.g. mass, energy, etc.), and usually depends on constants appearing in Gagliardo-Nirenberg, Sobolev, and Strichartz estimates.

[2]This is not the "sharp" constant in Strichartz inequalities but rather the one which will govern scattering.

1. *If* $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|u_Q\|_{L^2}\|\nabla u_Q\|_{L^2}$*, then* $\|u_0\|_{L^2}\|\nabla u(t)\|_{L^2} < \|u_Q\|_{L^2}\|\nabla u_Q\|_{L^2}$ *for all time, and thus, the solution is global.*[3]

2. *If* $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} > \|u_Q\|_{L^2}\|\nabla u_Q\|_{L^2}$*, then* $\|u_0\|_{L^2}\|\nabla u(t)\|_{L^2} > \|u_Q\|_{L^2}\|\nabla u_Q\|_{L^2}$ *on the maximal time interval of existence. If we further assume finite variance* $\|xu_0\|_{L^2} < \infty$ *or radial symmetry of the solution, then the solution blows-up in finite time.*

If (1.3) *fails to hold, then we have no information to conclude global existence or finite-time blow-up.*

We note that if the energy is negative, then via the sharp Gagliardo-Nirenberg inequality we automatically have $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} > \|u_Q\|_{L^2}\|\nabla u_Q\|_{L^2}$, and the second of the two cases in the theorem applies. In §2, we prove a generalized version of Theorem 1.1, and explain that it provides a bridge between similar known results for the L^2-critical (by Weinstein [27]) and $\dot{H}^1$-critical (by Kenig-Merle [11]) NLS equations.

Now, suppose blow-up in finite time occurs (for a solution of any energy), and let us restrict attention to the radially symmetric case. What can be said about the behavior of the solution as $t \to T$, where T is the blow-up time? As mentioned above, there is a lower bound (1.2) on the blow-up rate. Merle-Raphaël [17] have recently shown that the scale-invariant norm $\dot{H}^{1/2}$ has a divergent lower-bound:

$$\|u(\cdot,t)\|_{\dot{H}^{1/2}} \geq c\,|\log(T-t)|^{1/12}.$$

This is in contrast to the L^2-critical problem, where the scale invariant norm is, of course, constant. Merle-Raphaël do not prove any upper bound for general solutions on the rate of divergence of this norm.[4] The second result we present in this note describes two possibilities for the rate of concentration of the L^3 norm for radial finite-time blow-up solutions. We find it more convenient in this analysis to work with the critical Lebesgue norm L^3, rather than the critical Sobolev norm $\dot{H}^{1/2}$, since the former is more easily localized. In the physics or numerics literature (see Sulem-Sulem [23]), this type of concentration phenomena is termed "weak concentration" to distinguish it from concentration in the L^2 norm, which is called "strong concentration".

[3]Although, as far as we are aware, the threshold for scattering *might* be strictly smaller than $\|u_Q\|_{L^2}\|\nabla u_Q\|_{L^2}$.

[4]Indeed, the heuristic analysis of contracting sphere blow-up solutions we provide in this article suggests the existence of a solution for which $\|u(\cdot,t)\|_{\dot{H}^{1/2}} \sim (T-t)^{-1/3}$, far larger than the general lower bound of [17]. Merle-Raphaël do show, however, that if equality is achieved in (1.2), one can obtain an upper bound $\|u(t)\|_{\dot{H}^{1/2}} \leq |\log(T-t)|^{3/4}$.

Theorem 1.2. *Suppose the radial H^1 solution u to* (1.1) *blows-up at time $T<\infty$. Then either there is a non-absolute*[5] *constant $c_1 \gg 1$ such that, as $t \to T$*

$$\int_{|x|\le c_1^2\|\nabla u(t)\|_{L^2}^{-2}} |u(x,t)|^3\,dx \ge c_1^{-1} \tag{1.4}$$

or there exists a sequence of times $t_n \to T$ such that, for an absolute constant c_2

$$\int_{|x|\le c_2\|u_0\|_{L^2}^{3/2}\|\nabla u(t)\|_{L^2}^{-1/2}} |u(x,t_n)|^3\,dx \to \infty\,. \tag{1.5}$$

These two cases are not mutually exclusive. From the lower bound (1.2), we have that the concentration window in (1.4) satisfies $\|\nabla u(t)\|_{L^2}^{-2} \le c\,(T-t)^{1/2}$, and the concentration window in (1.5) satisfies $\|\nabla u(t)\|_{L^2}^{-1/2} \le c\,(T-t)^{1/8}$. The argument combines the radial Gagliardo-Nirenberg estimate (as we learned from J. Colliander, private communication) and the argument in the proof of Proposition 7 in Hmidi-Keraani [10]. It may be that a more refined analysis using the successive extraction of weak limits technique in [10] could yield more precise information, although we have decided not to explore this for the moment. This result can be compared with the mass concentration result for L^2 critical equations, see Merle-Tsutsumi [18] for the first results in this direction (radial case) and the recent paper of Hmidi-Keraani [10] for references and a simplified proof in the general (nonradial) case.

This analysis led us to consider: What type of blow-up solution would display the behavior described in (1.4), and what type would display the behavior described in (1.5)? There are currently no analytical results describing the specifics of the profile of the solution as $t \to T$ for finite-time blow-up solutions, although there have been several numerical studies. We mention, however, the construction by Raphaël [21] of a solution to the two-dimensional quintic NLS (also mass-supercritical) that blows-up on the unit circle. Raphaël's result [21] draws upon a large body of breakthrough work by Merle-Raphaël [12]-[17] [21] (see also Fibich-Merle-Raphaël [6]) on the blow-up problem for the L^2 critical NLS. Raphaël's construction [21], and the numerical study of Fibich-Gavish-Wang [5], have inspired our inquiry into the "contracting sphere" solutions that we describe through a heuristic analysis below. First, however, we call attention to the numerical results (see Sulem-Sulem [23] for references) describing the existence of self-similar radial blow-up solutions of the form

$$u(x,t) \approx \frac{1}{\lambda(t)} U\left(\frac{x}{\lambda(t)}\right) e^{i\log(T-t)} \text{ with } \lambda(t) = \sqrt{2b(T-t)} \tag{1.6}$$

for some parameter $b>0$ and some stationary profile $U=U(x)$ satisfying the nonlinear elliptic equation

$$\Delta U - U + ib(U + y\cdot\nabla U) + |U|^2U = 0.$$

[5] This means it depends on the solution but not on time.

It is expected that (nontrivial) zero-energy solutions U to this equation exist, although fail to belong to L^3 (and thus also $\dot{H}^{1/2}$) due to a logarithmic growth at infinity.[6] Thus, for solutions in H^1, the interpretation of $\approx$ in (1.6) is that one should introduce a time-dependent truncation of the profile u, where the size of the truncation enlarges as $t \to T$. The resulting u described by (1.6) will then display at least the logarithmic $\dot{H}^{1/2}$ divergence that must necessarily occur by the work of Merle-Raphaël [17].[7] We note that this type of solution would also display the L^3 concentration properties described in (1.4) in Theorem 1.2, and in fact, would also satisfy (1.5).

We are then led to consider: Could there be a solution for which (1.5) holds but (1.4) does not hold? Such a solution would have to concentrate on a contracting sphere of radius $r_0(t)$, where $r_0(t) \to 0$. That is, essentially no part of the solution sits directly on top of the origin $|x| \leq \frac{1}{2}r_0(t)$, and the whole blow-up action is taking place inside the spherical annulus $\frac{1}{2}r_0(t) < |x| < \frac{3}{2}r_0(t)$. Conservation of mass dictates that in fact the rate of contraction of the solution towards the sphere must far exceed the rate of contraction of the sphere itself. Specifically we seek a solution of the form (in terms of amplitudes)

$$|u(r,t)| \approx \frac{1}{\lambda(t)}\, P\left(\frac{r-r_0(t)}{\lambda(t)}\right),$$

where P is some one-dimensional profile and r is the three-dimensional radial coordinate. Then, we must have $\lambda(t) \sim r_0(t)^2$. By studying all the conservation laws and allowing for a little more generality, we provide a heuristic argument suggesting that such solutions do exist, but only with the following specific features.

Blow-up Scenario. *With blow-up time $0 < T < \infty$, we define for $t < T$ the radial position and focusing factor*

$$r_0(t) = \frac{3^{1/6}M[u]^{1/3}}{2\pi^{1/3}}(T-t)^{1/3}, \qquad \lambda(t) = \frac{18^{1/3}}{M[u]^{1/3}}(T-t)^{2/3}$$

and the rescaled time

$$s(t) = \frac{3M[u]^{2/3}}{18^{2/3}}(T-t)^{-1/3}$$

(so that, in particular, $s(t) \to +\infty$ as $t \to T$). Label the constant $\kappa = \frac{4}{3}\left(\frac{3}{32\pi}\right)^{2/3}$, and take θ as an arbitrary phase shift. Then

$$u(r,t) \approx e^{i\theta}e^{i\kappa^2 s}\exp\left[i\kappa\left(\frac{r-r_0(t)}{2\,\lambda(t)}\right)\right]\frac{1}{\lambda(t)}\,P\left(\frac{r-r_0(t)}{\lambda(t)}\right), \tag{1.7}$$

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[6]The only rigorous results on the existence of such U are for mass-supercritical NLS equations that scale in $\dot{H}^{s_c}$, for $s_c > 0$ close to 0 – see Rottschäfer-Kaper [22].

[7]We thank J. Colliander for supplying in private communication this interpretation derived from discussions with C. Sulem.

where

$$P(y) = \sqrt{\frac{3}{2}}\,\kappa\,\operatorname{sech}\left(\frac{\sqrt{3}}{2}\,\kappa y\right),$$

is a blow-up scenario that is consistent with all conservation laws.

A crucial component of this analysis is the observation that the inclusion of the spatial phase-shift gives a profile of zero energy, which is essential since the rescaling of the solution through the focusing factor $\lambda(t)$ would cause a nonzero energy to diverge to $+\infty$. We believe, however, that it is possible for the actual solution u to have nonzero energy, since it will in fact be represented in the form $u(x,t) =$ (above profile) $+\,\tilde{u}(x,t)$, where $\tilde{u}(x,t)$ is an error, and $\tilde{u}(x,t)$ could introduce nonzero energy by itself or through interaction with the profile (1.7).

The analysis demonstrates that the mass conservation and the virial identity act in conjunction to drive the blow-up: start with an initial configuration of the form (1.7), then the virial identity will tend to push $r_0(t)$ inward, and the mass conservation will force the solution to compensate by driving $\lambda(t)$ smaller (focusing the solution further), which in turn will feed back into the virial identity to push $r_0(t)$ yet smaller. We emphasize that we do not actually prove here that such blow-up solutions exist (we are currently working on such a rigorous construction which will be presented elsewhere), we only provide evidence through heuristic reasoning in this note. The numerical observation of solutions of this type was reported in Fibich-Gavish-Wang [5], although most of the specifics of the dynamic were not mentioned – only the relationship $\lambda(t) \sim r_0(t)^2$. The authors did remark that they intended to follow-up with a more detailed analysis of this supercritical problem, and devoted most of the [5] paper to the analysis of a similar phenomenon in the L^2 critical context.

For the hypothetical contracting sphere solutions, we have, as $t \to T$,

$$\begin{aligned}
\|u(t)\|_{L^3} &\sim \frac{r_0(t)^{2/3}}{\lambda(t)^{2/3}} \sim M[u_0]^{4/9}(T-t)^{-2/9},\\
\|u(t)\|_{\dot{H}^{1/2}} &\sim \frac{r_0(t)}{\lambda(t)} \sim M[u_0]^{2/3}(T-t)^{-1/3},\\
\|\nabla u(t)\|_{L^2} &\sim \frac{r_0(t)}{\lambda(t)^{3/2}} \sim M[u_0]^{5/6}(T-t)^{-2/3}.
\end{aligned} \tag{1.8}$$

Thus, the concentration window in (1.5) is $\sim (T-t)^{1/3}$ (which coincides with $r_0(t)$) and in (1.4) is $\sim (T-t)^{4/3}$. This type of solution, if it exists, should satisfy (1.5) without satisfying (1.4). Another remark in regard to (1.8) is that (1.7) is only an asymptotic description for t close to T, where this closeness depends on $M[u_0]$ (and potentially other factors), and thus, there is no contradiction with the above formula for $\|u(t)\|_{\dot{H}^{1/2}}$ and the small data $\dot{H}^{1/2}$ scattering theory. A similar comment applies to the other scale invariant quantity $\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}$ discussed earlier. (In addition, there is no clear constraint on the energy of such solutions – they may satisfy (1.3) or may not.)

Since the contracting sphere blow-up solutions contract at the rate $\sim (T-t)^{1/3}$, and the solutions (1.6) that blow-up on top of the origin focus at a faster rate $\sim (T-t)^{1/2}$, we propose the possibility of a solution that blows-up simultaneously in both manners – we see from the rates that a decoupling should occur between the two components of the solution. One could also speculate that multi-contracting sphere solutions, with or without a contracting blob at the origin itself, are possible. We further conjecture, on the basis of the article [5], that the contracting sphere blow-up solutions are stable under small radial perturbations but unstable under small nonradial (symmetry breaking) perturbations[8].

We remark that for the defocusing nonlinearity ($+|u|^2u$ changed to $-|u|^2u$ in (1.1)), one always has global existence, since the energy is then a positive definite quantity and thus automatically provides an *a priori* bound on the $\dot{H}^1$ norm. Furthermore, it has been shown by Ginibre-Velo [8], using a Morawetz estimate, that there is scattering. The argument of Ginibre-Velo has been simplified using a new "interaction Morawetz" identity by Colliander-Keel-Staffilani-Takaoka-Tao [4]. Thus, at least as far as H^1 data is concerned, the dynamics of this problem are comparatively well-understood.

The format for this note is as follows. In §2, we study a general version of the focusing NLS equation which is energy subcritical and prove the generalized version of Theorem 1.1; there we also discuss the blow up criterion for $u_0 \in H^1$ which includes positive energies and not necessarily finite variance. In §3, we prove Theorem 1.2 on L^3 norm concentration. In §4, we present the heuristic analysis of the conjectured contracting sphere solutions. This is followed in §5 by an analysis with somewhat more precision, indicating, in particular, a cancelation between second-order approximations to two different "error" terms. We conclude in §6 by noting that the ideas presented here for the specific problem (1.1) can be adapted to more general radial nonlinear Schrödinger equations. Interestingly, there is the possibility that for septic nonlinearity $|u|^6u$, one could have blow-up on an *expanding* sphere, but we have not conducted a thorough analysis of this hypothetical situation.

Acknowledgments. We would like to thank the organizers of the Fall 2005 MSRI program "Nonlinear Dispersive Equations." We met for the first time at this program and began work on this project there. Also, we are grateful to MSRI for providing accommodations for S.R. during a May 2006 visit to U.C. Berkeley. Finally, we are indebted to Jim Colliander for mentorship and encouragement. J.H. is partially

[8]Stability under small H^1 radial perturbations was also proved for the singularity formation on a constant ring in [21], and it was asked there whether the non-radial perturbations will affect the stability.

supported by an NSF postdoctoral fellowship. S.R. is partially supported by NSF grant DMS-0531337.

2. Dichotomy for the energy subcritical NLS

In this section we study a more general version of the focusing nonlinear Schrödinger equation $\mathrm{NLS}_p(\mathbb{R}^N)$ which is mass supercritical and energy subcritical, i.e.

$$\left\{\begin{array}{l} i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ u(x,0) = u_0(x), \end{array}\right. \tag{2.1}$$

with the choice of nonlinearity p and the dimension N such that

$$0 < s_c < 1, \quad \text{where} \quad s_c = \frac{N}{2} - \frac{2}{p-1}.$$

In other words, we consider $\dot{H}^{s_c}$-critical NLS equations with $0 < s_c < 1$. In this case the initial value problem with $u_0 \in H^1(\mathbb{R}^N)$ is locally well-posed, see [7]. Denote by $I = (-T_*, T^*)$ the maximal interval of existence of the solution u (e.g., see [3]). This implies that either $T^* = +\infty$ or $T^* < +\infty$ and $\|\nabla u(t)\|_{L^2} \to \infty$ as $t \to T^*$ (similar properties for T_*).

The solutions to this problem satisfy mass and energy conservation laws

$$M[u](t) = \int |u(x,t)|^2\, dx = M[u_0],$$

$$E[u](t) = \frac{1}{2}\int |\nabla u(x,t)|^2 - \frac{1}{p+1}\int |u(x,t)|^{p+1}\, dx = E[u_0],$$

and the Sobolev $\dot{H}^{s_c}$ norm and Lebesgue L^{p_c} norm, $p_c = \frac{N}{2}(p-1)$, are invariant under the scaling $u \mapsto u_\lambda(x,t) = \lambda^{2/(p-1)}u(\lambda x, \lambda^2 t)$. (Note that u_λ is a solution of $\mathrm{NLS}_p(\mathbb{R}^N)$, if u is.)

We investigate other scaling invariant quantities besides the above norms. Since

$$\|u_\lambda\|_{L^2(\mathbb{R}^N)} = \lambda^{-s_c}\|u\|_{L^2(\mathbb{R}^N)} \quad \text{and} \quad \|\nabla u_\lambda\|_{L^2(\mathbb{R}^N)} = \lambda^{-s_c+1}\|\nabla u\|_{L^2(\mathbb{R}^N)},$$

the quantity (or any power of it)

$$\|\nabla u_0\|^{s_c}_{L^2(\mathbb{R}^N)} \cdot \|u_0\|^{1-s_c}_{L^2(\mathbb{R}^N)}$$

is scaling invariant. Another scaling invariant quantity is

$$\Lambda_0 := E[u_0]^{s_c}\, M[u_0]^{1-s_c}.$$

Next, recall the Gagliardo-Nirenberg inequality from [27] which is valid for values p and N such that $0 \le s_c < 1$[9]:

$$\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \le c_{GN}\, \|\nabla u\|^{\frac{N(p-1)}{2}}_{L^2(\mathbb{R}^N)}\, \|u\|^{2-\frac{(N-2)(p-1)}{2}}_{L^2(\mathbb{R}^N)}, \tag{2.2}$$

[9]It is also valid for $s_c = 1$ becoming nothing else but Sobolev embedding, see Remark 2.4.

where $c_{GN} = c_{GN}(p,N) = \dfrac{p+1}{2\,\|Q\|^{p-1}_{L^2(\mathbb{R}^N)}}$ and Q is the ground state solution (positive solution of minimal L^2 norm) of the following equation[10]

$$\frac{N(p-1)}{4}\,\triangle Q - \left(1 - \frac{(N-2)(p-1)}{4}\right) Q + Q^p = 0. \tag{2.3}$$

(See [27] and references therein for the discussion on the existence of positive solutions of class $H^1(\mathbb{R}^N)$ to this equation.)

Define

$$u_Q(x,t) = e^{i\lambda t} Q(\alpha x)$$

with $\alpha = \frac{\sqrt{N(p-1)}}{2}$ and $\lambda = 1 - \frac{(N-2)(p-1)}{4}$. Then u_Q is a soliton solution of $i\partial_t u + \triangle u + |u|^{p-1}u = 0$. Since the Gagliardo-Nirenberg inequality is optimized by Q, we get

$$\|Q\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = \frac{p+1}{2}\,\|\nabla Q\|^{\frac{N(p-1)}{2}}_{L^2(\mathbb{R}^N)}\,\|Q\|^{2-\frac{N(p-1)}{2}}_{L^2(\mathbb{R}^N)}. \tag{2.4}$$

Multiplying (2.3) by Q and integrating, we obtain

$$-\lambda\|Q\|^2_{L^2(\mathbb{R}^N)} - \alpha^2\|\nabla Q\|^2_{L^2(\mathbb{R}^N)} + \|Q\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = 0. \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\frac{p+1}{2}\,\|Q\|^2_{L^2(\mathbb{R}^N)}\,\|\nabla Q\|^{\frac{N(p-1)}{2}}_{L^2(\mathbb{R}^N)} = \lambda\,\|Q\|^{2+\frac{N(p-1)}{2}}_{L^2(\mathbb{R}^N)} + \alpha^2\,\|\nabla Q\|^2_{L^2(\mathbb{R}^N)}\,\|Q\|^{\frac{N(p-1)}{2}}_{L^2(\mathbb{R}^N)}.$$

Note that $\|Q\|_{L^2(R^N)} = 0$ is the trivial solution of the above equation and we exclude it from further consideration. Denote by $z = \dfrac{\|\nabla Q\|_{L^2(R^N)}}{\|Q\|_{L^2(R^N)}}$. Now the above equation becomes

$$\frac{p+1}{2}\,z^{\frac{N(p-1)}{2}} - \frac{N(p-1)}{4}z^2 + \frac{(N-2)(p-1)}{4} - 1 = 0.$$

The equation has only one real root $z = 1$ which gives

(Q.1) $\|\nabla Q\|_{L^2(\mathbb{R}^N)} = \|Q\|_{L^2(\mathbb{R}^N)}$.

Substituting (Q.1) into (2.4), we also obtain

(Q.2) $\|Q\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = \frac{p+1}{2}\,\|Q\|^2_{L^2(\mathbb{R}^N)}$.

We are now ready to state the main result of this section.

Theorem 2.1. *Consider $NLS_p(\mathbb{R}^N)$ with $u_0 \in H^1(\mathbb{R}^N)$ and $0 < s_c < 1$. Let $u_Q(x,t)$ be as above and denote $\sigma_{p,N} = \left(\dfrac{4}{N(p-1)}\right)^{\frac{1}{p-1}} \|Q\|_{L^2(\mathbb{R}^N)}$. Suppose that*

$$\Lambda_0 := E[u_0]^{s_c}\,M[u_0]^{1-s_c} < E[u_Q]^{s_c}\,M[u_Q]^{(1-s_c)} \equiv \left(\frac{s_c}{N}\right)^{s_c}(\sigma_{p,N})^2, \quad E[u_0] \geq 0. \tag{2.6}$$

[10] We use the notation from Weinstein [27]; one can rescale Q so it solves $\triangle Q - Q + Q^p = 0$.

If

$$\|\nabla u_0\|_{L^2}^{s_c} \cdot \|u_0\|_{L^2}^{1-s_c} < \|\nabla u_Q\|_{L^2}^{s_c} \cdot \|u_Q\|_{L^2}^{1-s_c} \equiv \sigma_{p,N}, \tag{2.7}$$

then $I = (-\infty, +\infty)$, i.e. the solution exists globally in time, and for all time $t \in \mathbb{R}$

$$\|\nabla u(t)\|_{L^2}^{s_c} \cdot \|u_0\|_{L^2}^{1-s_c} < \|\nabla u_Q\|_{L^2}^{s_c} \cdot \|u_Q\|_{L^2}^{1-s_c} \equiv \sigma_{p,N}. \tag{2.8}$$

If

$$\|\nabla u_0\|_{L^2(\mathbb{R}^N)}^{s_c} \cdot \|u_0\|_{L^2(\mathbb{R}^N)}^{1-s_c} > \|\nabla u_Q\|_{L^2}^{s_c} \cdot \|u_Q\|_{L^2}^{1-s_c} \equiv \sigma_{p,N}, \tag{2.9}$$

then for $t \in I$

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^{s_c} \cdot \|u_0\|_{L^2(\mathbb{R}^N)}^{1-s_c} > \|\nabla u_Q\|_{L^2}^{s_c} \cdot \|u_Q\|_{L^2}^{1-s_c} \equiv \sigma_{p,N}. \tag{2.10}$$

Furthermore, if $|x|u_0 \in L^2(\mathbb{R}^N)$, then I is finite, and thus, the solution blows up in finite time. The finite-time blowup conclusion and (2.10) *also hold if, in place of* (2.6) *and* (2.9)*, we assume $E[u_0] < 0$.*

Remark 2.2. It is easy to check the equivalence on the right-hand side of (2.6) - (2.10):

$$\begin{aligned} E[u_Q]^{s_c} M[u_Q]^{(1-s_c)} &= \left(\frac{\alpha^2}{2} - \frac{1}{2}\right)^{s_c} \alpha^{-N} \|Q\|_{L^2(\mathbb{R}^N)}^2 \\ &= \left(\frac{N(p-1)-4}{8}\right)^{s_c} \left(\frac{4}{N(p-1)}\right)^{\frac{N}{2}} \|Q\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

On the other hand,

$$\left(\frac{s_c}{N}\right)^{s_c} (\sigma_{p,N})^2 = \left(\frac{N(p-1)-4}{2N(p-1)}\right)^{s_c} \left(\frac{4}{N(p-1)}\right)^{\frac{2}{(p-1)}} \|Q\|_{L^2(\mathbb{R}^N)}^2,$$

which equals the previous expression when recalling that $s_c = \frac{N}{2} - \frac{2}{p-1}$.

Furthermore, since $\sigma_{p,N} = \left(\frac{4}{N(p-1)}\right)^{\frac{1}{(p-1)}} \|Q\|_{L^2(\mathbb{R}^N)}$, we also obtain that

$$\|\nabla u_Q\|_{L^2(\mathbb{R}^N)}^{s_c} \cdot \|u_Q\|_{L^2(\mathbb{R}^N)}^{(1-s_c)} = \alpha^{s_c - \frac{N}{2}} \|Q\|_{L^2(\mathbb{R}^N)} = \alpha^{-\frac{2}{p-1}} \|Q\|_{L^2(\mathbb{R}^N)} \equiv \sigma_{p,N}.$$

Remark 2.3. Observe that the second part of Theorem 2.1 shows that there are solutions of $\mathrm{NLS}_p(\mathbb{R}^N)$ with positive energy which blow up in finite time, thus, we extend the standard virial argument (e.g. see [9]) on the existence of blow up solutions with negative energy and finite variance. Moreover, using the localized version of the virial identity, this result can be extended to the functions with infinite variance, see Corollary 2.5 below.

Remark 2.4. This theorem provides a link between the mass critical NLS and energy critical NLS equations. Consider $s_c = 1$, then the theorem holds true by the work of Kenig-Merle [11, Section 3]: in this case $\Lambda_0 = E[u_0]$, the Gagliardo-Nirenberg inequality (2.2) becomes the Sobolev inequality with $c_N = (c_{GN})^{1/(p+1)}$, the condition

(2.6) becomes $E[u_0] < \frac{1}{N}(c_{GN})^{-\frac{2}{p-1}} = \frac{1}{N}(c_N)^{-N} = E[W]$, where W is the radial positive decreasing (class $\dot{H}^1(\mathbb{R}^N)$) solution of $\triangle W + |W|^{p-1}W = 0$, and the conditions (2.7) - (2.10) involve only the size of $\|\nabla u_0\|_{L^2}$ in correlation with $\sigma_{p,N} = (c_{GN})^{-\frac{1}{p-1}} = c_N^{-N/2} = \|\nabla W\|_{L^2(\mathbb{R}^N)}$.

In the case $s_c = 0$, the only relevant scaling invariant quantity is the mass: $\Lambda_0 = M[u_0]$, the condition (2.6) becomes $M[u_0] < \left(\frac{p+1}{2}\frac{1}{c_{GN}}\right)^{\frac{2}{p-1}} = \|Q\|^2_{L^2(\mathbb{R}^N)}$, and the conditions (2.7) - (2.10) involve also only the mass in relation with $\|Q\|^2_{L^2(\mathbb{R}^N)}$, in fact (2.7) (and (2.8)) coincides with (2.6) and the conclusion on the global existence holds; the condition (2.9) becomes $\|u_0\|_{L^2} > \|Q\|_{L^2}$, and thus, the complement of (2.6) holds, which is the last statement $E[u_0] < 0$, hence, implying the blow up. Thus, the statement of the theorem in the limiting case $s_c = 0$ connects with Weinstein's results [27].

Proof of Theorem 2.1. Using the definition of energy and (2.2), we have

$$\begin{aligned} E[u] &= \frac{1}{2}\|\nabla u\|^2_{L^2} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}} \\ &\geq \frac{1}{2}\|\nabla u\|^2_{L^2} - \frac{c_{GN}}{p+1}\|\nabla u\|_{L^2}^{\frac{N(p-1)}{2}}\|u_0\|_{L^2}^{2-\frac{(N-2)(p-1)}{2}}. \end{aligned}$$

Define $f(x) = \frac{1}{2}x^2 - \frac{c_{GN}}{p+1}\|u_0\|_{L^2}^{2-\frac{(N-2)(p-1)}{2}} x^{\frac{N(p-1)}{2}}$, observe that $\deg(f) \geq 2$, since $N(p-1) \geq 4$. Then

$$\begin{aligned} f'(x) &= x - \frac{N(p-1)}{2(p+1)}c_{GN}\|u_0\|_{L^2}^{2-\frac{(N-2)(p-1)}{2}} x^{\frac{N(p-1)}{2}-1} \\ &= x\left(1 - \frac{N(p-1)}{2(p+1)}c_{GN}\|u_0\|_{L^2}^{(p-1)(1-s_c)} x^{(p-1)s_c}\right), \end{aligned}$$

and thus, $f'(x) = 0$ when $x_0 = 0$ and $x_1 = (\sigma_{p,N})^{\frac{1}{s_c}}\|u_0\|_{L^2(\mathbb{R}^N)}^{-\frac{1-s_c}{s_c}}$. Note that $f(0) = 0$ and $f(x_1) = \frac{s_c}{N}x_1^2$. Thus, the graph of f has two extrema: a local minimum at x_0 and a local maximum at x_1. Hence, the condition (2.6) implies that $E[u_0] < f(x_1)$. Combining this with energy conservation, we have

$$f(\|\nabla u(t)\|_{L^2}) \leq E[u(t)] = E[u_0] < f(x_1). \tag{2.11}$$

If initially $\|\nabla u_0\|_{L^2} < x_1$, i.e. the condition (2.7) holds, then by (2.11) and the continuity of $\|\nabla u(t)\|_{L^2}$ in t, we have $\|\nabla u(t)\|_{L^2} < x_1$ for all time $t \in I$ which gives (2.8). In particular, the $\dot{H}^1$ norm of the solution u is bounded, which proves global existence in this case.

If initially $\|\nabla u_0\|_{L^2} > x_1$, i.e. the condition (2.9) holds, then by (2.11) and the continuity of $\|\nabla u(t)\|_{L^2}$ in t, we have $\|\nabla u(t)\|_{L^2} > x_1$ for all time $t \in I$ which gives

(2.10). Now if u has the finite variance, we recall the virial identity

$$\partial_t^2 \int |x|^2 \, |u(x,t)|^2 \, dx = 4N(p-1)E[u_0] - 2(N(p-1)-4)\|\nabla u(t)\|_{L^2}^2.$$

Multiplying both sides by $M[u_0]^\theta$ with $\theta = \frac{1-s_c}{s_c}$ and applying inequalities (2.6) and (2.10), we obtain

$$\begin{aligned} M[u_0]^\theta \, \partial_t^2 \int |x|^2 \, |u(x,t)|^2 \, dx &= 4N(p-1)\Lambda_0^{\frac{1}{s_c}} - 2(N(p-1)-4)\|\nabla u(t)\|_{L^2}^2 \, \|u_0\|_{L^2}^{2\theta} \\ &< 4(p-1)s_c(\sigma_{p,N})^{\frac{2}{s_c}} - 2(N(p-1)-4)(\sigma_{p,N})^{\frac{2}{s_c}} = 0, \end{aligned}$$

and thus, I must be finite, which implies that in this case blow up occurs in finite time.[11] □

Corollary 2.5. *Suppose that all conditions of Theorem 2.1 hold except for finite variance, and now assume that the solution u is radial. Consider $N \geq 2$ and $1 + \frac{4}{N} < p < \min\{1 + \frac{4}{N-2}, 5\}$*[12]*. Also suppose that there exists $\delta > 0$ such that*

$$\Lambda_0 \leq (1-\delta)^{s_c} \left(\frac{s_c}{N}\right)^{s_c} (\sigma_{p,N})^2. \tag{2.12}$$

If (2.9) holds, then there exists $\tilde\delta = \tilde\delta(\delta) > 0$ such that

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^N)}^{s_c} \cdot \|u_0\|_{L^2(\mathbb{R}^N)}^{1-s_c} \geq (1+\tilde\delta)^{s_c} \, \sigma_{p,N}. \tag{2.13}$$

Furthermore, the maximal interval of existence I is finite.

Proof. The inequality (2.13) follows from the proof of the theorem by applying a refined version (2.12), so we concentrate on the second implication. We use a localized version of the virial identity (e.g. [11]), let $\varphi \in C_0^\infty(\mathbb{R}^N)$, then

$$\begin{aligned} \partial_t^2 \int \varphi(x) \, |u(x,t)|^2 \, dx = 4 \sum_{j,k} \mathrm{Re} \int \partial_{x_j} \partial_{x_k} \varphi \, \partial_{x_j} u \, \partial_{x_k} \bar u - \int \triangle^2 \varphi \, |u|^2 \\ - 4\left(\frac{1}{2} - \frac{1}{p+1}\right) \int \triangle \varphi \, |u|^{p+1}. \end{aligned}$$

Choose $\varphi(x) = \varphi(|x|)$ to be a radially symmetric function that is constant for large r and such that $\partial_r^2 \varphi(r) \leq 2$ for all $r \geq 0$ and $\varphi(r) = r^2$ for $0 \leq r \leq 1$. Taking $\varphi_m(r) = m^2 \varphi(r/m)$, and following the proof of the main theorem in Ogawa-Tsutsumi

[11] To be more accurate, in order to obtain the finite-time blow-up one in fact needs to deduce from (2.11) that $\|\nabla u(t)\|_{L^2} \geq x_1 + \delta_1$ and consequently that $M[u_0]^\theta \, \partial_t^2 \int |x|^2 \, |u(x,t)|^2 \, dx \leq -\delta_2 < 0$, where $\delta_1 > 0$ can be determined in terms of $f(x_1) - E[u_0] > 0$ and $\delta_2 > 0$ in terms of δ_1.

[12] This is a technical restriction.

[19][13], we obtain that for any large $m>0$ and $\gamma=(N-1)(p-1)/2$, we have

$$\partial_t^2 \int \varphi_m(|x|)\,|u(x,t)|^2\,dx \le 4N(p-1)E[u_0] - (2N(p-1)-8)\int |\nabla u|^2 + \frac{c_1}{m^\gamma}\,\|u\|_{L^2}^{(p+3)/2}\,\|\nabla u\|_{L^2}^{(p-1)/2} + \frac{c_2}{m^2}\int_{m<|x|} |u|^2.$$

Choose

$$0<\epsilon<(2N(p-1)-8)\left(1-\frac{1-\delta}{(1+\tilde\delta)^2}\right).$$

Use Young's inequality in the third term on the right hand side to separate the L^2-norm and gradient term and then absorb the gradient term into the second term with the chosen ϵ. Then multiplying the above expression by $M[u_0]^\theta$ and using (2.12) and (2.13), we get

$$\begin{aligned}
&M[u_0]^\theta\,\partial_t^2 \int \varphi_m(|x|)\,|u(x,t)|^2\,dx\\
&\quad\le 4N(p-1)\,E[u_0]M[u_0]^\theta - (2N(p-1)-8-\epsilon)\|\nabla u\|_{L^2}^2\|u\|_{L^2}^{2\theta}\\
&\qquad + \frac{c(\epsilon,N)}{m^{4\gamma/(5-p)}}\,\|u\|_{L^2}^{2(p+3)/(5-p)+2\theta} + \frac{c_2}{m^2}\,\|u\|_{L^2}^{2+2\theta}\\
&\quad\le 4N(p-1)\,(1-\delta)\,\frac{s_c}{N}\,(\sigma_{p,N})^{\frac{2}{s_c}} - (2N(p-1)-8-\epsilon)(1+\tilde\delta)^2\,(\sigma_{p,N})^{\frac{2}{s_c}}\\
&\qquad + \frac{c(\epsilon,N)}{m^{4\gamma/(5-p)}}\,\|u\|_{L^2}^{2(p+3)/(5-p)+2\theta} + \frac{c_2}{m^2}\,\|u\|_{L^2}^{2+2\theta}\\
&\quad< -c(\epsilon,N,p)
\end{aligned}$$

by choosing $m=m(\epsilon,\delta,\tilde\delta,N,p,M[u_0])$ large enough, where $c(\epsilon,N,p)>0$. This implies that the maximal interval of existence I is finite. □

3. Critical norm concentration phenomenon

This section is devoted to a proof of Theorem 1.2. We will prove Proposition 3.1 below, and then indicate how Theorem 1.2 is an easy corollary of this proposition.

To state the proposition, we need some notation for spatial and frequency localizations. Let $\phi(x)\in C_c^\infty(\mathbb{R}^3)$ be a radial function so that $\phi(x)=1$ for $|x|\le 1$ and $\phi(x)=0$ for $|x|\ge 2$, and then define the inner and outer spatial localizations of $u(x,t)$ at radius $R(t)>0$ as $u_1(x,t)=\phi(x/R(t))u(x,t)$, $u_2(x,t)=(1-\phi(x/R(t)))u(x,t)$. Let $\chi(x)\in C_c^\infty(\mathbb{R}^3)$ be a radial function so that $\chi(x)=1$ for $|x|\le 1/8\pi$ and $\chi(x)=0$ for $|x|\ge 1/2\pi$, and furthermore, $\hat\chi(0)=1$ and define the inner and outer frequency localizations at radius $\rho(t)$ of u_1 as $\hat u_{1L}(\xi,t)=\hat\chi(\xi/\rho(t))\hat u_1(\xi,t)$

[13] This proof uses the radial Gagliardo-Nirenberg estimate, and hence, we have the radial restriction in Cor 2.5.

and $\hat{u}_{1H}(\xi,t) = (1-\hat{\chi}(\xi/\rho(t)))\hat{u}_1(\xi,t)$.[14] Note that the frequency localization of $u_1 = u_{1L}+u_{1H}$ is inexact, although crucially we have

$$|1-\hat{\chi}(\xi)| \leq c\min(|\xi|,1)\,. \tag{3.1}$$

Proposition 3.1. *Let u be an H^1 radial solution to* (1.1) *that blows-up in finite $T>0$. Let $R(t) = c_1\|u_0\|_{L^2}^{3/2}\|\nabla u(t)\|_{L_x^2}^{-1/2}$ and $\rho(t) = c_2\|\nabla u(t)\|_{L_x^2}^2$ (for absolute constants c_1 and c_2), and decompose $u = u_{1L}+u_{1H}+u_2$ as described in the paragraph above.*

(1) *There exists an absolute constant $c>0$ such that*

$$\|u_{1L}(t)\|_{L_x^3} \geq c \quad \text{as} \quad t\to T. \tag{3.2}$$

(2) *Suppose that there exists a constant c^* such that $\|u_1(t)\|_{L^3}\leq c^*$. Then*

$$\|u_1(t)\|_{L_x^3(|x-x_0(t)|\leq \rho(t)^{-1})} \geq \frac{c}{(c^*)^3} \quad \text{as} \quad t\to T \tag{3.3}$$

for some absolute constant $c>0$, where $x_0(t)$ is a position function such that $|x_0(t)|/\rho(t)^{-1}\leq c\cdot(c^*)^6$.

We mention two Gagliardo-Nirenberg estimates for functions on $\mathbb{R}^3$ that will be applied in the proof. The first is an "exterior" estimate, applicable to radially symmetric functions only, originally due to W. Strauss:

$$\|v\|_{L^4_{\{|x|>R\}}}^4 \leq \frac{c}{R^2}\|v\|^3_{L^2_{\{|x|>R\}}}\|\nabla v\|_{L^2_{\{|x|>R\}}}, \tag{3.4}$$

where c is an absolute constant (in particular, independent of $R>0$). The second is generally applicable: For any function v,

$$\|v\|_{L^4(\mathbb{R}^3)}^4 \leq c\|v\|_{L^3(\mathbb{R}^3)}^2\|\nabla v\|_{L^2(\mathbb{R}^3)}^2\,. \tag{3.5}$$

Proof of Prop. 3.1. Since, by (1.2), $\|\nabla u(t)\|_{L_x^2}\to+\infty$ as $t\to T$, by energy conservation, we have $\|u(t)\|_{L_x^4}^4/\|\nabla u(t)\|_{L_x^2}^2 \to 2$. Thus, for t sufficiently close to T,

$$\|\nabla u\|_{L_x^2}^2 \leq \|u\|_{L_x^4}^4 \leq \|u_{1L}\|_{L_x^4}^4 + \|u_{1H}\|_{L_x^4}^4 + \|u_2\|_{L_x^4}^4\,. \tag{3.6}$$

By (3.4), the choice of $R(t)$, and mass conservation

$$\|u_2\|_{L_x^4}^4 \leq \frac{c}{R^2}\|u_0\|_{L_x^2}^3\|\nabla u\|_{L_x^2} \leq \frac{1}{4}\|\nabla u\|_{L_x^2}^2, \tag{3.7}$$

[14]The $1/8\pi$ and $1/2\pi$ radii are chosen to be consistent with the assumption $\hat{\chi}(0)=1$, since $\hat{\chi}(0) = \int_{\mathbb{R}^3}\chi(x)dx$. In actuality, this is for convenience only–the argument is easily adapted to the case where $\hat{\chi}(0)$ is any number $\neq 0$.

where the constant c_1 in the definition of $R(t)$ has been chosen to obtain the factor $\frac{1}{4}$ here. By Sobolev embedding, (3.1), and the choice of $\rho(t)$,

$$\begin{aligned}
\|u_{1H}\|_{L_x^4}^4 &\le c\,\|u_{1H}\|_{\dot H_x^{3/4}}^4 = c\,\||\xi|^{3/4}(1-\hat\chi(\xi/\rho))\hat u_1(\xi)\|_{L_\xi^2}^4 \\
&\le c\,\rho^{-1}\||\xi|\hat u_1(\xi)\|_{L_\xi^2}^4 \le c\,\rho^{-1}\|\nabla u_1\|_{L_x^2}^4 \le c\,\rho^{-1}\|\nabla u\|_{L_x^2}^4 \\
&\le \frac{1}{4}\|\nabla u\|_{L_x^2}^2, \qquad (3.8)
\end{aligned}$$

where the constant c_2 in the definition of $\rho(t)$ has been chosen to obtain the factor $\frac{1}{4}$ here. Combining (3.6), (3.7), and (3.8), we obtain

$$\|\nabla u\|_{L_x^2}^2 \le c\|u_{1L}\|_{L_x^4}^4. \qquad (3.9)$$

By (3.9) and (3.5), we obtain (3.2), completing the proof of part (1) of the proposition.
To prove part (2), we assume $\|u_1\|_{L^3} \le c^*$. By (3.9),

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 &\le c\|u_{1L}\|_{L_x^3}^3\|u_{1L}\|_{L_x^\infty} \le c\cdot(c^*)^3\|u_{1L}\|_{L_x^\infty} \\
&\le c\cdot(c^*)^3\sup_{x\in\mathbb{R}^3}\left|\int\rho^3\chi(\rho(x-y))u_1(y)\,dy\right|.
\end{aligned}$$

There exists $x_0 = x_0(t)\in\mathbb{R}^3$ for which at least $\frac{1}{2}$ of this supremum is attained. Thus,

$$\begin{aligned}
\|\nabla u\|_{L^2}^2 &\le c\cdot(c^*)^3\left|\int\rho^3\chi(\rho(x_0-y))u_1(y)\,dy\right| \\
&\le c\cdot(c^*)^3\rho^3\int_{|x_0(t)-y|\le\rho^{-1}}|u_1(y)|\,dy \\
&\le c\cdot(c^*)^3\rho\left(\int_{|x_0(t)-y|\le\rho^{-1}}|u_1(y)|^3\,dy\right)^{1/3}
\end{aligned}$$

with Hölder's inequality used in the last step. By the choice of ρ, we obtain (3.3). To complete the proof, it remains to obtain the stated control on $x_0(t)$, which will be a consequence of the radial assumption and the assumed bound $\|u_1\|_{L^3}\le c^*$. Suppose

$$\frac{|x_0(t_n)|}{\rho(t_n)^{-1}} \gg (c^*)^6$$

along a sequence of times $t_n \to T$. Consider the spherical annulus

$$A = \{\, x\in\mathbb{R}^3 \,:\, |x_0|-\rho^{-1} \le |x| \le |x_0|+\rho^{-1}\,\}$$

and inside A place $\sim \frac{4\pi|x_0|^2}{\pi(\rho^{-1})^2}$ disjoint balls, each of radius ρ^{-1}, centered on the sphere at radius $|x_0|$. By the radiality assumption, on each ball B, we have $\|u_1\|_{L_B^3} \ge c/(c^*)^3$, and hence on the annulus A,

$$\|u_1\|_{L_A^3}^3 \ge \frac{c}{(c^*)^9}\frac{|x_0|^2}{(\rho^{-1})^2} \gg (c^*)^3,$$

which contradicts the assumption $\|u_1\|_{L^3}\le c^*$. $\square$

Now we indicate how to obtain Theorem 1.2 as a consequence.

Proof of Theorem 1.2. By part (1) of Prop. 3.1 and the standard convolution inequality:

$$c \leq \|u_{1L}\|_{L^3_x} = \|\rho^3 \chi(\rho\,\cdot) * u_1\|_{L^3_x} \leq \|u_1\|_{L^3}.$$

Now, if $\|u_1(t)\|_{L^3}$ is not bounded above, then there exists a sequence of times $t_n \to T$ such that $\|u_1(t_n)\|_{L^3} \to +\infty$. Since $\|u(t_n)\|_{L^3(|x|\leq 2R)} \geq \|u_1(t_n)\|_{L^3}$, we have (1.5) in Theorem 1.2. If, on the other hand, $\|u_1(t)\|_{L^3} \leq c^*$, for some c^*, as $t \to T$, we have (3.3) of Prop. 3.1. Since $|x_0(t)| \leq c(c^*)^6 \rho(t)^{-1}$, we have

$$\frac{c}{(c^*)^3} \leq \|u_1(t)\|_{L^3(|x-x_0(t)|\leq \rho(t)^{-1})} \leq \|u_1(t_n)\|_{L^3(|x|\leq c(c^*)^6\rho(t)^{-1})},$$

which gives (1.4) in Theorem 1.2. □

4. Heuristic analysis of contracting sphere blow-up solutions

In this section, we develop a heuristic analysis of hypothetical contracting sphere solutions of (1.1) with blow-up time $T > 0$. The results of this analysis are summarized in the introduction §1 and we now present the argument itself. We assume radial symmetry of the solution and let $r = |x|$ denote the radial coordinate. In radial coordinates, (1.1) becomes

$$i\partial_t u + \frac{2}{r}\partial_r u + \partial_r^2 u + |u|^2 u = 0. \tag{4.1}$$

We define the radial position $r_0(t)$ as

$$r_0^2(t) = \frac{\int_0^{+\infty} r^4 |u(r,t)|^2\, dr}{\int_0^{+\infty} r^2 |u(r,t)|^2\, dr} = \frac{V[u](t)}{M[u](t)}, \tag{4.2}$$

where $V[u](t)$ is the first virial quantity

$$V[u](t) = 4\pi \int_0^{+\infty} r^4 |u(r,t)|^2\, dr,$$

and $M[u](t) = M[u_0]$ is the conserved mass of u. In terms of $r_0(t)$, we define the focusing factor $\lambda(t)$ via the relation

$$\frac{r_0^2(t)}{\lambda^3(t)} = \|\nabla u(t)\|_{L^2}^2. \tag{4.3}$$

The motivation for these definitions is that if there were some fixed profile $\phi(y)$ (where $y \in \mathbb{R}$) and $u(r,t)$ had approximately the form

$$|u(r,t)| \approx \frac{1}{\lambda_1(t)}\, \phi\left(\frac{r - r_1(t)}{\lambda(t)}\right),$$

then the formulas (4.2) and (4.3) would give $r_0(t) \approx r_1(t)$ and $\lambda(t) \approx \lambda_1(t)$. The convenience of these definitions is first that they are *always* defined (on the maximal existence interval $[0,T)$), and thus, we can distinguish between contracting sphere and non-contracting sphere solutions in terms of their behavior, and second that they immediately relate the parameters $r_0(t)$ and $\lambda(t)$ to quantities appearing in conservation laws for (1.1).

Let $s(t)$ denote the rescaled time

$$s(t) = \int_0^t \frac{d\sigma}{\lambda(\sigma)^2}$$

and take $S = s(T)$ to be the rescaled blow-up time (which could, and in fact, will be shown to be $+\infty$). Since $s(t)$ is monotonically increasing, we can also work with $t(s)$. Then $r_0(t)$ and $\lambda(t)$ become functions of s and thus we can speak of $\lambda(s) \equiv \lambda(t(s))$ and $r_0(s) \equiv r_0(t(s))$. Denote by y the rescaled radial position

$$y(r,t) = \frac{r - r_0(t)}{\lambda(t)}.$$

We now introduce the one-dimensional auxiliary quantity $w(y,s)$, defined by

$$u(r,t) = \frac{1}{\lambda(t)}\, w\left(\frac{r-r_0(t)}{\lambda(t)}, s(t)\right), \tag{4.4}$$

or equivalently,

$$w(y,s) = \lambda(s)\, u(\lambda(s)r + r_0(s), t(s)).$$

It is checked by direct computation that w so defined solves an NLS-type equation

$$i\partial_s w + \frac{2\lambda}{r}\partial_y w + \partial_y^2 w - i\frac{\lambda_s}{\lambda}\Lambda w - i\frac{(r_0)_s}{\lambda}\partial_y w + |w|^2 w = 0, \tag{4.5}$$

where $\Lambda w = (1 + y\partial_y)w$. We expect the term $-i\frac{(r_0)_s}{\lambda}\partial_y w$ to play a more significant role in our computation than the analogous term in Raphaël's two-dimensional quintic result [21]. Now, with these definitions we make

Spherical contraction assumption (SCA). Suppose that $\lambda(t)/r_0(t) \to 0$, and that $w(y,s)$ remains well-localized near $y = 0$ for all s as $s \to S$.

SCA has, in particular, the effect of driving $y_L \equiv -r_0(t)/\lambda(t)$, which corresponds to the position $r = 0$ in the original coordinates, to $-\infty$ as $t \to T$ (or $s \to S$). By SCA, the term $\frac{2\lambda}{r}\partial_y w$ in (4.5) should be negligible as $s \to S$. We now make a second simplifying assumption that we later confirm is consistent.

Focusing rate assumption (FRA). Suppose that $\lambda_s/\lambda \to 0$. Note that $\lambda_s/\lambda = \lambda_t\lambda$, and if $\lambda(t)$ is given by a power-type expression $\lambda(t) = (T-t)^\alpha$, then since $\lambda_t\lambda = \alpha(T-t)^{2\alpha-1}$, this assumption is valid if $\alpha > \frac{1}{2}$.

We have no solid justification for introducing FRA, only that it makes the analysis more tractable. We shall show that FRA gives rise to one self-consistent scenario (see the comment at the end of §4.2).[15] By FRA, we have license to neglect the term $-i\frac{\lambda_s}{\lambda}\Lambda w$ in (4.5). This leads to the following simplified equation in $w(y,s)$

$$i\partial_s w + \partial_y^2 w - i\frac{(r_0)_s}{\lambda}\partial_y w + |w|^2 w = 0. \tag{4.6}$$

4.1. **Asymptotically conserved quantities for w.** Assuming w solves (4.6), we derive "asymptotically conserved" quantities for w. The conservation is only approximate, since (4.6) is only approximate. Moreover, in the calculations, we will routinely ignore the boundary term $y_L = -r_0(s)/\lambda(s)$ corresponding to $r=0$ in the integration by parts computations. (An interpretation of the following computations is that "asymptotically conserved" quantities like $M[w](t)$ converge to specific values as $t \to T$, i.e. behave like $M[w](t) = M[w](T) + \mathcal{O}(T-t)^\mu$ for some $\mu > 0$.)

Define the following quantities (here, y is one-dimensional):

$$\begin{aligned} M[w] &= \int |w|^2\, dy, \\ P[w] &= \operatorname{Im} \int w\, \overline{\partial_y w}\, dy, \\ E[w] &= \frac{1}{2}\int |\partial_y w|^2\, dy - \frac{1}{4}\int |w|^4\, dy, \end{aligned}$$

the mass, momentum, and energy of w. Now we show that the mass and momentum of w are asymptotically conserved and the energy satisfies an *a priori* time dependent equation.

Pair the equation (4.6) with $\bar{w}$ and take $2\times$ the imaginary part:

$$\partial_s \int |w|^2\, dy = \frac{2(r_0)_s}{\lambda} \operatorname{Re} \int \partial_y w\, \bar{w}\, dy = 0.$$

Thus, we have $M[w]$ is approximately constant. By direct substitution of the equation we have

$$\partial_s P[w] = 0,$$

and thus $P[w]$ is approximately constant.

The next step is to study the energy of w. For this we introduce

$$\tilde{w}(y,s) = e^{-i\frac{(r_0)_s}{2\lambda}y} w(y,s) \tag{4.7}$$

and from (4.6) we see that $\tilde{w}$ solves

$$i\partial_s \tilde{w} - \left(\frac{(r_0)_s}{2\lambda}\right)_s y\tilde{w} + \left(\frac{(r_0)_s}{2\lambda}\right)^2 \tilde{w} + \partial_y^2 \tilde{w} + |\tilde{w}|^2 \tilde{w} = 0.$$

[15] Locating plausible scenarios is really the objective here anyhow – an analysis addressing all possibilities seems too ambitious at this point.

Pair this equation with $\overline{\partial_s\tilde{w}}$ and take the real part to get that $E[\tilde{w}]$ is approximately constant. Substituting the definition of $\tilde{w}$, we get that the quantity

$$\frac{1}{2}\left(\frac{(r_0)_s}{2\lambda}\right)^2 M[w] + \frac{1}{2}\frac{(r_0)_s}{\lambda}P[w] + E[w] = E[\tilde{w}] = \text{approx. const.} \tag{4.8}$$

Note that this only says that $E[w]$ is constant if we were to know that $(r_0)_s/\lambda$ is constant, a point we discuss next.

4.2. **Consequences of the asymptotic conservation laws.** Now we work out a consequence of the mass conservation of u and the mass conservation of w.

$$\begin{aligned} M[u] &= 4\pi \int_0^\infty |u|^2\, r^2 dr \\ &= \frac{4\pi}{\lambda(t)^2}\int \left| w\left(\frac{r-r_0(t)}{\lambda(t)}\right)\right|^2 r^2 dr. \end{aligned}$$

Now by SCA, we have

$$M[u] \approx \frac{4\pi r_0^2(t)}{\lambda(t)}\int |w(y)|^2\, dy = \frac{4\pi r_0^2(t)}{\lambda(t)} M[w]. \tag{4.9}$$

An immediate consequence of this is that

$$r_0(t) \approx \lambda(t)^{1/2}. \tag{4.10}$$

Next we study the consequence of energy conservation of both the initial solution u and the rescaled version w.

$$\begin{aligned} E[u] &= \frac{1}{2}\int |\nabla u|^2\, dx - \frac{1}{4}\int |u|^4\, dx = 4\pi\left(\frac{1}{2}\int |\partial_r u|^2 r^2\, dr - \frac{1}{4}\int |u|^4 r^2 dr\right) \\ &= \frac{4\pi}{\lambda^3(t)}\left(\frac{1}{2}\int |\partial_y w|^2 r^2\, dy - \frac{1}{4}\int |w|^4 r^2\, dy\right) \approx 4\pi\, \frac{r_0^2(t)}{\lambda^3(t)}\, E[w]. \end{aligned}$$

Since $r_0(t)$ contracts at a slower rate than $\lambda(t)$, we have $\dfrac{r_0^2(t)}{\lambda^3(t)} \to \infty$ as $t \to T$, and hence, we must have $E[w](t) \to 0$.

Consider the equation (4.8) again: the condition $E[w](t) \to 0$ together with $M[w]$, $P[w]$ and $E[\tilde{w}]$ being constant forces either $\frac{(r_0)_s}{\lambda}$ to be a constant, denote it by κ, or $\frac{(r_0)_s}{\lambda} \to 0$ as $t \to T$. If $\lambda(t) \sim (T-t)^\gamma$, then by (4.10) we have $r_0(t) \sim (T-t)^{\gamma/2}$, and thus, $\kappa = (r_0)_t\lambda = c\,(T-t)^{\gamma/2-1+\gamma}$ implies $\gamma = 2/3$ if $\kappa \neq 0$ and $\gamma > 2/3$ if $\kappa = 0$. The latter case is ruled out by the virial identity in the next section. Thus,

$$r_0(t) \sim (T-t)^{1/3} \quad \text{and} \quad \lambda(t) \sim (T-t)^{2/3}. \tag{4.11}$$

Note that under these conditions and SCA the second term in (4.5) has the coefficient $\frac{\lambda}{r} \approx \frac{\lambda}{r_0} \sim (T-t)^{1/3}$, and thus, the decision to drop it in the analysis close to the blow-up time was, at least, self-consistent. Similarly, the fourth term coefficient in (4.5) $\frac{\lambda_s}{\lambda} = \lambda_t\lambda \sim (T-t)^{1/3}$ and becomes negligible near the blow-up time as well.

4.3. **Application of the virial identities.** By (4.4) and SCA,

$$\|\nabla u(t)\|_{L^2}^2 = 4\pi \int |\partial_r u|^2 r^2 \, dr \approx 4\pi \frac{r_0^2(t)}{\lambda^3(t)} \int |\partial_y w|^2 \, dy.$$

Substituting (4.3), we obtain $\|\partial_y w\|_{L^2}^2 \approx \frac{1}{4\pi}$. The virial identities are

$$\partial_t \int r^4 |u(r,t)|^2 \, dr = 4 \operatorname{Im} \int r^3 \, \bar{u} \, \partial_r u \, dr \tag{4.12}$$

and

$$4\pi \partial_t^2 \int r^4 |u(r,t)|^2 \, dr = 24 E[u] - 4 \|\nabla u\|_{L^2_{xyz}}^2. \tag{4.13}$$

The equation (4.13) produces the approximate relation

$$\partial_t^2 (r_0^2(t)) = \frac{1}{M[u]} \left(24 E[u] - \frac{4 r_0^2(t)}{\lambda^3(t)} \right). \tag{4.14}$$

Observe that $\gamma > 2/3$ would contradict (4.14); similarly we cannot have lower order corrections in $(T-t)^\gamma$ (e.g. $(T-t)^{1/3} \log^{\gamma_1}(T-t)$). We write

$$r_0(t) = \alpha \, (T-t)^{1/3}, \qquad \lambda(t) = \beta (T-t)^{2/3} \tag{4.15}$$

with, as yet, undetermined coefficients α, β. The relation (4.14) forces one relation

$$\beta = \left(\frac{18}{M[u]} \right)^{1/3}.$$

To pursue this further, we incorporate the quantities $M[w]$, $P[w]$, $E[w]$, and $E[\tilde{w}]$ into the analysis. The first of the virial relations (4.12) gives

$$(r_0^2(t))' \approx -\frac{16\pi}{M[u]} \frac{r_0^3(t)}{\lambda^2(t)} P[w],$$

which produces the relation

$$P[w] = \frac{M[u]}{24\pi} \frac{\beta^2}{\alpha} = \frac{(12 M[u])^{1/3}}{8\pi\alpha}. \tag{4.16}$$

The mass conservation from (4.9) gives

$$M[w] = \frac{M[u]}{4\pi} \frac{\beta}{\alpha^2} = \frac{18^{1/3} \, M[u]^{2/3}}{4\pi\alpha^2}. \tag{4.17}$$

By the energy conservation from the previous section we have

$$E[w] = \frac{\lambda^3(t)}{r_0^2(t)} \, E[u] = \frac{\beta^2}{\alpha^2} (T-t)^{4/3} E[u] \to 0.$$

We now have three quantities: α, $M[w]$, $P[w]$, and two equations (4.16) and (4.17). We substitute these values for $P[w]$, $M[w]$ and $E[w]$ into (4.8) and obtain

$$E[\tilde{w}] = -\frac{1}{16\pi}.$$

Observe that we are still free to choose α as it does not affect any of the conservation properties – this flexibility will be used in the next section.

4.4. **Asymptotic profile.** Recall from (4.7) and (4.15) that

$$\tilde{w}(y,s) = e^{-i\kappa y/2}\, w(y,s) \quad \text{with} \quad \kappa = -\alpha\beta/3$$

satisfies

$$i\partial_s \tilde{w} + \left(\frac{\kappa}{2}\right)^2 \tilde{w} + \partial_y^2 \tilde{w} + |\tilde{w}|^2 \tilde{w} = 0.$$

On the grounds that $\tilde{w}(y,s)$ is approximately a global-in-time solution to the one-dimensional cubic NLS, and is well localized at the origin, the only reasonable asymptotic configuration is a stationary soliton (see Zakharov-Shabat [28])[16]. Thus, we assume that as $s \to +\infty$,

$$\tilde{w}(y,s) = e^{i\theta_0} e^{i\nu s} P(y + y_0) \tag{4.18}$$

for some fixed phase shift θ_0 and spatial shift y_0 (since $y_0 \neq 0$ amounts to a lower-order modification in $r_0(t)$, we might as well drop it and take $y_0 = 0$) and where ν is to be chosen later. Then P satisfies

$$-\sigma P + P'' + |P|^2 P = 0, \quad \sigma = \nu - \frac{\kappa^2}{4}. \tag{4.19}$$

The solution of this equation is

$$P(y) = e^{i\theta}\sqrt{2\sigma}\,\mathrm{sech}(\sqrt{\sigma} y),$$

here, θ is arbitrary. By (4.18) and $y_0 = 0$, we have $w(y,s) \approx e^{i\theta} e^{i\nu s} e^{i\kappa y/2} P(y)$. The analysis from the previous section gave

$$E[\tilde{w}] = -\frac{1}{16\pi}, \tag{4.20}$$

$$E[w] = 0, \tag{4.21}$$

$$M[w] = M[\tilde{w}] = \frac{18^{1/3} M[u]^{2/3}}{4\pi\alpha^2}, \tag{4.22}$$

$$P[w] = \frac{(12M[u])^{1/3}}{8\pi\alpha}. \tag{4.23}$$

We choose ν such that $E[e^{i\kappa y/2}P] = 0$. This implies that $\frac{1}{8}\kappa^2 M[P] + E[P] = 0$ and using $E[P] = -\frac{2}{3}\sigma\sqrt{\sigma}$ and $M[P] = 4\sqrt{\sigma}$, we obtain

$$\sigma = \frac{3}{4}\kappa^2, \quad \text{and hence,} \quad \nu = \kappa^2.$$

[16] Also possible are the envelope solitons or "breathers" solutions described in [28], although we choose not to investigate this possibility here since they are unstable and the small corrections to the w equation would likely cause them to collapse to decoupled solitons moving away from each other.

The equations (4.20) and $E[P] = -\frac{2}{3}\sigma^{3/2}$ give

$$\sigma = \left(\frac{3}{32\pi}\right)^{2/3}.$$

The equations (4.22) and $M[P] = 4\sigma^{1/2}$ give

$$\alpha = \frac{3^{1/6}}{2\pi^{1/3}} M[u]^{1/3}.$$

Now we conclude with two consistency checks: The values of κ obtained together with the definition $\kappa = -\alpha\beta/3$, and the formula for β, are consistent with the value of α obtained here. The value in (4.23) and $P[e^{i\kappa y/2}P(y)] = -2\kappa\sigma^{1/2}$ is consistent with the value of α obtained here.

Pulling all of this information together, we obtain the description given in the introduction.

5. Consistency with higher precision computations

As a consistency check, we show that the result obtained in the previous section regarding the "approximate conservation" of the mass, momentum, and energy of w stands up to a second-level of precision. To do this, we consider (4.5) with only the approximation $r \approx r_0$ in the second term and we leave the fourth term as is (in the previous section, we completely dropped the second and fourth terms):

$$i\partial_s w + \frac{2\lambda}{r_0}\partial_y w + \partial_y^2 w - i\frac{\lambda_s}{\lambda}\Lambda w - i\frac{(r_0)_s}{\lambda}\partial_y w + |w|^2 w = 0. \tag{5.1}$$

Pairing this equation with $\bar{w}$ and taking the imaginary part, we have

$$\frac{1}{2}\,\partial_s \int |w|^2 + \frac{2\lambda}{r_0}\,\mathrm{Im}\int \partial_y w\,\bar{w} - \frac{\lambda_s}{\lambda}\,\mathrm{Re}\int \Lambda w\,\bar{w} = 0,$$

which simplifies to

$$\frac{1}{2}\,\partial_s \int |w|^2 - \frac{2\lambda}{r_0}\,P[w] - \frac{\lambda_s}{2\lambda}\,M[w] = 0.$$

Applying (4.15), (4.16) and (4.17) to the second and third terms (note that they cancel each other out), we obtain the conservation of mass $M[w]$.

To calculate the refined momentum, substitute (5.1) into the definition of $P[w]$ to get

$$\partial_s P[w] - \frac{4\lambda}{r_0}\int |\partial_y w|^2 - \frac{2\lambda_s}{\lambda}P[w] = 0,$$

from which we obtain

$$\partial_s P[w] = \frac{\beta}{\alpha\pi}(T-t)^{1/3} - \frac{\beta^4}{18\pi\alpha}M[u](T-t)^{1/3} = 0,$$

and thus, the momentum of w is also preserved.

Note that in this more precise approximation we obtain the cancelation of the "error" terms in (5.1) as it was claimed in the introduction.

The calculation of the refined energy from (5.1) doesn't produce any similar cancelation, however, it confirms (4.8). We outline it next: first substitute

$$w(y,s) = e^{i\frac{(r_0)_s}{2\lambda}y}\tilde{w}(y,s)$$

to remove $i\dfrac{(r_0)_s}{\lambda}\partial_y w$ term and obtain

$$i\partial_s\tilde{w} + i\left(\frac{(r_0)_s}{\lambda}\right)_s\tilde{w} + \frac{2\lambda}{r_0}\,\partial_y\tilde{w} + i\frac{(r_0)_s}{r_0}\,\tilde{w} + \partial_y^2\tilde{w} + \left(\frac{(r_0)_s}{2\lambda}\right)^2\tilde{w} - i\frac{\lambda_s}{\lambda}\,\Lambda\tilde{w}$$
$$+\frac{\lambda_s}{\lambda}\frac{(r_0)_s}{2\lambda}\,y\tilde{w} + |\tilde{w}|^2\tilde{w} = 0.$$

Next we substitute

$$\tilde{w}(y,s) = e^{i\frac{\lambda_s}{4\lambda}y^2}v(y,s)$$

into the previous equation which results in

$$i\partial_s v + \frac{2\lambda}{r_0}\partial_y v + \partial_y^2 v + v\left(\left(\frac{(r_0)_s}{2\lambda}\right)^2 + i\left[\left(\frac{(r_0)_s}{\lambda}\right)_s + \frac{(r_0)_s}{r_0} - \frac{\lambda_s}{2\lambda}\right]\right) \tag{5.2}$$
$$+yv\left(\frac{\lambda_s}{\lambda}\frac{(r_0)_s}{2\lambda} + \frac{\lambda_s}{r_0}\right) + y^2v\left(\frac{\lambda_s}{2\lambda}\right)^2 + |v|^2v = 0.$$

We examine coefficients in front of v, yv and y^2v. Observe that $\left(\frac{(r_0)_s}{2\lambda}\right)^2$ is the largest coefficient by (4.15) and in fact, is a constant; all other coefficients have the order of $(T-t)^{1/3}$ or $(T-t)^{2/3}$, and therefore, we drop corresponding terms from further analysis. Thus, we obtain

$$i\partial_s v + \frac{2\lambda}{r_0}\,\partial_y v + \partial_y^2 v + \left(\frac{(r_0)_s}{2\lambda}\right)^2 v + |v|^2 v \approx 0.$$

We proceed further and make the substitution

$$v(y,s) = e^{-\frac{\lambda}{r_0}y}\,\tilde{v}(y,s)$$

to remove the $\frac{2\lambda}{r_0}\partial_y v$ term in (5.2) and obtain

$$i\partial_s\tilde{v} + \partial_y^2\tilde{v} + \left[\left(\frac{(r_0)_s}{2\lambda}\right)^2 - \frac{\lambda^2}{r_0^2}\right]\tilde{v} + |\tilde{v}|^2\tilde{v}\,e^{-2\frac{\lambda}{r_0}y} \approx 0.$$

Again finding the order of the coefficients and expanding the exponent in the nonlinear term, we drop $\frac{\lambda^2}{r_0^2}$ as well as all positive powers of $-2\frac{\lambda}{r_0}y$ to get

$$i\partial_s\tilde{v} + \partial_y^2\tilde{v} + \left(\frac{(r_0)_s}{2\lambda}\right)^2\tilde{v} + |\tilde{v}|^2\tilde{v} \approx 0.$$

This produces $E[\tilde{v}] \approx \text{const} + O((T-t)^{\gamma})$, $\gamma > 0$. Revealing all substitutions, we obtain $E[\tilde{w}] \approx \text{const} + O((T-t)^{\gamma_1})$, $\gamma_1 > 0$, and expressing the last approximate identity in terms of w we end up with (4.8).

6. General case $\mathrm{NLS}_p(\mathbb{R}^N)$

Consider the mass supercritical focusing $\mathrm{NLS}_p(\mathbb{R}^N)$ equation with $p > 1 + \frac{4}{N}$ nonlinearity

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \tag{6.1}$$

for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with Schwartz class initial-data $u_0 \in \mathcal{S}(\mathbb{R}^N)$. Then the following scaling of the solution is itself a solution:

$$u_\lambda(x,t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t).$$

The scale invariant Sobolev norm is $\dot{H}^{s_c}$, where $s_c = \frac{N}{2} - \frac{2}{p-1}$ (since $p > 1 + \frac{4}{N}$, we have $s_c > 0$). If $s_c > 1$ (prototypical case $N = 3$ and $p = 7$), then we do not have a local theory in H^1. [17] One does, however, have local well-posedness in H^s for $s > s_c$ by the Strichartz estimates on a maximal time interval $[0, T_s)$ with $\lim_{t\to T_s} \|u(t)\|_{H^s} = +\infty$. It would appear that if $s_1 > s_2 > s_c$, then it might be possible for $T_{s_1} < T_{s_2}$. However, a *persistence of regularity* result also follows from the Strichartz estimates and gives that $T_{s_1} = T_{s_2}$. Thus, even though an H^1 local theory is absent, there is a clear distinction between global solutions and finite-time blow-up solutions, and it still makes sense to speak of "the blow-up time."

A heuristic similar to the one presented in §4 for a radial blow-up solution of (6.1) results in the following estimation of parameters. Let

$$u(r,t) = \frac{1}{\lambda(t)^{2/(p-1)}}\, w\left(\frac{r - r_0(t)}{\lambda(t)}, t\right).$$

Using the conservation of mass as in §4.2, we obtain

$$M[u] \approx \frac{r_0(t)^{(N-1)}}{\lambda(t)^{\frac{5-p}{p-1}}}\, |\,\mathbb{S}^{N-1}|\, M[w],$$

and thus, $r_0(t) \sim \lambda(t)^{\frac{5-p}{(p-1)(N-1)}}$. This means that for all quintic nonlinearity mass supercritical problems the radial solution would blow up on a constant radius sphere

[17] Indeed, it was observed by Birnir-Kenig-Ponce-Svanstedt-Vega [1] that one can take a finite-time radial H^1-blow-up solution (whose existence is guaranteed by the virial identity) and suitably rescale it to obtain a solution initially arbitrarily small in H^1 that blows-up in an arbitrarily short interval of time.

as in Raphaël's construction [21][18]; for $p = 7$, the solution would blow up on a sphere with radius $r_0(t) \nearrow \infty$, i.e. on an *expanding* sphere.

Using the virial identities as in §4.3, we obtain

$$\lambda(t) \sim (T-t)^{\gamma} \quad \text{with} \quad \gamma = \frac{(p-1)(N-1)}{(p-1)(N-1)+5-p},$$

and correspondingly,

$$r_0(t) \sim (T-t)^{\frac{5-p}{(p-1)(N-1)+(5-p)}}.$$

References

[1] B. Birnir, C.E. Kenig, G. Ponce, N. Svanstedt, L. Vega, *On the ill-posedness of the IVP for the generalized Korteweg-de Vries and nonlinear Schrödinger equations*. J. London Math. Soc. (2) 53 (1996), no. 3, 551–559.

[2] J. Bourgain, W. Wang, *Construction of blowup solutions for the nonlinear Schrödinger equation with critical nonlinearity. Dedicated to Ennio De Giorgi*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. 1-2, 197–215 (1998).

[3] T. Cazenave, *Semilinear Schrödinger equations*. Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323 pp. ISBN: 0-8218-3399-5.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on* $\mathbb{R}^3$, Comm. Pure Appl. Math. 57 (2004), no. 8, pp. 987–1014.

[5] G. Fibich, N. Gavish, X-P. Wang, *New singular solutions of the nonlinear Schrödinger equation*, Phys. D 211 (2005), no. 3-4, pp. 193–220.

[6] G. Fibich, F. Merle, and P. Raphaël, *Proof of a spectral property related to the singularity formation for the L^2 critical nonlinear Schrödinger equation*, to appear in Physica D.

[7] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equation. I. The Cauchy problems; II. Scattering theory, general case*, J. Func. Anal. 32 (1979), 1-32, 33-71.

[8] J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of nonlinear Schrdinger equations*, J. Math. Pures Appl. (9) 64 (1985), no. 4, pp. 363–401.

[9] Glassey, R. T., *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equation*, J. Math. Phys., 18, 1977, 9, 1794–1797.

[10] T. Hmidi, S. Keraani, *Blowup theory for the critical nonlinear Schrödinger equations revisited*, Int. Math. Res. Not. 2005, no. 46, pp. 2815–2828.

[11] F. Merle, K. Kenig, *Global well-posedness, scattering, and blow-up for the energy-critical focusing nonlinear Schrödinger equation in the radial case*, preprint.

[12] F. Merle, P. Raphaël, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Ann. of Math. (2) 161 (2005), no. 1, 157–222.

[13] F. Merle, P. Raphaël, *Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation*, Geom. Funct. Anal. 13 (2003), no. 3, 591–642.

[18]Note that this does not contradict Remark 5.20 in [11] where it is remarked that for $N \geq 4$ if $\|\nabla u(t)\|_{L^2_x} \to \infty$ as $t \to T$, then the blow up necessarily occurs at the origin. Together with our analysis, it suggests that blow up can occur simultaneously at the origin and on the sphere of constant radius.

[14] F. Merle, P. Raphaël, *On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation*, Invent. Math. 156 (2004), no. 3, 565–672.

[15] F. Merle, P. Raphaël, *On a sharp lower bound on the blow-up rate for the L^2 critical nonlinear Schrödinger equation*, J. Amer. Math. Soc. 19 (2006), no. 1, 37–90.

[16] F. Merle, P. Raphaël, *Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation*, Comm. Math. Phys. 253 (2005), no. 3, 675–704.

[17] F. Merle, P. Raphaël, *Blow-up of the critical norm for some radial L^2 supercritical nonlinear Schrödinger equations*, arxiv.org preprint `math.AP/0605378`.

[18] F. Merle and Y. Tsutsumi, *L^2 concentration of blow-up solutions for the nonlinear Schrödinger equation with critical power nonlinearity*, J. Differential Equations 84 (1990), no. 2, pp. 205–214.

[19] T. Ogawa and Y. Tsutsumi, *Blow-Up of H^1 solution for the Nonlinear Schrödinger Equation*, J. Diff. Eq. 92 (1991), 317-330.

[20] P. Raphaël, *Stability of the log-log bound for blow up solutions to the critical non linear Schrödinger equation*, Math. Ann. 331 (2005), no. 3, 577–609.

[21] P. Raphaël, *Existence and stability of a solution blowing up on a sphere for an L^2 supercritical non linear Schrödinger equation*, Duke Math J. 134 (2006), pp. 199-258.

[22] V. Rottschäfer and T.J. Kaper, *Blowup in the nonlinear Schrödinger equation near critical dimension*, J. Math. Anal. Appl. 268 (2002), no. 2, pp. 517–549.

[23] C. Sulem, P-L. Sulem, *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences, 139. Springer-Verlag, New York, 1999. xvi+350 pp.

[24] T. Tao, *On the asymptotic behavior of large radial data for a focusing non-linear Schrödinger equation*, Dyn. Partial Differ. Equ. 1 (2004), no. 1, pp. 1–48.

[25] T. Tao, *Nonlinear dispersive equations. Local and global analysis.* CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp. ISBN: 0-8218-4143-2

[26] T. Tao, *A (concentration-)compact attractor for high-dimensional non-linear Schrödinger equations*, arxiv.org preprint math.AP/0611402.

[27] M. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. 87 (1982/83), no. 4, 567–576.

[28] V.E. Zakharov and A.B. Shabat. *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media.* Soviet Physics JETP 34 (1972), no. 1, 62–69.

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