# On the 2d Zakharov system with $L^2$ Schrödinger data

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**Abstract.** We prove local in time well-posedness for the Zakharov system in two space dimensions with large initial data in $L^2 \times H^{-1/2} \times H^{-3/2}$. This is the space of optimal regularity in the sense that the data-to-solution map fails to be smooth at the origin for any rougher pair of spaces in the $L^2$-based Sobolev scale. Moreover, it is a natural space for the Cauchy problem in view of the subsonic limit equation, namely the focusing cubic nonlinear Schrödinger equation. The existence time we obtain depends only upon the corresponding norms of the initial data – a result which is false for the cubic nonlinear Schrödinger equation in dimension two – and it is optimal because Glangetas–Merle's solutions blow up at that time.

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## 1. Introduction and main result

We study the initial-value problem for the Zakharov system in two spatial dimensions:

$$\begin{aligned} &i\partial_t u + \Delta u = nu, \\ &\partial_t^2 n - \Delta n = \Delta |u|^2, \end{aligned} \tag{1.1}$$

where $u : \mathbb{R}^{2+1} \to \mathbb{C}$ and $n : \mathbb{R}^{2+1} \to \mathbb{R}$, with initial data

$$(u|_{t=0}, n|_{t=0}, \partial_t n|_{t=0}) = (u_0, n_0, n_1).$$

This system was introduced by Zakharov [22] as a model for the propagation of Langmuir waves in a plasma.

We address the question of local well-posedness of (1.1) for large data in low regularity Sobolev spaces. For $k, \ell \in \mathbb{R}$ we define the space

$$\mathbf{H}^{k,\ell} := H^k(\mathbb{R}^2; \mathbb{C}) \times H^\ell(\mathbb{R}^2; \mathbb{R}) \times H^{\ell-1}(\mathbb{R}^2; \mathbb{R})$$

with the natural norm. By $\mathbf{X}_T^{k,\ell}$ we denote the space of all tempered distributions $(u, n)$ on $(0, T) \times \mathbb{R}^2$ such that

$$\begin{aligned} &u \in C([0, T]; H^k(\mathbb{R}^2; \mathbb{C})), \\ &n \in C([0, T]; H^\ell(\mathbb{R}^2; \mathbb{R})) \cap C^1([0, T]; H^{\ell-1}(\mathbb{R}^2; \mathbb{R})). \end{aligned}$$

with the standard norm, see (2.4). For $0 < r \le R$ we also define

$$\mathbf{H}^{k,\ell}_{R,r} := \{(u_0, n_0, n_1) \in \mathbf{H}^{k,\ell} : \ \|(u_0, n_0, n_1)\|_{\mathbf{H}^{k,\ell}} \le R; \|u_0\|_{L^2} \le r\}$$

as a metric subspace of $\mathbf{H}^{k,\ell}$.

Our main result is the local well-posedness of (1.1) in $\mathbf{H}^{0,-\frac{1}{2}}$, which was phrased as an open problem by Merle [19, p. 58, ll. 14–15].

**Theorem 1.1.** *For every $0 < r \le R$ and initial data $(u_0, n_0, n_1) \in \mathbf{H}^{0,-\frac{1}{2}}_{R,r}$ and time $T \lesssim \min\{\langle R\rangle^{-2} r^{-2}, 1\}$, there exists a subspace $\mathbf{X}_T \subset \mathbf{X}_T^{0,-\frac{1}{2}}$ and a unique solution $(u, n) \in \mathbf{X}_T$ of the Cauchy problem (1.1). The map*

$$\mathbf{H}^{0,-\frac{1}{2}}_{R,r} \longrightarrow \mathbf{X}_T^{0,-\frac{1}{2}} : \quad (u_0, n_0, n_1) \mapsto (u, n)$$

*is locally Lipschitz-continuous.*

*Remark* 1. Note that a-priori the nonlinear system (1.1) is not well-defined for rough distributions. The precise notion of *solution* in Theorem 1.1 is explained in Section 3. The auxiliary space $\mathbf{X}_T$ is based on generalized Fourier restriction spaces.

*Remark* 2. Notice that Theorem 1.1 implies in particular that locally the flow map for smooth data extends continuously to initial data in $\mathbf{H}^{0,-\frac{1}{2}}$. The uniqueness claim in Theorem 1.1 is restricted to the subspace $\mathbf{X}_T$ of $X_T^{0,-\frac{1}{2}}$, which ensures that $(u, n)$ is the unique limit of smooth solutions.

Local well-posedness of (1.1) in the low-regularity setting has been previously considered by many authors: Bourgain–Colliander [6] proved local well-posedness in spaces which comprise the energy space and established global well-posedness in the energy space under a smallness condition. The local result has been improved later by Ginibre–Tsutsumi–Velo [12]. Both aforementioned approaches are based on the Fourier restriction norm method. For previous well-posedness results we refer the reader to the references in [6, 12]. In [12] Ginibre–Tsutsumi–Velo obtain local well-posedness of (1.1) in the case of space dimension $d = 2$ in the space $H^k \times H^\ell \times H^{\ell-1}$ for $(k, \ell)$ confined to the strip $\ell \geq 0$, $2k \geq \ell + 1$. The optimal corner of this strip occurs at $H^{\frac{1}{2}} \times L^2 \times H^{-1}$, one-half a derivative away from the result in Theorem 1.1.

One motivation for considering the space $L^2 \times H^{-1/2} \times H^{-3/2}$ is the connection to the cubic nonlinear Schrödinger equation in two spatial dimensions

$$i\partial_t u + \Delta u + |u|^2 u = 0. \tag{1.2}$$

Consider the Zakharov system with wave speed $\lambda > 0$:

$$\begin{aligned} &i\partial_t u + \Delta u = nu, \\ &\frac{1}{\lambda^2}\partial_t^2 n - \Delta n = \Delta |u|^2. \end{aligned} \tag{1.3}$$

Then formally (1.3) converges to (1.2) as $\lambda \to \infty$ in the sense that for fixed initial data $u_\lambda \to u$, where $(u_\lambda, n_\lambda)$ solves (1.3) and $u$ solves (1.2) with the same initial data. Rigorous results of this type in a high regularity setting were obtained by Schochet–Weinstein [21], Added–Added [1], Ozawa–Tsutsumi [20], see also the recent work by Masmoudi–Nakanishi [18] on this issue in 3d.

Local well-posedness in $L^2$ of (1.2) was obtained by Cazenave–Weissler [7]. However, in this version of well-posedness, the time interval of existence depends directly upon the initial data, not just on the $L^2$ norm of the initial data. Indeed, via the pseudoconformal transformation, it can be shown that a result giving the maximal time of existence in terms of the $L^2$ norm alone is not possible‡.

*Remark* 3. Our result gives local well-posedness of (1.3) with a time of existence depending on the $L^2$ norm of $u_0$, but also on the $H^{-1/2} \times H^{-3/2}$ norm of the wave data $(n_0, n_1)$ as well as the wave speed. Indeed, this claim follows by combining the rescaling

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad v_\lambda(t, x) = \lambda^2 v(\lambda^2 t, \lambda x)$$

and Theorem 1.1. However, note that the lower bound on the maximal time of existence obtain by this method tends to zero as the wave speed goes to infinity.

Global well-posedness of (1.1) is known for initial data in the energy space $H^1 \times L^2 \times H^{-1}$ with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, see [6, 13]; see also [10] regarding bounds on higher order Sobolev norms. Recently, the imposed regularity assumption has been

‡ Note that Killip-Tao-Visan [16] have recently obtained global well-posedness for (1.2) if $u_0 \in L^2$ is radial and $\|u_0\|_{L^2} < \|Q\|_{L^2}$, see (1.4).

slightly weakened in [11]. Here, $Q$ is the ground state solution for (1.2), i.e. $Q$ is the unique solution to

$$-Q + \Delta Q + |Q|^2 Q = 0,\ Q > 0,\ Q(x) = Q(|x|),\ Q \in \mathcal{S}(\mathbb{R}^2) \tag{1.4}$$

of minimal $L^2$ mass. This gives rise to a blow-up solution of (1.2) by the pseudoconformal transformation. This idea is exploited in [14], where Glangetas–Merle construct a family of blow-up solutions for (1.1) of the form

$$\begin{aligned} u(t,x) &= \frac{\omega}{T-t} e^{i\left(\theta + \frac{\omega^2}{T-t} - \frac{|x|^2}{4(T-t)}\right)} P_\omega\left(\frac{x\omega}{T-t}\right) \\ n(t,x) &= \left(\frac{\omega}{T-t}\right)^2 N_\omega\left(\frac{x\omega}{T-t}\right) \end{aligned} \tag{1.5}$$

for parameters $\theta \in \mathbb{S}^1$, $T > 0$, and $\omega \gg 1$, such that $P_\omega \in H^1$ is smooth and radially symmetric, $N_\omega \in L^2$ is a radially symmetric Schwartz function, and $(P_\omega, N_\omega) \to (Q, -Q^2)$ in $H^1 \times L^2$ as $\omega \to \infty$. In particular, this implies the necessity of the smallness assumption $\|u_0\|_{L^2} \le \|Q\|_{L^2}$ for any global existence result for (1.1).

We prove Theorem 1.1 by the contraction method in a suitably defined Fourier restriction norm space, which gives a certain lower bound on the time of existence. By adapting the argument of Colliander–Holmer–Tzirakis [9] using the $L^2$ conservation of $u(t)$ and iteration, we are able to show that this time can in fact be extended to the longer lifespan given in Theorem 1.1. In summary, the time of existence we obtain is based on

(i) sharp multilinear estimates

(ii) the $L^2$ conservation law for the Schrödinger part.

Reviewing the solutions (1.5) constructed by Glangetas–Merle we observe that Theorem 1.1 contains the optimal§ lifespan for Schrödinger data with fixed $L^2$ norm larger than the ground state mass.

**Theorem 1.2** (follows from [13, 14])**.** *For each $r > \|Q\|_{L^2}$ there exists $c > 0$ such that for every $R \ge r$ there exists a smooth solution $(u, n)$ with initial datum $(u(0), n(0), \partial_t n(0)) \in \mathbf{H}^{0,-\frac{1}{2}}_{R,r}$ which blows up at time $T := cR^{-2}$, i.e.*

$$\|n(t)\|_{H^{-\frac{1}{2}}} + \|\partial_t n(t)\|_{H^{-\frac{3}{2}}} \to \infty \quad (t \to T). \tag{1.6}$$

The absence of the $L^2$ norm of $u$ in (1.6) is due to the $L^2$ conservation law. We refer the reader to [13, 14] for further properties of the blow-up solutions such as $L^2$ norm concentration for $u$. Finally, we state a result which shows the optimality of the imposed regularity assumptions in Theorem 1.1.

**Theorem 1.3.** *Assume there exists $0 < r \le R$ and $T > 0$ such that the flow map $u_0 \mapsto u$ for smooth data extends continuously to a map*

$$\mathbf{H}^{k,\ell}_{r,R} \to \mathbf{X}^{k,\ell}_T$$

§ up to the implicit multiplicative constant *for some $\ell - 2k + \frac{1}{2} > 0$ or $\ell < -\frac{1}{2}$. Then this map fails to be $C^2$ at the origin with respect to these norms.*

The rest of the paper is organized as follows: In Section 2, we set up the notation and introduce function spaces which we will use in the sequel. In Section 3 we outline the standard procedure (cp. [12]) for reducing (1.1) to a first order (in time) system. Section 4 is devoted to the crucial multilinear estimates which are the main ingredients in the proof of Theorem 1.1. Section 5 contains estimates for the linear group and the conclusion of the proof of Theorem 1.1. The counterexamples which lead to the sharpness result of Theorem 1.3 are constructed in Section 6, along with a proof of Theorem 1.2 (which is based on the results from [13, 14]). In the Appendix we give an alternative proof of Proposition 4.4 which keeps the paper self-contained.

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## 2. Notation and function spaces

We write $A \lesssim B$ if there is a harmless constant $c > 0$ such that $A \le cB$. Moreover, we write $A \gtrsim B$ iff $B \lesssim A$. and $A \sim B$ iff $A \lesssim B$ and $A \gtrsim B$. Throughout this work we will denote dyadic numbers $2^n$ for $n \in \mathbb{N}$ by capital letters, e.g. $N = 2^n, L = 2^l, \ldots$.

Let $\psi \in C_0^\infty((-2,2))$ be an even, non-negative function with the property $\psi(r) = 1$ for $|r| \le 1$. We use it to define a partition of unity in $\mathbb{R}$,

$$1 = \sum_{N \ge 1} \psi_N, \ \psi_1 = \psi, \ \psi_N(r) = \psi\left(\frac{r}{N}\right) - \psi\left(\frac{2r}{N}\right), \ N = 2^n \ge 2.$$

Thus $\operatorname{supp}\psi_1 \subset [-2,2]$ and $\operatorname{supp}\psi_N \subset [-2N, -N/2] \cup [N/2, 2N]$ for $N \ge 2$. For $f : \mathbb{R}^2 \to \mathbb{C}$ we define the dyadic frequency localization operators $P_N$ by

$$\mathcal{F}_x(P_N f)(\xi) = \psi_N(|\xi|)\mathcal{F}_x f(\xi).$$

For $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ we define $(P_N u)(x,t) = (P_N u(\cdot,t))(x)$. We will often write $u_N = P_N u$ for brevity. We denote the space-time Fourier support of $P_N$ by the corresponding Gothic letter

$$\begin{aligned} \mathfrak{P}_1 &= \left\{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} \mid |\xi| \le 2\right\}, \\ \mathfrak{P}_N &= \left\{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} \mid N/2 \le |\xi| \le 2N\right\}. \end{aligned}$$

Moreover, for dyadic $L \geq 1$ we define the modulation localization operators

$$\begin{aligned} \mathcal{F}(S_L u)(\tau,\xi) &= \psi_L(\tau+|\xi|^2)\mathcal{F}u(\tau,\xi) \qquad \text{(Schrödinger case)}, \qquad (2.1)\\ \mathcal{F}(W_L^{\pm} u)(\tau,\xi) &= \psi_L(\tau\pm|\xi|)\mathcal{F}u(\tau,\xi) \qquad \text{(Wave case)}, \qquad (2.2)\end{aligned}$$

and the corresponding space-time Fourier supports

$$\begin{aligned} \mathfrak{S}_1 &= \left\{(\xi,\tau)\in\mathbb{R}^2\times\mathbb{R} \mid |\tau+|\xi|^2|\leq 2\right\},\\ \mathfrak{S}_L &= \left\{(\xi,\tau)\in\mathbb{R}^2\times\mathbb{R} \mid L/2\leq|\tau+|\xi|^2|\leq 2L\right\},\end{aligned}$$

respectively

$$\begin{aligned} \mathfrak{W}_1^{\pm} &= \left\{(\xi,\tau)\in\mathbb{R}^2\times\mathbb{R} \mid |\tau\pm|\xi||\leq 2\right\},\\ \mathfrak{W}_L^{\pm} &= \left\{(\xi,\tau)\in\mathbb{R}^2\times\mathbb{R} \mid L/2\leq|\tau\pm|\xi||\leq 2L\right\}.\end{aligned}$$

We also define an equidistant partition of unity in $\mathbb{R}$,

$$1=\sum_{j\in\mathbb{Z}}\beta_j, \qquad \beta_j(s)=\psi(s-j)\left(\sum_{k\in\mathbb{Z}}\psi(s-k)\right)^{-1}.$$

Finally, for $A\in\mathbb{N}$ we define an equidistant partition of unity on the unit circle,

$$1=\sum_{j=0}^{A-1}\beta_j^A, \qquad \beta_j^A(\theta)=\beta_j\left(\frac{A\theta}{\pi}\right)+\beta_{j-A}\left(\frac{A\theta}{\pi}\right)$$

We observe that $\operatorname{supp}(\beta_j^A)\subset\Theta_j^A$, where

$$\Theta_j^A:=\left[\frac{\pi}{A}(j-2),\frac{\pi}{A}(j+2)\right]\cup\left[-\pi+\frac{\pi}{A}(j-2),-\pi+\frac{\pi}{A}(j+2)\right].$$

Next we introduce the angular frequency localization operators $Q_j^A$,

$$\mathcal{F}_x(Q_j^A f)(\xi)=\beta_j^A(\theta)\mathcal{F}_x f(\xi), \quad \text{where } \ \xi=|\xi|(\cos\theta,\sin\theta).$$

For $u:\mathbb{R}^2\times\mathbb{R}\to\mathbb{C}$, $(x,t)\mapsto u(x,t)$ we set $(Q_j^A u)(x,t)=(Q_j^A u(\cdot,t))(x)$. These operators localize functions in frequency to the sets

$$\mathfrak{Q}_j^A=\left\{(|\xi|\cos(\theta),|\xi|\sin(\theta),\tau)\in\mathbb{R}^2\times\mathbb{R} \mid \theta\in\Theta_j^A\right\}.$$

For $A\in\mathbb{N}$ we can now decompose $u:\mathbb{R}^2\times\mathbb{R}\to\mathbb{C}$ as

$$u=\sum_{j=0}^{A-1}Q_j^A u.$$

Next we turn our attention to defining the spaces which play a crucial role in our analysis. As explained in the introduction, for $k,\ell\in\mathbb{R}$ and $T>0$ we define the space $\mathbf{X}_T^{k,\ell}$ as the Banach space of all pairs of space-time distributions $(u,n)$

$$\begin{aligned} &u\in C([0,T];H^k(\mathbb{R}^2;\mathbb{C})),\\ &n\in C([0,T];H^\ell(\mathbb{R}^2;\mathbb{R}))\cap C^1([0,T];H^{\ell-1}(\mathbb{R}^2;\mathbb{R})),\end{aligned} \qquad (2.3)$$

endowed with the standard norm defined via

$$\|(u,n)\|_{\mathbf{X}_T^{k,\ell}}^2=\|u\|_{L^\infty([0,T];H_x^k)}^2+\|n\|_{L^\infty([0,T];H_x^\ell)}^2+\|\partial_t n\|_{L^\infty([0,T];H_x^{\ell-1})}^2. \qquad (2.4)$$

Let $\sigma, b \in \mathbb{R}$, $1 \leq p < \infty$. In connection to the operator $i\partial_t + \Delta$ we define the Bourgain space $X^S_{\sigma,b,p}$ of all $u \in \mathcal{S}'(\mathbb{R}^2 \times \mathbb{R})$ for which the norm

$$\|u\|_{X^S_{\sigma,b,p}} = \left( \sum_{N \geq 1} N^{2\sigma} \left( \sum_{L \geq 1} L^{pb} \|S_L P_N u\|^p_{L^2} \right)^{\frac{2}{p}} \right)^{\frac{1}{2}}$$

is finite. Similarly, to the half-wave operators $i\partial_t \pm \langle \nabla \rangle$ we associate the Bourgain spaces $X^{W\pm}_{\sigma,b,p}$ of all $v \in \mathcal{S}'(\mathbb{R}^2 \times \mathbb{R})$ for which the norm

$$\|v\|_{X^{W\pm}_{\sigma,b,p}} = \left( \sum_{N \geq 1} N^{2\sigma} \left( \sum_{L \geq 1} L^{pb} \|W^{\pm}_L P_N u\|^p_{L^2} \right)^{\frac{2}{p}} \right)^{\frac{1}{2}}$$

is finite. For $p = \infty$ we modify the definition as usual:

$$\begin{aligned} \|v\|_{X^{W\pm}_{\sigma,b,\infty}} &= \left( \sum_{N \geq 1} N^{2\sigma} \sup_{L \geq 1} L^{2b} \|W^{\pm}_L P_N u\|^2_{L^2} \right)^{\frac{1}{2}}, \\ \|u\|_{X^S_{\sigma,b,\infty}} &= \left( \sum_{N \geq 1} N^{2\sigma} \sup_{L \geq 1} L^{2b} \|S_L P_N u\|^2_{L^2} \right)^{\frac{1}{2}}. \end{aligned}$$

In cases where the Schwartz space $\mathcal{S}(\mathbb{R}^2 \times \mathbb{R})$ is not dense in $X^{W\pm}_{\sigma,b,p}$ or $X^S_{\sigma,b,p}$, respectively, we redefine the spaces and take the closure of $\mathcal{S}(\mathbb{R}^2 \times \mathbb{R})$ instead. Therefore, it is enough to prove all estimates in these spaces for Schwartz functions.

Notice that a change of $\tau \pm |\xi|$ to $\tau \pm \langle \xi \rangle$ in (2.2) would lead to equivalent norms.

Finally, we define $X^W_{\sigma,b,p}$ similar to $X^{W\pm}_{\sigma,b,p}$ by means of replacing $\tau \pm |\xi|$ in (2.2) by $|\tau| - |\xi|$. $X^W_{\sigma,b,p}$ will only be used to describe the regularity for solutions of the full wave equation in Theorem 1.1.

For a normed space $B \subset \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}; \mathbb{C})$ of space-time distributions we denote by $\overline{B}$ the space of complex conjugates with the induced norm.

A calculation shows that $\overline{X^{W\pm}_{s,b,p}} = X^{W\mp}_{s,b,p}$. By duality,

$$(\overline{X^S_{s,b,p}})^* = X^S_{-s,-b,p'}\ , \tag{2.5}$$

$$(\overline{X^{W\pm}_{s,b,p}})^* = X^{W\pm}_{-s,-b,p'}\ , \tag{2.6}$$

for $1 \leq p < \infty$, $s, b \in \mathbb{R}$.

For $T > 0$ we define the space $B(T)$ of restrictions of distributions in $B$ to the set $\mathbb{R}^n \times (0, T)$ with the induced norm

$$\|u\|_{B(T)} = \inf\{\|\tilde{u}\|_B : \ \tilde{u} \in B \text{ is an extension of } u \text{ to } \mathbb{R}^n \times \mathbb{R}\}.$$

## 3. The reduced system

For the Zakharov system there is a standard procedure to factor the wave operator in order to derive a first order system. In this section we outline the approach described in [12].

Suppose that $(u, n)$ is a sufficiently regular solution to (1.1). We define $\langle\nabla\rangle = (1-\Delta)^{\frac{1}{2}}$ and $v = n + i\langle\nabla\rangle^{-1}\partial_t n$ and obtain the system

$$\begin{aligned} &i\partial_t u + \Delta u = (\operatorname{Re} v)u, \\ &i\partial_t v - \langle\nabla\rangle v = -\frac{\Delta}{\langle\nabla\rangle}|u|^2 - \langle\nabla\rangle^{-1}\operatorname{Re} v. \end{aligned} \tag{3.1}$$

Given a solution $(u, v)$ to (3.1) with initial data $(u_0, v_0)$, we obtain a solution to the original system (1.1) by setting $n = \operatorname{Re} v$.

In the following sections we will study the system (3.1) and prove a well-posedness result for this system since it is slightly more convenient to iterate the reduced system (3.1) instead of (1.1) for symmetry reasons.

We call a pair of distributions $(u, n)$ a solution to (1.1) if

$$(u, n + i\langle\nabla\rangle^{-1}\partial_t n) \tag{3.2}$$

is a solution of (3.1) in the sense of the integral equation (5.10). The uniqueness class $\mathbf{X}_T$ in the statement of Theorem 1.1 can be chosen as all $(u, n)$ such that $u \in X^S_{0,\frac{1}{2},1}(T)$, $n \in X^W_{-\frac{1}{2},\frac{1}{2},1}(T)$ and $\partial_t n \in X^W_{-\frac{3}{2},\frac{1}{2},1}(T)$, see Section 2 for definitions.

We now reformulate the statement of Theorem 1.1 into a similar statement about the reduced system (3.1).

From the above relation (3.2) between $v$ and $n$ and the definitions it follows that if $v \in X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)$ is a solution to (3.1) then we have $n = \operatorname{Re} v \in X^W_{-\frac{1}{2},\frac{1}{2},1}(T)$; since $\partial_t n = \langle\nabla\rangle\operatorname{Im} v$ it also follows that $\partial_t n \in X^W_{-\frac{3}{2},\frac{1}{2},1}(T)$. Conversely, if $n \in X^W_{-\frac{1}{2},\frac{1}{2},1}(T)$ and $\partial_t n \in X^W_{-\frac{3}{2},\frac{1}{2},1}(T)$ then a straightforward computation shows that $v \in X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)$.

The above considerations allow us to claim the statement of Theorem 1.1 by a proving a similar statement about the reduced system (3.1) with initial data $(u_0, v_0) \in L^2 \times H^{-\frac{1}{2}}$. Obviously, in the context of (3.1) we adjust the definition of $\mathbf{X}_T$ to $X^S_{0,\frac{1}{2},1}(T) \times X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)$.

We finish the section with a simple remark. According to the linear part of the equation of $v$ in (3.1), the corresponding $X^{W+}_{s,b,p}$ spaces should have been defined with the weight $\tau + \langle\xi\rangle$ instead of $\tau + |\xi|$. However, a direct computation shows that the two spaces are the same. The reason behind it is that we deal with local theory $T \le 1$ and inhomogeneous norms.

In the sequel of the paper we will restrict our attention to the reduced system (3.1).

## 4. Multilinear estimates

This section is devoted to the proof of the following Theorem.

**Theorem 4.1.** *For all $0 < T \le 1$ and for all functions $u, u_1, u_2 \in X^S_{0,\frac{5}{12},1}(T)$ and $v \in X^{W+}_{-\frac{1}{2},\frac{5}{12},1}(T)$ the following estimates hold true:*

$$\|uv\|_{X^S_{0,-\frac{5}{12},\infty}(T)} \lesssim \|u\|_{X^S_{0,\frac{5}{12},1}(T)}\|v\|_{X^{W+}_{-\frac{1}{2},\frac{5}{12},1}(T)}, \tag{4.1}$$

$$\|u\bar{v}\|_{X^S_{0,-\frac{5}{12},\infty}(T)} \lesssim \|u\|_{X^S_{0,\frac{5}{12},1}(T)} \|v\|_{X^{W+}_{-\frac{1}{2},\frac{5}{12},1}(T)}, \tag{4.2}$$

$$\left\| \frac{\Delta}{\langle\nabla\rangle}(u_1\bar{u}_2) \right\|_{X^{W+}_{-\frac{1}{2},-\frac{5}{12},\infty}(T)} \lesssim \|u_1\|_{X^S_{0,\frac{5}{12},1}(T)} \|u_2\|_{X^S_{0,\frac{5}{12},1}(T)}. \tag{4.3}$$

We introduce the notation

$$I(f,g_1,g_2) = \int f(\zeta_1-\zeta_2) g_1(\zeta_1) g_2(\zeta_2) d\zeta_1 d\zeta_2,$$

where $\zeta_i = (\xi_i, \tau_i)$, $i = 1,2$. Using (2.5) and (2.6) and the fact that $\overline{\mathcal{F}u} = \mathcal{F}\overline{u}(-\cdot)$, we can reduce Theorem 4.1 to the following trilinear estimate:

**Proposition 4.2.** *For all $v, u_1, u_2 \in \mathcal{S}(\mathbb{R}^2 \times \mathbb{R})$ it holds*

$$|I(\mathcal{F}v, \mathcal{F}u_1, \mathcal{F}u_2)| \lesssim \|u_1\|_{X^S_{0,\frac{5}{12},1}} \|u_2\|_{X^S_{0,\frac{5}{12},1}} \|v\|_{X^{W\pm}_{-\frac{1}{2},\frac{5}{12},1}}. \tag{4.4}$$

The proof of Proposition 4.2 is given at the end of this section. As building blocks we provide a number of preliminary estimates first. These are concerned with functions which are dyadically localized in frequency and modulation. In some cases we additionally differentiate frequencies by their angular separation.

We start this analysis by recalling the well-known bilinear generalization of the linear $L^4$ Strichartz estimate for the Schrödinger equation due to Bourgain [5, Lemma 111], see (4.5) below. We observe that a similar estimate is true for a Wave-Schrödinger interaction.

**Proposition 4.3** (Bilinear Strichartz estimates)**.**

*(i) Let $v_1, v_2 \in L^2(\mathbb{R}^3)$ be dyadically Fourier-localized such that*

$$\operatorname{supp} \mathcal{F}v_i \subset \mathfrak{P}_{N_i} \cap \mathfrak{S}_{L_i}$$

*for $L_1, L_2 \geq 1$, $N_1, N_2 \geq 1$. Then the following estimate holds:*

$$\|v_1 v_2\|_{L^2(\mathbb{R}^3)} \lesssim \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|v_1\|_{L^2} \|v_2\|_{L^2}. \tag{4.5}$$

*(ii) Let $u, v \in L^2(\mathbb{R}^3)$ be such that*

$$\operatorname{supp} \mathcal{F}u \subset C \times \mathbb{R} \cap \mathfrak{W}^{\pm}_L, \ \operatorname{supp} \mathcal{F}v \subset \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1}$$

*for $L, L_1 \geq 1$, $N_1 \geq 1$ and a cube $C \subset \mathbb{R}^2$ of side length $d \geq 1$. Then the following estimate holds:*

$$\|uv\|_{L^2(\mathbb{R}^3)} \lesssim \left(\frac{\min\{d, N_1\}}{N_1}\right)^{\frac{1}{2}} L^{\frac{1}{2}} L_1^{\frac{1}{2}} \|u\|_{L^2} \|v\|_{L^2}. \tag{4.6}$$

*In particular, if*

$$\operatorname{supp} \mathcal{F}u \subset \mathfrak{P}_N \cap \mathfrak{W}^{\pm}_L, \ \operatorname{supp} \mathcal{F}v \subset \mathfrak{P}_{N_1} \cap \mathfrak{S}_{L_1}$$

*for $L, L_1 \geq 1$, $N, N_1 \geq 1$, it follows*

$$\|uv\|_{L^2(\mathbb{R}^3)} \lesssim \left(\frac{\min\{N, N_1\}}{N_1}\right)^{\frac{1}{2}} L^{\frac{1}{2}} L_1^{\frac{1}{2}}. \|u\|_{L^2} \|v_1\|_{L^2}. \tag{4.7}$$

*On the left hand side of (4.5), (4.7) and (4.5) we may replace each function with its complex conjugate.*

*Proof.* As remarked above the estimate (4.5) is provided by [5, Lemma 111], so it remains to show (4.6) and (4.7). With $f = \mathcal{F}u$ and $g = \mathcal{F}v$ it follows

$$\left\| \int f(\xi_1,\tau_1)g(\xi-\xi_1,\tau-\tau_1)d\xi_1 d\tau_1 \right\|_{L^2_{\xi,\tau}} \lesssim \sup_{\xi,\tau} |E(\xi,\tau)|^{\frac{1}{2}} \|f\|_{L^2}\|g\|_{L^2}$$

by the Cauchy-Schwarz inequality, where

$$E(\xi,\tau) = \{(\xi_1,\tau_1) \in \operatorname{supp} f \mid (\xi-\xi_1,\tau-\tau_1) \in \operatorname{supp} g\} \subset \mathbb{R}^3.$$

With $\underline{l} = \min\{L, L_1\}$ and $\bar{l} = \max\{L, L_1\}$ the volume of this set can be estimated as

$$|E(\xi,\tau)| \leq \underline{l} \cdot |\{\xi_1 \mid |\tau \pm |\xi_1| + |\xi-\xi_1|^2| \lesssim \bar{l}, \xi_1 \in C, |\xi-\xi_1| \sim N_1\}|,$$

by Fubini's theorem. The latter subset of $\mathbb{R}^2$ is contained in a cube of side length $m$, where $m \sim \min\{d, N_1\}$, so if $N_1 = 1$ the estimate follows. If $N_1 \geq 2$ and the first component $\xi_{1,1}$ is fixed, then the second component $\xi_{1,2}$ is confined to an interval of length $m$, and vice versa. In the subset where $|(\xi-\xi_1)_2| \gtrsim N_1$ we observe that $|\partial_{\xi_{1,2}}(\tau \pm |\xi_1| + |\xi-\xi_1|^2)| \gtrsim N_1$, and similarly in the subset where $|(\xi-\xi_1)_1| \gtrsim N_1$ we observe that $|\partial_{\xi_{1,1}}(\tau \pm |\xi_1| + |\xi-\xi_1|^2)| \gtrsim N_1$. This shows that

$$|\{\xi_1 \mid |\tau \pm |\xi_1| + |\xi-\xi_1|^2| \lesssim \bar{l}, \xi_1 \in C, |\xi-\xi_1| \sim N_1\}| \lesssim N_1^{-1}\bar{l}m,$$

and the claim (4.6) follows. This also implies the claim (4.7) because the dyadic annulus of radius $N$ is contained in a cube of side length $d \sim N$. □

Let $\angle(\xi_1,\xi_2) \in [0, \frac{\pi}{2}]$ denote the (smaller) angle between the lines spanned by $\xi_1, \xi_2 \in \mathbb{R}^2$. For dyadic numbers $64 \leq A \leq M$ we consider the following angular decomposition

$$\begin{aligned} \mathbb{R}^2 \times \mathbb{R}^2 &= \left\{ \angle(\xi_1,\xi_2) \leq \frac{16\pi}{M} \right\} \cup \bigcup_{64 \leq A \leq M} \left\{ \frac{16\pi}{A} \leq \angle(\xi_1,\xi_2) \leq \frac{32\pi}{A} \right\} \\ &= \bigcup_{\substack{0 \leq j_1, j_2 \leq M-1 \\ |j_1 - j_2| \leq 16}} \mathfrak{Q}^M_{j_1} \times \mathfrak{Q}^M_{j_2} \cup \bigcup_{64 \leq A \leq M} \bigcup_{\substack{0 \leq j_1, j_2 \leq A-1 \\ 16 \leq |j_1 - j_2| \leq 32}} \mathfrak{Q}^A_{j_1} \times \mathfrak{Q}^A_{j_2} \end{aligned} \tag{4.8}$$

Therefore, we consider for each dyadic $A \in [64, M]$ slices of angular aperture $\sim A^{-1}$ with an angular separation of size $\sim A^{-1}$, and additionally slices which are of angular aperture less than $M^{-1}$. This is a dyadic, angular Whitney type decomposition with threshold $M$.

**Proposition 4.4** (Transverse high-high interactions, low modulation)**.** *Let $f, g_1, g_2 \in L^2$ with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ and*

$$\operatorname{supp}(f) \subset \mathfrak{W}^{\pm}_L \cap \mathfrak{P}_N, \quad \operatorname{supp}(g_k) \subset \mathfrak{Q}^A_{j_k} \cap \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2).$$

*where the frequencies $N, N_1, N_2$ and modulations $L, L_1, L_2$ satisfy*

$$64 \leq N \lesssim N_1 \sim N_2, \qquad L_1, L_2, L \lesssim N_1^2$$

*while the angular localization parameters $A$ and $j_1, j_2$ satisfy*

$$64 \leq A \ll N_1, \qquad 16 \leq |j_1 - j_2| \leq 32$$

*Then the following estimate holds*

$$|I(f, g_1, g_2)| \lesssim \frac{1}{N_1^{\frac{1}{2}}} \left(\frac{A}{N_1}\right)^{\frac{1}{2}} (L_1 L_2 L)^{\frac{1}{2}}. \tag{4.9}$$

The following proof of Proposition 4.4 is based on a quantitative, nonlinear version of the classical Loomis-Whitney-inequality [17].

**Proposition 4.5** (see [2])**.** *Let $C_1, C_2, C_3$ be cubes in $\mathbb{R}^3$ of diameter $2R > 0$. Consider two paraboloids in $\mathbb{R}^3$ which are graphs of $\phi_1, \phi_2 \in C^{1,1}$ within $C_1, C_2$ and a cone in $\mathbb{R}^3$ which is a graph of $\phi_3 \in C^{1,1}$ within $C_3$, such that the homogeneous semi-norms satisfy $[\phi_j]_{C^{1,1}} \lesssim 1$. Moreover, assume that they are transversal in the sense that the determinant of every triple of unit normals to points on the surfaces within these cubes is at least of size $\theta > 0$ and suppose that $R \lesssim \theta$. Now, for given subsets $\Sigma_1, \Sigma_2, \Sigma_3$ of the above surfaces which are contained in the $\frac{1}{2}$-shrinked cubes with same center and for each $f \in L^2(\Sigma_1)$ and $g \in L^2(\Sigma_2)$ the restriction of the convolution $f * g$ to $\Sigma_3$ is a well-defined $L^2(\Sigma_3)$-function which satisfies*

$$\|f * g\|_{L^2(\Sigma_3)} \leq \frac{C}{\sqrt{\theta}} \|f\|_{L^2(\Sigma_1)} \|g\|_{L^2(\Sigma_2)}. \tag{4.10}$$

This follows from [2, Corollary 1.6]. We also refer the interested reader to the earlier paper [3] which contains a version of the aforementioned inequality in broader generality under slightly more restrictive and non-scalable assumptions. To keep the paper self-contained, we provide an independent proof of Proposition 4.4 in Appendix A which is based on elementary geometric considerations and orthogonality.

*Proof.* We abuse notation and replace $g_2$ by $g_2(-\cdot)$ and change variables $\zeta_2 \mapsto -\zeta_2$ to obtain the usual convolution structure. From now on it holds $|\tau_2 - |\xi_2|^2| \sim L_2$ within the support of $g_2$. We consider only the case $\operatorname{supp}(f) \subset \mathfrak{W}_L^-$ since in the case $\operatorname{supp}(f) \subset \mathfrak{W}_L^+$ the same arguments apply.

For fixed $\xi_1, \xi_2$ we change variables $c_1 = \tau_1 + |\xi_1|^2$, $c_2 = \tau_2 - |\xi_2|^2$. By decomposing $f$ into $\sim L$ pieces and applying the Cauchy-Schwarz inequality, it suffices to prove

$$\begin{aligned} &\left| \int g_1(\phi_{c_1}^-(\xi_1)) g_2(\phi_{c_2}^+(\xi_2)) f(\phi_{c_1}^-(\xi_1) + \phi_{c_2}^+(\xi_2)) d\xi_1 d\xi_2 \right| \\ &\lesssim \frac{A^{\frac{1}{2}}}{N_1} \|g_1 \circ \phi_{c_1}^-\|_{L^2_\xi} \|g_2 \circ \phi_{c_2}^+\|_{L^2_\xi} \|f\|_{L^2} \end{aligned} \tag{4.11}$$

where $f$ is now supported in $c \leq \tau - |\xi| \leq c + 1$ and $\phi_{c_k}^{\pm}(\xi) = (\xi, \pm|\xi|^2 + c_k)$, $k = 1, 2$, and the implicit constant is independent of $c, c_1, c_2$.

We refine the localization on $\xi$ and $\tau$ components by orthogonality methods, see also Lemma Appendix A.1. Since the support of $f$ in the $\tau$ direction is confined to an interval of length $\lesssim N_1$, $|\xi_2|^2 - |\xi_1|^2$ is localized in a specific interval of length $\sim N_1$ which in turn localizes $|\xi_2| - |\xi_1|$ in an interval of size $\sim 1$. By decomposing the plane into annuli of size $\sim 1$ and using the Cauchy-Schwarz inequality, we reduce (4.11) further to the additional assumption that the support of $g_1 \circ \phi^-_{c_1}$ and $g_2 \circ \phi^+_{c_2}$ is an interval of length $\sim 1 \lesssim N_1 A^{-1}$. Recalling the additional angular localization, we can assume that $g_1, g_2$ and $f$ are each localized in cubes of size $N_1 A^{-1}$ with respect to the $\xi$ variables.

We use the parabolic scaling $(\xi, \tau) \mapsto (N_1 \xi, N_1^2 \tau)$ to define

$$\tilde{f}(\xi,\tau) = f(N_1\xi, N_1^2\tau),\ \tilde{g}_k(\xi_k,\tau_k) = g_k(N_1\xi_k, N_1^2\tau_k),\ k=1,2.$$

If we set $\tilde{c}_k = c_k N_k^{-2}$, equation (4.11) reduces to

$$\begin{aligned} &\left| \int \tilde{g}_1(\phi^-_{\tilde{c}_1}(\xi_1))\tilde{g}_2(\phi^+_{\tilde{c}_2}(\xi_2))\tilde{f}(\phi^-_{\tilde{c}_1}(\xi_1)+\phi^+_{\tilde{c}_2}(\xi_2))d\xi_1 d\xi_2 \right| \\ \lesssim\ & \frac{A^{\frac{1}{2}}}{N_1} \|\tilde{g}_1 \circ \phi^-_{\tilde{c}_1}\|_{L^2_\xi} \|\tilde{g}_2 \circ \phi^+_{\tilde{c}_2}\|_{L^2_\xi} \|\tilde{f}\|_{L^2}, \end{aligned} \tag{4.12}$$

where now $\tilde{g}_k$ is supported in a cube of size $\sim A^{-1}$ with $|\xi_k| \sim 1$ and the supports are separated by $\sim A^{-1}$. $\tilde{f}$ is supported in a neighborhood of size $N_1^{-2}$ of the surface $S_3$ parametrized by $(\xi, \psi_{N_1}(\xi))$ for $\psi_{N_1}(\xi) = \frac{|\xi|}{N_1} + \frac{c}{N_1^2}$. Let us put $\varepsilon = N_1^{-2}$ and denote this neighborhood by $S_3(\varepsilon)$. The separation of $\xi_1$ and $\xi_2$ above implies also that in the support of $\tilde{f}$ we have $|\xi| \gtrsim A^{-1} \geq N_1^{-1}$.

By density and duality it is enough to consider continuous $\tilde{g}_1, \tilde{g}_2$ and we can further rewrite the above estimate as

$$\|\tilde{g}_1|_{S_1} * \tilde{g}_2|_{S_2}\|_{L^2(S_3(\varepsilon))} \lesssim A^{\frac{1}{2}}\varepsilon^{\frac{1}{2}} \|\tilde{g}_1\|_{L^2(S_1)} \|\tilde{g}_2\|_{L^2(S_2)} \tag{4.13}$$

where $S_i$, $i = 1, 2$ are parametrized by $\phi^\pm_{\tilde{c}_i}$. The above localization properties of the support of $\tilde{g}_i$ are inherited by $S_i$, which implies that the maximal diameter of the $S_1$, $S_2$ and $S_3$ is at most $R \sim A^{-1}$. Obviously, the parametrizations of the paraboloids $S_1$ and $S_2$ have $C^{1,1}$ semi-norm $\sim 1$. Concerning $S_3$ we estimate

$$|\nabla\psi_{N_1}(\xi) - \nabla\psi_{N_1}(\eta)| \lesssim N_1^{-1} \left|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\right| \lesssim |\xi - \eta|$$

where we have used that $|\xi|, |\eta| \geq N_1^{-1}$ in the base of $S_3$. Therefore, the $C^{1,1}$ semi-norm for our parametrization of $S_3$ is $\lesssim 1$.

Finally, we need to analyze the transversality properties of our surfaces. In other words, we need to determine a uniform lower bound $\theta$ on the size of the determinant $d$ of the matrix of three unit normal vector fields. Intuitively it is clear that – since the parabolically rescaled cone is almost flat – this is determined by the minimal angular separation $\sim A^{-1}$ between the $\xi$-supports of $g_1$ and $g_2$. In fact, we will show that $\theta \gtrsim A^{-1}$ below. In summary, we have $R \lesssim \theta$ and we invoke (4.10) to obtain (4.13).

Let us carefully verify the transversality condition $\theta \gtrsim A^{-1}$ indicated above: The determinant of any three unit normals to $S_1$, $S_2$, and $S_3$ is given by

$$d = \begin{vmatrix} \frac{2\xi_1}{\langle 2\xi\rangle} & \frac{2\eta_1}{\langle 2\eta\rangle} & \frac{\zeta_1}{|\zeta|\langle N_1\rangle} \\ \frac{2\xi_2}{\langle 2\xi\rangle} & \frac{2\eta_2}{\langle 2\eta\rangle} & \frac{\zeta_2}{|\zeta|\langle N_1\rangle} \\ \frac{1}{\langle 2\xi\rangle} & -\frac{1}{\langle 2\eta\rangle} & \frac{N_1}{\langle N_1\rangle} \end{vmatrix}$$

which we expand as $d = d_1 + d_2 + d_3$, with main contribution

$$d_1 = \frac{N_1}{\langle N_1\rangle}\begin{vmatrix} \frac{2\xi_1}{\langle 2\xi\rangle} & \frac{2\eta_1}{\langle 2\eta\rangle} \\ \frac{2\xi_2}{\langle 2\xi\rangle} & \frac{2\eta_2}{\langle 2\eta\rangle} \end{vmatrix}$$

and the error terms

$$d_2 = -\frac{\zeta_2}{|\zeta|\langle N_1\rangle}\begin{vmatrix} \frac{2\xi_1}{\langle 2\xi\rangle} & \frac{2\eta_1}{\langle 2\eta\rangle} \\ \frac{1}{\langle 2\xi\rangle} & -\frac{1}{\langle 2\eta\rangle} \end{vmatrix}, \quad d_3 = \frac{\zeta_1}{|\zeta|\langle N_1\rangle}\begin{vmatrix} \frac{2\xi_2}{\langle 2\xi\rangle} & \frac{2\eta_2}{\langle 2\eta\rangle} \\ \frac{1}{\langle 2\xi\rangle} & -\frac{1}{\langle 2\eta\rangle} \end{vmatrix}.$$

The contribution of the last two terms $d_2$ and $d_3$ is bounded by

$$|d_2| + |d_3| \lesssim \frac{|\zeta_1| + |\zeta_2|}{|\zeta|\langle N_1\rangle} \lesssim N_1^{-1}$$

The first determinant $d_1$ can be rewritten as

$$d_1 = \frac{N_1}{\langle N_1\rangle}\frac{2|\xi|}{\langle 2\xi\rangle}\frac{2|\eta|}{\langle 2\eta\rangle}\begin{vmatrix} \frac{\xi_1}{|\xi|} & \frac{\eta_1}{|\eta|} \\ \frac{\xi_2}{|\xi|} & \frac{\eta_2}{|\eta|} \end{vmatrix} = \frac{N_1}{\langle N_1\rangle}\frac{2|\xi|}{\langle 2\xi\rangle}\frac{2|\eta|}{\langle 2\eta\rangle}\sin\angle\left(\frac{\xi}{|\xi|}, \frac{\eta}{|\eta|}\right)$$

Recalling that $|\xi|, |\eta| \sim 1$ (since they are in the support of $g_1$, respectively $g_2$), it follows that $\frac{N_1}{\langle N_1\rangle}\frac{2|\xi|}{\langle 2\xi\rangle}\frac{2|\eta|}{\langle 2\eta\rangle} \gtrsim 1$. By the angular separation between $S_1$ and $S_2$ we obtain $|d_1| \gtrsim A^{-1}$ and by recalling that $A \gg N_1$ it follows that $|d| \gtrsim A^{-1}$. $\square$

In the case where the maximal modulation is high a different bound will be favourable.

**Proposition 4.6** (Transverse high-high interactions, high modulation)**.** *Let $f, g_1, g_2 \in L^2$, $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ such that*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \quad \operatorname{supp}(g_k) \subset \mathfrak{Q}_{j_k}^A \cap \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2),$$

*with $64 \le N \lesssim N_1 \sim N_2$ and $64 \le A \le N_1$. Moreover, assume that $16 \le |j_1 - j_2| \le 32$. Then*

$$|I(f, g_1, g_2)| \lesssim \frac{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} L^{\frac{1}{2}} N^{-\frac{1}{2}}}{\max\{L, L_1, L_2\}^{\frac{1}{2}}}\left(\frac{N_1}{A}\right)^{\frac{1}{2}} \tag{4.14}$$

*Remark* 4. The estimate (4.14) gives a better bound than (4.9) in the case where

$$\max\{L, L_1, L_2\} \ge \left(\frac{N_1}{A}\right)^2 \frac{N_1}{N}. \tag{4.15}$$

*Proof of Proposition 4.6.* After a rotation we may assume that $j_1 = 0$. Due to the localization of the wedges we observe that the integral vanishes unless $N \gtrsim N_1 A^{-1}$, since $|\xi_{2,2} - \xi_{1,2}| \sim N_1 A^{-1}$. We consider two cases:

(i) $N \sim N_1 A^{-1}$

(ii) $N \gg N_1 A^{-1}$.

In case (i) we start with the subcase where $\max\{L, L_1, L_2\} = L$. From the bilinear Strichartz estimate for the Schrödinger equation (4.5), using $N \sim A^{-1}N_1$, we obtain

$$|I(f, g_1, g_2)| \lesssim (L_1L_2)^{\frac{1}{2}} \|f\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2}.$$

The subcases where $\max\{L, L_1, L_2\} = L_i$ for $i = 1, 2$ follow in the same way by using (4.7) instead of (4.5).

In Case (ii) we also start with the subcase where $\max\{L, L_1, L_2\} = L$. Without any restriction in generality assume also that $L_1 \le L_2$. Denoting

$$\chi = 1_{\mathfrak{Q}^A_{j_1}\cap\mathfrak{P}_{N_1}\cap\mathfrak{S}_{L_1}} 1_{\mathfrak{Q}^A_{j_2}\cap\mathfrak{P}_{N_2}\cap\mathfrak{S}_{L_2}}$$

we use Cauchy-Schwarz to estimate

$$\begin{aligned}\left|\int f(\zeta_1-\zeta_2)g_1(\zeta_1)g_2(\zeta_2)d\zeta_1 d\zeta_2\right| &\lesssim \|\chi f(\zeta_1-\zeta_2)\|_{L^2}\|g_1(\zeta_1)g_2(\zeta_2)\|_{L^2}\\ &\lesssim \sup_{\zeta_0\in\mathfrak{P}_N\cap\mathfrak{W}^\pm_L} |B(\zeta_0)|^{\frac{1}{2}}\|f\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2}\end{aligned}$$

where

$$B(\zeta_0) = \{\zeta_1 \mid \zeta_1 \in \mathfrak{Q}^A_{j_1}\cap\mathfrak{P}_{N_1}\cap\mathfrak{S}_{L_1}; \zeta_1-\zeta_0 \in \mathfrak{Q}^A_{j_2}\cap\mathfrak{P}_{N_2}\cap\mathfrak{S}_{L_2}\}.$$

To bound the size of the set $B(\zeta_0)$ we observe that for $\zeta_0 = (\xi_0, \tau_0)$ and $\zeta_1 = (\xi_1, \tau_1)$ as above we must have $|\xi_{0,1}| \sim N$ and

$$|\tau_1 - \xi_1^2| \lesssim L_1, \qquad |\xi_{1,2}| \lesssim \frac{N_1}{A}, \qquad |\tau_1 - \tau_0 + |\xi_1 - \xi_0|^2| \sim L_2.$$

Since $\partial_{\xi_{1,1}}(|\xi_1|^2 - |\xi_1-\xi_0|^2) = 2\xi_{0,1}$ which has size $N$, it follows that

$$|B(\xi_0, \tau_0)| \lesssim L_1\frac{L_2}{N}\frac{N_1}{A} \tag{4.16}$$

and the conclusion of the Proposition follows.

Let us now assume that $\max\{L, L_1, L_2\} = L_1$; the subcase when $\max\{L, L_1, L_2\} = L_2$ is similar. Using Cauchy-Schwarz as above we obtain

$$|I(f, g_1, g_2)| \lesssim \sup_{\zeta_1\in\mathfrak{Q}^A_{j_1}\cap\mathfrak{P}_{N_1}\cap\mathfrak{S}_{L_1}} |C(\zeta_1)|^{\frac{1}{2}}\|f\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2},$$

where

$$C(\zeta_1) = \{\zeta_2 \mid \zeta_2 \in \mathfrak{Q}^A_{j_2}\cap\mathfrak{P}_{N_2}\cap\mathfrak{S}_{L_2}; \zeta_1-\zeta_2 \in \mathfrak{P}_N\cap\mathfrak{W}^\pm_L\}.$$

Setting $\underline{l} = \min\{L, L_2\}$ and $\bar{l} = \max\{L, L_2\}$, we observe that given $\xi_2$, $\tau_2$ can only range in an interval of size $\lesssim \underline{l}$. On the other hand, for $\xi_2$ we have the restrictions

$$|\xi_{2,2}| \lesssim \frac{N_1}{A}, \qquad |\tau_1 + |\xi_2|^2 \pm |\xi_1-\xi_2|| \lesssim \bar{l}.$$

Since $|\partial_{\xi_{2,1}}(|\xi_2|^2 \pm |\xi_1-\xi_2|)| = 2|\xi_{2,1}| \gtrsim N_1$, we obtain

$$|C(\zeta_1)| \lesssim \underline{l}\frac{\bar{l}}{N_1}\frac{N_1}{A} = \frac{LL_2}{A} \tag{4.17}$$

again concluding the proof of the Proposition. □

Next, we consider the case where the frequencies $\xi_1$ and $\xi_2$ are almost parallel. This can be viewed as an almost one-dimensional interaction.

**Proposition 4.7** (Parallel high-high interactions)**.** *Let $f, g_1, g_2 \in L^2$, $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ such that*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \quad \operatorname{supp}(g_k) \subset \mathfrak{Q}_{j_k}^A \cap \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k=1,2),$$

*with $1 \ll N \lesssim N_1 \sim N_2$. Assume that $A \sim N_1$ and $|j_1 - j_2| \leq 16$. Then for all $L, L_1, L_2 \geq 1$ we have*

$$|I(f, g_1, g_2)| \lesssim L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} \frac{1}{N^{\frac{1}{2}}} \left(\frac{N}{N_1}\right)^{\frac{1}{4}} \tag{4.18}$$

*Proof.* After a rotation we may assume that $j_1 = 0$. Due to the localization of the wedges we observe that $|\xi_{0,2}|, |\xi_{1,2}|, |\xi_{2,2}| \lesssim 1$. This shows that $|\xi_{1,1} - \xi_{2,1}| = |\xi_{0,1}| \sim N$, $|\xi_{1,1}|, |\xi_{2,1}| \sim N_1$. In addition, we must have

$$\left||\xi_1 - \xi_2| \pm (|\xi_1|^2 - |\xi_2|^2)\right| \lesssim \max\{L, L_1, L_2\}$$

If $N \ll N_1$ then the above left hand side must have size $NN_1$. Thus we have established the following dichotomy:

$$\text{either } N \sim N_1 \;\text{ or }\; NN_1 \lesssim \max\{L, L_1, L_2\}. \tag{4.19}$$

Then we can use the same argument as in Case (ii) of the proof of Proposition 4.6.

If $L = \max\{L_1, L_2, L\}$ then the bound (4.16) holds, and corresponding to the two cases in (4.19) we only need to compute

$$L_1 L_2 \frac{1}{N} \frac{N_1}{A} = L_1 L_2 \frac{1}{N} \lesssim L_1^{\frac{2}{3}} L_2^{\frac{2}{3}} L^{\frac{2}{3}} \frac{1}{N} \frac{N}{N_1}$$

respectively

$$L_1 L_2 \frac{1}{N} \frac{N_1}{A} = L_1^{\frac{5}{6}} L_2^{\frac{5}{6}} L^{\frac{1}{3}} \frac{1}{N} \lesssim L_1^{\frac{5}{6}} L_2^{\frac{5}{6}} L^{\frac{5}{6}} \frac{1}{N} \frac{1}{(NN_1)^{\frac{1}{2}}}$$

both of which are stronger than needed.

On the other hand if $L_1 = \max\{L, L_1, L_2\}$ then (4.17) holds, and we conclude as above taking into account the two cases in (4.19). The case $L_2 = \max\{L, L_1, L_2\}$ is similar. □

The next proposition covers the case of high-low interactions.

**Proposition 4.8** (high-low interactions)**.** *Let $f, g_1, g_2 \in L^2$ be functions with $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ such that*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \quad \operatorname{supp}(g_k) \subset \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k=1,2),$$

*with $1 \leq N_1 \ll N_2$ or $1 \leq N_2 \ll N_1$. Then, for all $L, L_1, L_2 \geq 1$ we have*

$$|I(f, g_1, g_2)| \lesssim L_1^{\frac{5}{12}} L_2^{\frac{5}{12}} L^{\frac{5}{12}} N^{-\frac{1}{2}} \min\left\{\frac{N_1}{N_2}, \frac{N_2}{N_1}\right\}^{\frac{1}{6}} \tag{4.20}$$

*Proof.* Assume first that $N_1 \ll N_2$. Then, the integral vanishes unless $N_2 \sim N$ and

$$\max\{L, L_1, L_2\} \gtrsim ||\xi_1|^2 - |\xi_2|^2 \pm |\xi_1 - \xi_2|| \gtrsim N_2^2. \tag{4.21}$$

We consider three cases:

*Case 1:* $L = \max\{L, L_1, L_2\}$. Then by the bilinear Strichartz estimate (4.5) we have

$$|I(f, g_1, g_2)| \lesssim \|f\|_{L^2} \|\mathcal{F}^{-1} g_1 \overline{\mathcal{F}^{-1} g_2}\|_{L^2} \lesssim L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}}$$

Then the claim follows due to (4.21).

*Case 2:* $L_1 = \max\{L, L_1, L_2\}$. Since $g_1$ is localized in frequency in a cube of size $N_1$, by orthogonality the estimate reduces to the case when $f$ and $g_2$ are frequency localized in cubes of size $N_1$. Then we use bilinear $L^2$ estimate (4.6) with $d = N_1$ to obtain

$$|I(f, g_1, g_2)| \lesssim \|g_1\|_{L^2} \|\mathcal{F}^{-1} f \mathcal{F}^{-1} g_2\|_{L^2} \lesssim L^{\frac{1}{2}} L_2^{\frac{1}{2}} \left(\frac{N_1}{N_2}\right)^{\frac{1}{2}}$$

and conclude again using (4.21).

*Case 3:* $L_2 = \max\{L, L_1, L_2\}$. On one hand, by (4.7) we obtain the bound

$$|I(f, g_1, g_2)| \lesssim \|g_2\|_{L^2} \|\overline{\mathcal{F}^{-1} f} \mathcal{F}^{-1} g_1\|_{L^2} \lesssim L^{\frac{1}{2}} L_1^{\frac{1}{2}}$$

which implies (4.20) if additionally $L_1 \leq N_1^2$ holds.

On the other hand, by Young's inequality we have

$$|I(f, g_1, g_2)| \leq \|g_2\|_{L^2} \|f\|_{L^2_\xi L^1_\tau} \|g_1\|_{L^1_\xi L^2_\tau} \lesssim L^{\frac{1}{2}} N_1.$$

which, combined with (4.21), suffices in the elliptic regime $L_1 > N_1^2$.

The case $N_1 \gg N_2$ follows by the same arguments. □

Finally, we deal with the case where the wave frequency is very small.

**Proposition 4.9** (Very small wave frequency)**.** *Let* $f, g_1, g_2 \in L^2$ *with* $\|f\|_{L^2} = \|g_1\|_{L^2} = \|g_2\|_{L^2} = 1$ *such that*

$$\operatorname{supp}(f) \subset \mathfrak{P}_N \cap \mathfrak{W}_L^{\pm}, \quad \operatorname{supp}(g_k) \subset \mathfrak{P}_{N_k} \cap \mathfrak{S}_{L_k} \quad (k = 1, 2),$$

*and assume that* $N \lesssim 1$*. Then,*

$$|I(f, g_1, g_2)| \lesssim L^{\frac{1}{3}} L_1^{\frac{1}{3}} L_2^{\frac{1}{3}}. \tag{4.22}$$

*Proof.* Depending on which of $L, L_1, L_2$ is maximal we apply the bilinear Strichartz refinements (4.5) or (4.7) and the result follows. □

We are ready to provide a proof of our main trilinear estimate (4.4).

*Proof of Proposition 4.2.* By definition of the norms it is enough to consider functions with non-negative Fourier transform. We dyadically decompose

$$u_i = \sum_{N_i, L_i \geq 1} S_{L_i} P_{N_i} u_i \ , \quad v = \sum_{N, L \geq 1} W_L^{\pm} P_N v.$$

Setting $g_i^{L_i,N_i} = \mathcal{F} S_{L_i} P_{N_i} u_i$ and $f^{L,N} = \mathcal{F} W_L^{\pm} P_N v$, we observe

$$I(\mathcal{F}v, \mathcal{F}u_1, \mathcal{F}u_2) = \sum_{N,N_1,N_2 \geq 1} \sum_{L,L_1,L_2 \geq 1} I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}).$$

*Case 1:* high-high-low interactions, i.e. $N_1 \sim N_2 \gtrsim N \geq 2^{10}$.
We fix $M = 2^{-4} N_1$ and use the decomposition (4.8) to write

$$\begin{aligned} I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2}) = & \sum_{\substack{0 \leq j_1, j_2 \leq M-1 \\ |j_1 - j_2| \leq 16}} I(f^{L,N}, g_1^{L_1,N_1,M,j_1}, g_2^{L_2,N_2,M,j_2}) \\ & + \sum_{64 \leq A \leq M} \sum_{\substack{0 \leq j_1, j_2 \leq A-1 \\ 16 \leq |j_1 - j_2| \leq 32}} I(f^{L,N}, g_1^{L_1,N_1,A,j_1}, g_2^{L_2,N_2,A,j_2}) \end{aligned}$$

where $g_i^{L_i,N_i,A,j_i} = g_i^{L_i,N_i}|_{\mathfrak{Q}_{j_i}^A}$. We apply Proposition 4.7 to the first term and use Cauchy-Schwarz to obtain

$$\begin{aligned} & \sum_{\substack{0 \leq j_1, j_2 \leq M-1 \\ |j_1 - j_2| \leq 16}} I(f^{L,N}, g_1^{L_1,N_1,M,j_1}, g_2^{L_2,N_2,M,j_2}) \\ \lesssim & \frac{(LL_1L_2)^{\frac{5}{12}}}{N^{\frac{1}{2}}} \left(\frac{N}{N_1}\right)^{\frac{1}{4}} \|f^{L,N}\|_{L^2} \sum_{\substack{0 \leq j_1, j_2 \leq M-1 \\ |j_1 - j_2| \leq 16}} \|g_1^{L_1,N_1,M,j_1}\|_{L^2} \|g_2^{L_2,N_2,M,j_2}\|_{L^2} \\ \lesssim & \frac{(LL_1L_2)^{\frac{5}{12}}}{N^{\frac{1}{2}}} \left(\frac{N}{N_1}\right)^{\frac{1}{4}} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}. \end{aligned}$$

Concerning the second term, we split the sum with respect to $A$ into two parts according to the quantity

$$\alpha := 2^{-4} \min \left\{ \left(\frac{N_1}{N}\right)^{\frac{1}{2}} N_1 \max\{L, L_1, L_2\}^{-\frac{1}{2}}, N_1 \right\}.$$

For the part where $64 \leq A \leq \alpha$ we apply Proposition 4.4 and obtain

$$\begin{aligned} S_1 := & \sum_{64 \leq A \leq \alpha} \sum_{\substack{0 \leq j_1, j_2 \leq A-1 \\ 16 \leq |j_1 - j_2| \leq 32}} I(f^{L,N}, g_1^{L_1,N_1,A,j_1}, g_2^{L_2,N_2,A,j_2}) \\ \lesssim & \left(\frac{LL_1L_2}{N_1}\right)^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \sum_{64 \leq A \leq \alpha} \frac{A^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \sum_{\substack{0 \leq j_1, j_2 \leq A-1 \\ 16 \leq |j_1 - j_2| \leq 32}} \|g_1^{L_1,N_1,A,j_1}\|_{L^2} \|g_2^{L_2,N_2,A,j_2}\|_{L^2}. \end{aligned}$$

Then, we use Cauchy-Schwarz with respect to $j_1, j_2$

$$S_1 \lesssim \left(\frac{LL_1L_2}{N_1}\right)^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2} \sum_{64 \leq A \leq \alpha} \frac{A^{\frac{1}{2}}}{N_1^{\frac{1}{2}}}$$

$$\lesssim (LL_1L_2)^{\frac{5}{12}} N^{-\frac{1}{2}} \left(\frac{N}{N_1}\right)^{\frac{1}{4}} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2},$$

due to the property of the dyadic sum $\sum_{64 \leq A \leq \alpha} A^{\frac{1}{2}} \lesssim \alpha^{\frac{1}{2}}$.

For the part where $\alpha \leq A \leq N_1$ we use Proposition 4.6 and obtain

$$S_2 := \sum_{\alpha \leq A \leq M} \sum_{\substack{0 \leq j_1, j_2 \leq A-1 \\ 16 \leq |j_1 - j_2| \leq 32}} I(f^{L,N}, g_1^{L_1,N_1,A,j_1}, g_2^{L_2,N_2,A,j_2})$$

$$\lesssim \frac{L^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|f^{L,N}\|_{L^2}}{\max\{L_1, L_2, L\}^{\frac{1}{2}} N^{\frac{1}{2}}} \sum_{\alpha \leq A \leq M} \frac{N_1^{\frac{1}{2}}}{A^{\frac{1}{2}}} \sum_{\substack{0 \leq j_1, j_2 \leq A-1 \\ 16 \leq |j_1 - j_2| \leq 32}} \|g_1^{L_1,N_1,A,j_1}\|_{L^2} \|g_2^{L_2,N_2,A,j_2}\|_{L^2}.$$

As above, we use Cauchy-Schwarz with respect to $j_1, j_2$ and obtain

$$S_2 \lesssim \frac{L^{\frac{1}{2}} L_1^{\frac{1}{2}} L_2^{\frac{1}{2}} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2}}{\max\{L_1, L_2, L\}^{\frac{1}{2}} N^{\frac{1}{2}}} \sum_{\alpha \leq A \leq M} \frac{N_1^{\frac{1}{2}}}{A^{\frac{1}{2}}}$$

$$\lesssim \frac{(LL_1L_2)^{\frac{5}{12}}}{N^{1/2}} \left(\frac{N}{N_1}\right)^{\frac{1}{4}} \|f^{L,N}\|_{L^2} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2},$$

because of $\sum_{\alpha \leq A \leq M} A^{-\frac{1}{2}} \lesssim \alpha^{-\frac{1}{2}}$.

*Case 2*: very small wave frequency, i.e. $N \lesssim 1$. In this case, either $N_1 \sim N_2$ or $N, N_1, N_2 \lesssim 1$ and we apply Proposition 4.9 and arrive at the bound (4.22)

*Case 3*: high-low interactions, i.e. $N_1 \ll N_2$ or $N_1 \gg N_2$. We apply Proposition 4.8 and obtain the bound (4.20).

To summarize, we obtain in any case the weakest of all three bounds, namely

$$I(f^{L,N}, g_1^{L_1,N_1}, g_2^{L_2,N_2})$$
$$\lesssim (LL_1L_2)^{\frac{5}{12}} \min\left\{\frac{N}{N_1}, \frac{N_1}{N_2}, \frac{N_2}{N_1}\right\}^{\frac{1}{6}} \frac{\|f^{L,N}\|_{L^2}}{N^{\frac{1}{2}}} \|g_1^{L_1,N_1}\|_{L^2} \|g_2^{L_2,N_2}\|_{L^2},$$

which we dyadically sum with respect to $L, L_1, L_2 \geq 1$. Then, we use that for non-vanishing contributions we must have $N \lesssim N_1 \sim N_2$ or $N_1 \lesssim N \sim N_2$ or $N_2 \lesssim N$ and the prefactor enables us to control the sum by the corresponding dyadic $\ell^2$-norms. □

## 5. Linear estimates and the proof of Theorem 1.1

Before we prove Theorem 1.1 we present some linear estimates which are well-known at least in the case of standard Bourgain spaces, see e.g. [12, Section 2].

We define the 1d inhomogeneous Besov norms

$$\|g\|_{B^b_{2,1}} = \sum_{L \geq 1} L^b \|P_L g\|_{L^2}, \quad \|g\|_{B^b_{2,\infty}} = \sup_{L \geq 1} L^b \|P_L g\|_{L^2}.$$

For $0 < T \leq 1$ we define a smooth cutoff function for the interval $[0,T]$ as $\psi_T(t) = \psi(t/T)$ and we define the Fourier localization operator $P_{\leq T^{-1}} := \sum_{1 \leq L \leq T^{-1}} P_L$, cp. Section 2.

**Lemma 5.1.** *Let $0 < b \leq \frac{1}{2}$. For all $g \in \mathcal{S}(\mathbb{R})$ and $T \in (0,1]$ we have*

$$\|g\psi_T\|_{B^b_{2,1}} \sim T^{-b}\|P_{\leq T^{-1}}(g\psi_T)\|_{L^2} + \sum_{L>T^{-1}} L^b \|P_L(g\psi_T)\|_{L^2}, \tag{5.1}$$

*where the implicit constants are independent of $T$ and $g$.*

*Proof.* On the one hand we have

$$\begin{aligned}\sum_{1\leq L\leq T^{-1}} L^b\|P_L(g\psi_T)\|_{L^2} &\leq 2\left(\sum_{1\leq L\leq T^{-1}} L^{2b}\right)^{\frac{1}{2}} \|P_{\leq T^{-1}}(g\psi_T)\|_{L^2}\\ &\lesssim T^{-b}\|P_{\leq T^{-1}}(g\psi_T)\|_{L^2},\end{aligned}$$

and on the other hand

$$T^{-b}\|P_{\leq T^{-1}}(g\psi_T)\|_{L^2} \leq T^{-b}\|g\psi_T\|_{L^2} \leq \|g\psi_T\|_{L^{\frac{2}{1-2b}}} \lesssim \|g\psi_T\|_{B^b_{2,1}}$$

where we have used the embedding $B^b_{2,1} \subset L^{\frac{2}{1-2b}}$ in the last step. □

In the following, let $X_{s,b,p}(T)$ denote either $X^S_{s,b,p}(T)$ or $X^{W\pm}_{s,b,p}(T)$.

**Proposition 5.2.** *Let $s, b \in \mathbb{R}$, $0 < b < \frac{1}{2}$. There exists a constant $C > 0$ such that for all $T \in (0,1]$ the estimate*

$$\|f\|_{X_{s,b,1}(T)} \leq CT^{\frac{1}{2}-b}\|f\|_{X_{s,\frac{1}{2},1}(T)} \tag{5.2}$$

*holds for all $f \in X_{s,\frac{1}{2},1}(T)$. Moreover, the embedding $X_{s,\frac{1}{2},1}(T) \subset C([0,T];H^s)$ is continuous, i.e. there exists a constant $C > 0$ such that for all $T \in (0,1]$ it holds*

$$\sup_{0\leq t\leq T}\|f(t)\|_{H^s} \leq C\|f\|_{X_{s,\frac{1}{2},1}(T)} \tag{5.3}$$

*for all $f \in X_{s,\frac{1}{2},1}(T)$.*

*Proof.* We show (5.2) first. By the definition of the restriction norm it suffices to prove

$$\|f\psi_T\|_{X_{s,b,1}} \leq CT^{\frac{1}{2}-b}\|f\|_{X_{s,\frac{1}{2},1}}$$

for all $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R})$. After conjugating $f$ with the linear group the claim is reduced to the estimate

$$\|g\psi_T\|_{B^b_{2,1}} \leq CT^{\frac{1}{2}-b}\|g\|_{B^{\frac{1}{2}}_{2,1}}$$

for $g \in \mathcal{S}(\mathbb{R})$. Then, with $g_T = g(\cdot T)$ we use (5.1) and obtain

$$\begin{aligned}\|g\psi_T\|_{B^b_{2,1}} &\lesssim T^{-b}\|P_{\leq T^{-1}}(g\psi_T)\|_{L^2} + \sum_{L>T^{-1}} L^b\|P_L(g\psi_T)\|_{L^2}\\ &\lesssim T^{\frac{1}{2}-b}\sum_{L\geq 1} L^b\|P_L(g_T\psi)\|_{L^2}\\ &\lesssim T^{\frac{1}{2}-b}(\|g_T\psi\|_{L^2} + \|g_T\psi\|_{\dot{H}^{\frac{1}{2}}})\end{aligned}$$

by rescaling. Obviously,

$$\|g_T\psi\|_{L^2} \leq \|g_T\|_{L^\infty} \lesssim \|g\|_{B^{\frac{1}{2}}_{2,1}}$$

and by the 1d Sobolev Multiplication Theorem

$$\|g_T\psi\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|g_T\|_{\dot{H}^{\frac{1}{2}}} + \|g_T\|_{L^\infty} \lesssim \|g\|_{B^{\frac{1}{2}}_{2,1}}.$$

The second claim, including formula (5.3), follows from the continuous embedding $B^{\frac{1}{2}}_{2,1} \subset C(\mathbb{R};\mathbb{R})$. □

For $f \in \mathcal{S}(\mathbb{R}^2 \times \mathbb{R})$ and $t \in \mathbb{R}$ let

$$\mathcal{I}^S(f)(t) \quad := \int_0^t e^{i(t-s)\Delta} f(s)ds, \tag{5.4}$$

$$\mathcal{I}^{W+}(f)(t) := \int_0^t e^{-i(t-s)\langle\nabla\rangle} f(s)ds. \tag{5.5}$$

The following Proposition corresponds to [12, Lemma 2.1].

**Proposition 5.3.** *Let $s \in \mathbb{R}$. There exists $C > 0$ such that for all $0 < T \leq 1$ and $\phi \in H^s$ the estimates*

$$\|e^{it\Delta}\phi\|_{X^S_{s,\frac{1}{2},1}(T)} \quad \leq C\|\phi\|_{H^s}, \tag{5.6}$$

$$\|e^{-it\langle\nabla\rangle}\phi\|_{X^{W+}_{s,\frac{1}{2},1}(T)} \leq C\|\phi\|_{H^s}, \tag{5.7}$$

*are true, and moreover for sufficiently smooth $f$ the estimates*

$$\|\mathcal{I}^S(f)\|_{X^S_{s,\frac{1}{2},1}(T)} \quad \leq CT^{\frac{1}{12}}\|f\|_{X^S_{s,-\frac{5}{12},\infty}(T)}, \tag{5.8}$$

$$\|\mathcal{I}^{W+}(f)\|_{X^{W+}_{s,\frac{1}{2},1}(T)} \leq CT^{\frac{1}{12}}\|f\|_{X^{W+}_{s,-\frac{5}{12},\infty}(T)}, \tag{5.9}$$

*are true. Therefore, $\mathcal{I}^S$ and $\mathcal{I}^{W+}$ can be extended to continuous linear operators on these spaces, which satisfy the same bounds.*

*Proof.* We use the notation as in Proposition 5.2 above.

First, (5.6) and (5.7) are proved as in [12, equation (2.19)] upon replacing the Sobolev space $H^{\frac{1}{2}}_t$ by the Besov space $B^{\frac{1}{2}}_{2,1}$.

Second, by choosing appropriate extensions and conjugating with the linear group, the estimates (5.8) and (5.9) easily reduce to the estimate

$$\|\psi_T\mathcal{I}(g)\|_{B^{\frac{1}{2}}_{2,1}} \leq CT^{\frac{1}{12}}\|g\|_{B^{-\frac{5}{12}}_{2,\infty}}$$

for all $g \in \mathcal{S}(\mathbb{R})$, $T \in (0,1]$, where $\mathcal{I}(g) = \int_0^t g(t')dt'$. With $g_T(t) = g(Tt)$ we calculate

$$(\psi_T I(g))(Tt) = T\psi(t)I(g_T)(t).$$

Now, (5.1) and rescaling yields

$$\|\psi_T\mathcal{I}(g)\|_{B^{\frac{1}{2}}_{2,1}} \leq CT\|\psi I(g_T)\|_{B^{\frac{1}{2}}_{2,1}}.$$

From estimate [12, formula (2.24)] with $T = 1$ and trivial embeddings we deduce

$$\|\psi I(g_T)\|_{B^{\frac{1}{2}}_{2,1}} \leq C\|g_T\|_{B^{-\frac{5}{12}}_{2,\infty}}.$$

Finally, rescaling shows that

$$\|g_T\|_{B^{-\frac{5}{12}}_{2,\infty}} \leq CT^{-\frac{11}{12}}\|g\|_{B^{-\frac{5}{12}}_{2,\infty}}$$

for all $0 < T \leq 1$, which concludes the proof. $\square$

**Definition 5.4.** We call $(u, v) \in X^S_{s,\frac{1}{2},1}(T) \times X^{W+}_{s',\frac{1}{2},1}(T)$ a solution of (3.1) with initial data $(u_0, v_0) \in H^s \times H^{s'}$, if it solves

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} e^{it\Delta}u_0 \\ e^{-it\langle\nabla\rangle}v_0 \end{pmatrix} - i \begin{pmatrix} \mathcal{I}^S(2\mathrm{Re}\,(v)u)(t) \\ \mathcal{I}^{W+}(-\frac{\Delta}{\langle\nabla\rangle}|u|^2 - \frac{1}{\langle\nabla\rangle}\mathrm{Re}\,v)(t) \end{pmatrix} \tag{5.10}$$

for all $t \in [0, T]$.

Now we are ready to proceed with the proof of our main result.

*Proof of Theorem 1.1.* Let $R = \|u_0\|_{L^2} + \|v_0\|_{H^{-\frac{1}{2}}}$. Since the time of existence claimed in Theorem 1.1 is smaller than 1 it is enough to discuss only the case $1 \lesssim R$.

The estimates (5.8) and (4.1), (4.2) yield

$$\begin{aligned}\|\mathcal{I}^S(2\mathrm{Re}\,(v)u)\|_{X^S_{0,\frac{1}{2},1}(T)} &\lesssim T^{\frac{1}{12}}(\|uv\|_{X^S_{0,-\frac{5}{12},\infty}(T)} + \|u\bar{v}\|_{X^S_{0,-\frac{5}{12},\infty}(T)}) \\ &\lesssim T^{\frac{1}{12}}\|u\|_{X^S_{0,\frac{5}{12},1}(T)}\|v\|_{X^{W+}_{-\frac{1}{2},\frac{5}{12},1}(T)},\end{aligned}$$

and (5.2) implies

$$\|\mathcal{I}^S(2\mathrm{Re}\,(v)u)\|_{X^S_{0,\frac{1}{2},1}(T)} \lesssim T^{\frac{1}{4}}\|u\|_{X^S_{0,\frac{1}{2},1}(T)}\|v\|_{X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)}. \tag{5.11}$$

In a similar manner, using (4.3), we estimate

$$\begin{aligned}\left\|\mathcal{I}^{W+}(\frac{\Delta}{\langle\nabla\rangle}|u|^2)\right\|_{X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)} &\lesssim T^{\frac{1}{12}}\left\|\frac{\Delta}{\langle\nabla\rangle}|u|^2\right\|_{X^{W+}_{-\frac{1}{2},-\frac{5}{12},\infty}(T)} \\ &\lesssim T^{\frac{1}{12}}\|u\|^2_{X^S_{0,\frac{5}{12},1}(T)}\end{aligned}$$

and obtain

$$\left\|\mathcal{I}^{W+}(\frac{\Delta}{\langle\nabla\rangle}|u|^2)\right\|_{X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)} \lesssim T^{\frac{1}{4}}\|u_1\|_{X^S_{0,\frac{1}{2},1}(T)}\|u_2\|_{X^S_{0,\frac{1}{2},1}(T)}. \tag{5.12}$$

Additionally we obtain

$$\begin{aligned}\|\mathcal{I}^{W+}(\langle\nabla\rangle^{-1}\mathrm{Re}\,v)\|_{X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)} &\lesssim \|\langle\nabla\rangle^{-1}\mathrm{Re}\,v\|_{L^2([0,T]\times\mathbb{R}^2)} \\ &\lesssim T^{\frac{1}{2}}\|v\|_{L^\infty_t H^{-\frac{1}{2}}_x} \lesssim T^{\frac{1}{2}}\|v\|_{X^{W+}_{-\frac{1}{2},\frac{1}{2},1}},\end{aligned} \tag{5.13}$$

which easily follows from (5.9). The analoguos estimates for differences can be shown by the same arguments. Using these nonlinear estimates and the linear estimates in Proposition 5.3, a standard iteration argument constructs a unique solution

$$(u,v) \in B_{X^S_{0,\frac{1}{2},1}(T)}(0, C\|u_0\|_{L^2}) \times B_{X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)}(0, C\|v_0\|_{H^{-\frac{1}{2}}})$$

for (5.10), provided that $T \sim R^{-4}$. In addition, one can show local Lipschitz continuity of the induced map $(u_0, v_0) \mapsto (u, v)$.

Next we seek to boost the time of existence based on the technique described in [9]. This is possible due to the $L^2$ norm conservation for $u$ and to the fact that the nonlinearity for $v$ depends only on $u$. We claim that the time of existence can be improved to $T \sim \min\{R^{-2}\|u_0\|_{L^2}^{-2}, 1\}$.

Without restricting the generality of the argument we can assume that $\|v_0\|_{H^{-\frac{1}{2}}} \geq \|u_0\|_{L^2}$. Then by the above argument we are able to construct solutions on the time interval $\delta \sim \|v_0\|^{-4}_{H^{-\frac{1}{2}}}$.

On the other hand, using (5.10), (5.12), (5.13) and that $e^{-it\langle\nabla\rangle}$ is unitary we obtain

$$\begin{aligned}
\|v\|_{L^\infty_t H_x^{-\frac{1}{2}}([0,\delta]\times\mathbb{R}^2)} &\leq \|v_0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} + C\delta^{\frac{1}{4}}\|u\|^2_{X^S_{0,\frac{1}{2},1}(\delta)} + \|v\|_{L^1_t H_x^{-\frac{1}{2}}([0,\delta]\times\mathbb{R}^2)} \\
&\leq \|v_0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} + C\delta^{\frac{1}{4}}\|u_0\|^2_{L^2} + \delta\|v\|_{L^\infty_t H_x^{-\frac{1}{2}}([0,\delta]\times\mathbb{R}^2)}.
\end{aligned}$$

This allows us to keep reiterating the problem on intervals $[j\delta, (j+1)\delta]$ for $j = 0, 1, \ldots, m$ until we double the size of the wave data, i.e. up to the first time when $\|v(t_0)\|_{H^{-\frac{1}{2}}} = 2\|v_0\|_{H^{-\frac{1}{2}}}$ (after this time the value of $\delta$ has to be adjusted). After $m$ iterations we obtain

$$\|v\|_{L^\infty_t H_x^{-\frac{1}{2}}([0,m\delta]\times\mathbb{R}^2)} \leq \|v_0\|_{H^{-\frac{1}{2}}(\mathbb{R}^2)} + Cm\delta^{\frac{1}{4}}\|u_0\|^2_{L^2} + m\delta\|v\|_{L^\infty_t H_x^{-\frac{1}{2}}([0,m\delta]\times\mathbb{R}^2)}.$$

A direct computation gives $m \sim \min\left(\|v_0\|_{H^{-\frac{1}{2}}}\delta^{-1/4}\|u_0\|^{-2}_{L^2}, \delta^{-1}\right)$ and this improves the time of existence for solutions to

$$m\delta \sim \min\left(\frac{R}{CR^{-1}\|u_0\|^2_{L^2}}R^{-4}, 1\right) \sim \min\left(R^{-2}\|u_0\|^{-2}_{L^2}, 1\right).$$

Therefore we are able to improve the life-span of solution to a time $T \sim \min\left(R^{-2}\|u_0\|^{-2}_{L^2}, 1\right)$ which implies the claim in Theorem 1.1.

Then a standard argument also establishes the uniqueness of solutions in $X^S_{0,\frac{1}{2},1}(T) \times X^{W+}_{-\frac{1}{2},\frac{1}{2},1}(T)$ and the Lipschitz dependence with respect to the initial data. □

## 6. Counterexamples

We first show that the time of existence provided in Theorem 1.1 is optimal up to the multiplicative constant.

*Proof of Theorem 1.2.* Fix $r > \|Q\|_{L^2}$. There exists $\omega \gg 1$ such that the Glangetas–Merle [14, 13] solution $P_\omega$, see (1.5), satisfies $\|P_\omega\|_{L^2} < r$. We fix such $\omega \gg 1$ and calculate for the corresponding solution (1.5)

$$\|u(t)\|_{L^2} = \|P_\omega\|_{L^2} < r,$$

and

$$\|n(t)\|_{H^{-\frac{1}{2}}} + \|\partial_t n(t)\|_{H^{-\frac{3}{2}}} \sim |T-t|^{-\frac{1}{2}}.$$

Theorem 1.2 follows. □

Next, we show that our multilinear estimates in Theorem 4.1 are sharp. We follow the approach which has been pioneered by Bourgain [4] to show non-smoothness of the flow map. We also refer the reader to [15] where related counterexamples in the 1d case have been constructed.

In order to avoid unnecessary technicalities, we write $X^S_{k,b}$ to denote $X^S_{k,b,2}$ and $X^{W\pm}_{\ell,b}$ to denote $X^{W\pm}_{\ell,b,2}$ and provide counterexamples for this scale of norms. We remark that the arguments remain valid for any choice of $1 \leq p \leq \infty$ instead of 2. The reason is that the norms $X^S_{k,b,p}$ for distinct $p$ are equivalent up to logarithms of the size of the modulation (same for $X^{W\pm}_{k,b,p}$), but our counterexamples will always involve powers of the modulation.

Moreover, in Proposition 6.1 we restrict the exposition to the case of the $X^{W+}_{k,b}$ space, i.e. the sharpness of (4.2); the case $X^{W+}_{k,b}$, i.e. the sharpness of (4.1), follows by the same argument up to obvious modifications.

**Proposition 6.1.** *The inequality*

$$\|uv\|_{X^S_{k,-b'}} \lesssim \|v\|_{X^{W+}_{\ell,b_1}} \|u\|_{X^S_{k,b_2}}$$

*is false in either of the following two situations:*

*(i) if* $\ell < -\frac{1}{2}$*, for any* $b'$*,* $b_1$*, and* $b_2$*,*

*(ii) if* $\ell = -\frac{1}{2}$ *and* $b' + b_1 + b_2 < \frac{5}{4}$*.*

This follows from applying Lemma 6.3 with $\sigma = -1$ to establish the first claim, and any $\sigma$ such that $-1 < \sigma < 0$ to establish the second claim.

**Proposition 6.2.** *The inequality*

$$\left\| \frac{\Delta}{\langle \nabla \rangle} (u\bar{w}) \right\|_{X^{W+}_{\ell,-b'}} \lesssim \|u\|_{X^S_{k,b_1}} \|w\|_{X^S_{k,b_2}}$$

*is false in either of the following two situations:*

*(i) if* $\ell - 2k + \frac{1}{2} > 0$ *for any* $b'$*,* $b_1$*, and* $b_2$*,*

*(ii) if* $\ell - 2k + \frac{1}{2} = 0$ *and* $b' + b_1 + b_2 < \frac{5}{4}$

This follows from applying Lemma 6.4 with $\sigma = -1$ to establish the first claim, and any $\sigma$ such that $-1 < \sigma < 0$ to establish the second claim.

**Lemma 6.3.** *For each $N \gg 1$, there exist $v_N$ and $u_N$ such that*

$$\frac{\|v_N u_N\|_{X^S_{k,-b'}}}{\|v_N\|_{X^{W+}_{\ell,b_1}}\|u_N\|_{X^S_{k,b_2}}} \gtrsim N^{-\ell-\frac{1}{2}+(1+\sigma)[\frac{5}{4}-(b'+b_1+b_2)]}$$

*for all $k,\ell \in \mathbb{R}$ and $b', b_1, b_2 \geq 0$, and any $-1 \leq \sigma < 0$, with the implicit constant independent of all of $k,\ell,b',b_1,b_2,\sigma$, and $N$.*

*Proof.* Denote $\xi = (\xi_1, \xi_2)$ (i.e. $\xi_j$ now denotes the $j$th component of $\xi$). Let $\hat{v} = \chi_E$, where $E$ is the rectangle centered at $(\xi_1, \xi_2, \tau) = (2N+1, 0, -2N-1)$ and width $N^\sigma \times N^{\frac{1}{2}(1+\sigma)} \times N^{1+\sigma}$, so that on $E$, we have $|\tau + |\xi|| \lesssim N^{1+\sigma}$. Let $\hat{u} = \chi_F$, where $F$ is the rectangle centered at $(\xi_1, \xi_2, \tau) = (-N, 0, -N^2)$ and width $N^\sigma \times N^{\frac{1}{2}(1+\sigma)} \times N^{1+\sigma}$, so that on $F$, we have $|\tau + |\xi|^2| \lesssim N^{1+\sigma}$. Then $\widehat{vu} \gtrsim N^{\frac{3}{2}+\frac{5}{2}\sigma}\chi_G$, where $G$ is a rectangle centered at $(\xi_1, \xi_2, \tau) = (N+1, 0, -(N+1)^2)$ and width $N^\sigma \times N^{\frac{1}{2}(1+\sigma)} \times N^{1+\sigma}$. Note that on $G$, we have $|\tau + |\xi|^2| \lesssim N^{1+\sigma}$. Then

$$\begin{aligned}\|vu\|_{X^S_{k,-b'}} &\gtrsim N^{\frac{3}{2}+\frac{5}{2}\sigma} N^k N^{-(1+\sigma)b'} \|\chi_G\|_{L^2} \\ &= N^{\frac{3}{2}+\frac{5}{2}\sigma} N^k N^{-(1+\sigma)b'} N^{\frac{1}{2}(\frac{3}{2}+\frac{5}{2}\sigma)} ,\end{aligned}$$

and

$$\begin{aligned}\|v\|_{X^{W+}_{\ell,b_1}} &\lesssim N^\ell N^{(1+\sigma)b_1} \|\chi_E\|_{L^2} = N^\ell N^{(1+\sigma)b_1} N^{\frac{1}{2}(\frac{3}{2}+\frac{5}{2}\sigma)} , \\ \|u\|_{X^S_{k,b_2}} &\lesssim N^k N^{(1+\sigma)b_2} \|\chi_F\|_{L^2} = N^k N^{(1+\sigma)b_2} N^{\frac{1}{2}(\frac{3}{2}+\frac{5}{2}\sigma)} ,\end{aligned}$$

which proves the claim. □

**Lemma 6.4.** *For each $N \gg 1$, there exists $u_N$ and $w_N$ such that*

$$\frac{\|\frac{\Delta}{\langle\nabla\rangle}(u_N \bar{w}_N)\|_{X^{W+}_{\ell,-b'}}}{\|u_N\|_{X^S_{k,b_1}}\|w_N\|_{X^S_{k,b_2}}} \gtrsim N^{\ell-2k+\frac{1}{2}+(1+\sigma)[\frac{5}{4}-(b'+b_1+b_2)]}$$

*for all $k,\ell \in \mathbb{R}$, any $b', b_1, b_2 \geq 0$, and any $-1 \leq \sigma < 0$, with the implicit constant independent of all of $k,\ell,b',b_1,b_2,\sigma$, and $N$.*

*Proof.* Let $\hat{u}_N = \chi_E$, where $E$ is the rectangle centered at $(\xi_1, \xi_2, \tau) = (N+1, 0, -(N+1)^2)$ with width $N^\sigma \times N^{\frac{1}{2}(1+\sigma)} \times N^{1+\sigma}$, so that on $E$, the quantity $|\tau + |\xi|^2| \leq N^{1+\sigma}$. Let $\hat{w}_N = \chi_F$, where $F$ is the rectangle centered at $(\xi_1, \xi_2, \tau) = (-N, 0, -N^2)$ with width $N^\sigma \times N^{\frac{1}{2}(1+\sigma)} \times N^{1+\sigma}$, so that on $F$, the quantity $|\tau + |\xi|^2| \leq N^{1+\sigma}$. Then $\widehat{u_N \bar{w}_N} \gtrsim N^{\frac{3}{2}+\frac{5}{2}\sigma}\chi_G$, where $G$ is the rectangle centered at $(2N+1, 0, -2N-1)$ and width $N^\sigma \times N^{\frac{1}{2}(1+\sigma)} \times N^{1+\sigma}$ so that on $G$, the quantity $|\tau + |\xi|| \leq N^{1+\sigma}$. Thus,

$$\begin{aligned}\left\|\frac{\Delta}{\langle\nabla\rangle}(u_N \bar{w}_N)\right\|_{X^{W+}_{\ell,-b'}} &\gtrsim N^{\frac{3}{2}+\frac{5}{2}\sigma} N^{\ell+1} N^{-(1+\sigma)b'} \|\chi_G\|_{L^2} \\ &= N^{\frac{3}{2}+\frac{5}{2}\sigma} N^{\ell+1} N^{-(1+\sigma)b'} N^{\frac{1}{2}(\frac{3}{2}+\frac{5}{2}\sigma)} ,\end{aligned}$$

and

$$\|u_N\|_{X^S_{k,b_1}} \lesssim N^k N^{(1+\sigma)b_1} \|\chi_E\|_{L^2} = N^k N^{(1+\sigma)b_1} N^{\frac{1}{2}(\frac{3}{2}+\frac{5}{2}\sigma)} ,$$

$$\|w_N\|_{X^S_{k,b_2}} \lesssim N^k N^{(1+\sigma)b_2}\|\chi_F\|_{L^2} = N^k N^{(1+\sigma)b_2} N^{\frac{1}{2}(\frac{3}{2}+\frac{5}{2}\sigma)}\,,$$

which proves the claim. □

*Remark* 5. Alternatively, the optimality of our choice of $b_1 = b_2 = b_3 = \frac{5}{12}$ can be seen by an indirect argument: If it was possible to choose smaller $b$'s, we would be able to improve the time of existence by the iterative argument given in Section 5 above and would obtain a contradiction to the blow-up of the Glangetas–Merle solutions constructed in [13, 14].

The following proposition is based on a variant of the example from the proof of Proposition 6.1 and contains a slightly stronger conclusion.

**Proposition 6.5.** *Fix $0 < T \leq 1$. For all $N \gg T^{-1}$ there exists $u_N \in H^k_x$ and $v_N \in H^\ell_x$ such that*

$$\sup_{|t|\leq T}\left\|\int_0^t e^{i(t-t')\Delta}\left(e^{it'\Delta}u_N \mathrm{Re}\,\left(e^{-it'\langle\nabla\rangle}v_N\right)\right)dt'\right\|_{H^k_x} \gtrsim \frac{\|u_N\|_{H^k_x}\|v_N\|_{H^\ell_x}}{N^{\ell+\frac{1}{2}}}\,,$$

*where the constant is independent of $N$.*

*Proof.* Set $\hat{u}_N := \chi_A$, where $A$ is the rectangle where $\xi = (\xi_1, \xi_2)$ satisfies

$$-N - N^{-1} \leq \xi_1 \leq -N + N^{-1} \quad\text{and}\quad -1 \leq \xi_2 \leq 1,$$

such that $\|u_N\|_{H^k} \sim N^{k-\frac{1}{2}}$. Similarly, define $v_N := \chi_B + \chi_{-B}$ for the rectangle $B$ where

$$2N + 1 - 2N^{-1} \leq \xi_1 \leq 2N + 1 + 2N^{-1} \quad\text{and}\quad -2 \leq \xi_2 \leq 2.$$

Note that $v_N$ is real-valued and $\|v_N\|_{H^\ell_x} \sim N^{\ell-\frac{1}{2}}$. We observe that

$$\widehat{u_N v_N}(\xi) \gtrsim N^{-1}, \tag{6.1}$$

whenever $\xi = (\xi_1, \xi_2)$ satisfies

$$N + 1 - N^{-1} \leq \xi_1 \leq N + 1 + N^{-1} \quad\text{and}\quad -1 \leq \xi_2 \leq 1. \tag{6.2}$$

We write $2\mathrm{Re}\,(e^{-it'\langle\nabla\rangle}v_N) = (e^{-it'\langle\nabla\rangle} + e^{it'\langle\nabla\rangle})v_N$. For $\xi$ satisfying (6.2) and $N^{-1} \ll |t| \ll T$ it holds

$$\begin{aligned}&\left|\mathcal{F}_x\left(\int_0^t e^{i(t-t')\Delta}\left(e^{it'\Delta}u_N(e^{-it'\langle\nabla\rangle} + e^{it'\langle\nabla\rangle})v_N\right)dt'\right)(\xi)\right|\\ =&\left|\int\int_0^t e^{it'(|\xi|^2-|\eta|^2)}(e^{-it'\langle\xi-\eta\rangle} + e^{it'\langle\xi-\eta\rangle})dt'\widehat{u_N}(\eta)\widehat{v_N}(\xi-\eta)d\eta\right| \gtrsim |t|N^{-1}\end{aligned}$$

by (6.1) and because the first phase factor $|\xi|^2 - |\eta|^2 - \langle\xi-\eta\rangle$ is bounded whenever $\eta \in A$ and (6.2) holds for $\xi$, whereas the second phase factor $|\xi|^2 - |\eta|^2 + \langle\xi-\eta\rangle$ is of size $N$ in this region.

Integrating over this region (6.2) gives

$$\left\|\int_0^t e^{i(t-t')\Delta}\left(e^{it'\Delta}u_N \mathrm{Re}\,\left(e^{-it'\langle\nabla\rangle}v_N\right)\right)dt'\right\|_{H^k_x} \gtrsim |t|N^{k-\frac{3}{2}},$$

which implies the claim. □

The following proposition is based on a variant of the example from the proof of Proposition 6.2.

**Proposition 6.6.** *Fix $0 < T \leq 1$. For all $N \gg 1$ there exists $u_N \in H^k_x$ such that*

$$\sup_{|t|\leq T} \left\| \int_0^t e^{-i(t-t')\langle\nabla\rangle} \frac{\Delta}{\langle\nabla\rangle} \left( e^{it'\Delta} u_N \overline{e^{it'\Delta} u_N} \right) dt' \right\|_{H^\ell_x} \gtrsim N^{\ell-2k+\frac{1}{2}} \|u_N\|^2_{H^k_x} ,$$

*where the constant is independent of $N$.*

*Proof.* Set $\hat{u}_N := \chi_{D_1} + \chi_{D_2}$, where $D_1$ is the rectangle where $\xi = (\xi_1, \xi_2)$ satisfies

$$N + 1 - N^{-1} \leq \xi_1 \leq N + 1 + N^{-1} \quad \text{and} \quad -1 \leq \xi_2 \leq 1,$$

and $D_2$ is the rectangle where

$$-N - 2N^{-2} \leq \xi_1 \leq -N + 2N^{-1} \quad \text{and} \quad -2 \leq \xi_2 \leq 2.$$

Then, $\|u_N\|_{H^k} \sim N^{k-\frac{1}{2}}$. We observe that

$$\widehat{u_N \bar{u}_N}(\xi) \gtrsim N^{-1}, \tag{6.3}$$

whenever $\xi = (\xi_1, \xi_2)$ satisfies

$$2N + 1 - N^{-1} \leq \xi_1 \leq 2N + 1 + N^{-1} \quad \text{and} \quad -1 \leq \xi_2 \leq 1. \tag{6.4}$$

Therefore, for such $\xi$ and $|t| \ll 1$ it holds

$$\begin{aligned} &\left| \mathcal{F}_x \left( \int_0^t e^{i(t-t')\langle\nabla\rangle} \frac{\Delta}{\langle\nabla\rangle} \left( e^{it'\Delta} u_N \overline{e^{it'\Delta} u_N} \right) dt' \right) (\xi) \right| \\ \sim |\xi| &\left| \int_0^t \int e^{it'(\langle\xi\rangle - |\eta|^2 + |\xi-\eta|^2)} \widehat{u_N}(\eta) \widehat{\overline{u_N}}(\xi - \eta) d\eta dt' \right| \gtrsim |t| \end{aligned}$$

by (6.3), $|\xi| \sim N$ and because the phase factor $\langle\xi\rangle - |\eta|^2 + |\xi - \eta|^2$ is bounded whenever (6.4) holds. Integrating over this region (6.4) gives

$$\left\| \int_0^t e^{-i(t-t')\langle\nabla\rangle} \frac{\Delta}{\langle\nabla\rangle} \left( e^{it\Delta} u_N \overline{e^{it\Delta} u_N} \right) dt' \right\|_{H^\ell_x} \gtrsim |t| N^{\ell-\frac{1}{2}}$$

and the claim follows. □

Finally, we indicate how we use Propositions 6.5 and 6.6 to prove Theorem 1.3.

*Proof of Theorem 1.3.* Proposition 6.5 shows that for $\ell < -\frac{1}{2}$ the first component of the directional (Fréchet) derivative of second order of the flow map to the reduced system (3.1) at 0 with respect to the direction $(u_0, v_0) = (u_N, v_N)$ is unbounded.

Proposition 6.6 shows that for $\ell - 2k + \frac{1}{2} > 0$ the second component of the directional derivative of second order of the flow map to the reduced system (3.1) at 0 with respect to the direction $(u_0, v_0) = (u_N, 0)$ is unbounded.

If the flow map to the original system (1.1) were $C^2$ then we could conclude that the flow map for the reduced system is $C^2$ by the arguments in Section 3. But this contradicts to the assertions above. □

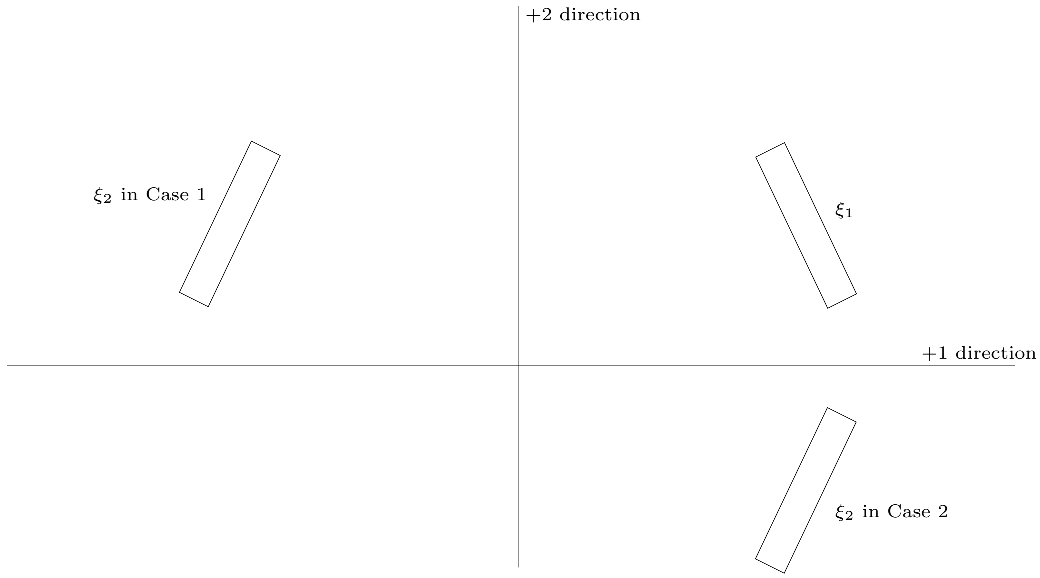


**Figure A1.** After rotation, the two possible positions of $\xi_2$ in the proof of Prop. 4.4, labeled as Case 1 and Case 2.

## Appendix A. Alternative proof of Proposition 4.4

Here we present an alternate proof of Proposition 4.4 that does not make use of the restriction theorem from [2]. The main source of technique for the proof that follows is Colliander–Delort–Kenig–Staffilani [8].

*Proof.* We abuse notation and replace $g_2$ by $g_2(-\cdot)$ and change variables $\zeta_2 \mapsto -\zeta_2$ to obtain the usual convolution structure. From now on it holds $|\tau_2 - |\xi_2|^2| \sim L_2$ within the support of $g_2$.

By the change of variables $\tau_1 = -|\xi_1|^2 + c_1$, $\tau_2 = |\xi_2|^2 + c_2$ and by applying the Cauchy-Schwarz inequality with respect to $c_1$ and $c_2$ it suffices to consider the trilinear expression

$$T(g_{1,c_1}, g_{2,c_2}, f) = \int g_{1,c_1}(\xi_1) g_{2,c_2}(\xi_2) f(\xi_1 + \xi_2, |\xi_2|^2 - |\xi_1|^2 + c_1 + c_2) d\xi_1 d\xi_2$$

where $g_{k,c_k}(\xi) = g_k(\xi, (-1)^k|\xi|^2 + c_k)$ for $k = 1,2$ and $f$ is localized in the region $|\tau - |\xi|| \leq L$, and prove that

$$|T(g_{1,c_1}, g_{2,c_2}, f)| \lesssim \frac{A^{1/2} L^{1/2}}{N_1} \|g_{1,c_1}\|_{L^2_\xi} \|g_{2,c_2}\|_{L^2_\xi} \|f\|_{L^2}. \tag{A.1}$$

We exploit the geometry of the problem in order to better localize the interacting elements. Taking into account the angular localization and separation of $\xi_1$ and $\xi_2$ which is $\sim A^{-1}$ and their size localization, it follows that after a rotation we may assume that $\xi_{1,1} > 0$, $\xi_{1,2} > 0$ with $\xi_{1,1} \sim N_1$ and $\xi_{1,2} \sim N_1 A^{-1}$, and that either Case 1 or Case 2 below holds (see Figure A1).

*Case 1.* $\xi_{2,1} < 0$, $\xi_{2,2} > 0$ with $|\xi_{2,1}| \sim N_1$ and $|\xi_{2,2}| \sim N_1 A^{-1}$.

*Case 2.* $\xi_{2,1} > 0$, $\xi_{2,2} < 0$ with $|\xi_{2,1}| \sim N_1$ and $|\xi_{2,2}| \sim N_1 A^{-1}$.

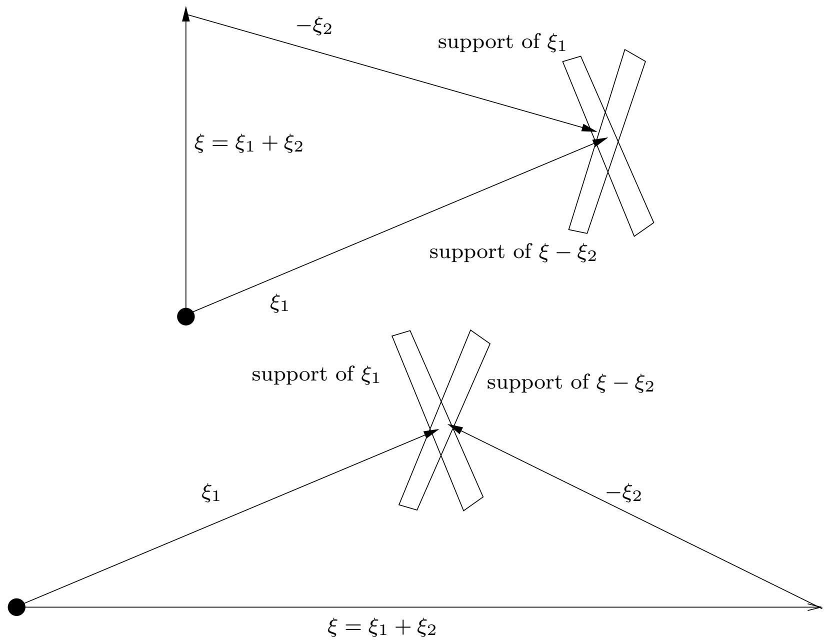


**Figure A2.** Depiction of Case 1 (top) and Case 2 (bottom). $\xi_1$ is supported in an annular ring $D_1$ of thickness $L/N_1$ and $\xi_2$ is supported in an annular ring $D_2$ of thickness $L/N_1$. For fixed $\xi = \xi_1 + \xi_2$, $\xi_1$ is confined to $D_1$ and also to $\xi - D_2$. These two sets have thickness $L/N_1$ but also meet at an angle $A^{-1}$, and thus $\xi_{1,1}$ is confined to an interval of size $L/N_1$ and $\xi_{1,2}$ is confined to an interval of size $LA/N_1$

In addition we consider the following two cases separately, Case A: $L \geq N$ and Case B: $L \leq N$.

*Case A.* Suppose that $L \geq N$. Since $|\tau - |\xi|| \leq L$, we have that $|\xi_2|^2 - |\xi_1|^2$ is confined to an interval of size $L$, and thus $|\xi_2| - |\xi_1|$ is confined to an interval of size $L/N_1$. By the "orthogonality" Lemma Appendix A.1 below, and Cauchy-Schwarz, we might as well assume that $|\xi_1|$ and $|\xi_2|$ are confined to fixed intervals of size $L/N_1$. Note that in the two cases outlined above, we have (see Fig. A2)

*Case A1.* $\xi_{1,2} + \xi_{2,2} \sim N_1 A^{-1}$ and if $\xi = \xi_1 + \xi_2$ is fixed, then $\xi_{1,1}$ is contained in an interval of size $LN_1^{-1}$.

*Case A2.* $\xi_{1,1} + \xi_{2,1} \sim N_1$ and if $\xi = \xi_1 + \xi_2$ is fixed, then $\xi_{1,2}$ is contained in an interval of size $LAN_1^{-1}$.

Let $\mu = \xi_1 + \xi_2$, $\nu = -|\xi_1|^2 + |\xi_2|^2 + c_1 + c_2$, and in Case 1 let $\sigma = \xi_{1,1}$, but in Case 2 let $\sigma = \xi_{1,2}$. Denote by $J$ the Jacobian determinant. We have

$$J = \begin{array}{c|cccc|} & \xi_{1,1} & \xi_{1,2} & \xi_{2,1} & \xi_{2,2} \\ \hline \mu_1 & 1 & 0 & 1 & 0 \\ \mu_2 & 0 & 1 & 0 & 1 \\ \nu & -2\xi_{1,1} & -2\xi_{1,2} & 2\xi_{2,1} & 2\xi_{2,2} \\ \sigma & * & * & 0 & 0 \end{array}$$

and thus

$$|J| = \begin{cases} 2|\xi_{2,2}+\xi_{1,2}| & \text{in Case 1} \\ 2|\xi_{2,1}+\xi_{1,1}| & \text{in Case 2} \end{cases} \sim \begin{cases} N_1 A^{-1} & \text{in Case 1} \\ N_1 & \text{in Case 2} \end{cases} .$$

So, $|J|$ is essentially constant over the region of integration, and can be removed from the integration. We obtain

$$T(g_{1,c_1}, g_{2,c_2}, f) = \int g_{1,c_1}(\xi_1) g_{2,c_2}(\xi_2) f(\mu,\nu) |J|^{-1} \, d\mu \, d\nu \, d\sigma \le |J|^{-1/2} I_1 I_2$$

where

$$I_1 = \left( \int_{\mu,\nu,\sigma} |J|^{-1} |g_{1,c_1}(\xi_1) g_{2,c_2}(\xi_2)|^2 \, d\mu \, d\nu \, d\sigma \right)^{1/2} = \|g_{1,c_1}\|_{L^2_{\xi_1}} \|g_{2,c_2}\|_{L^2_{\xi_2}}$$

and

$$I_2 = \left( \int_{\mu,\nu} |f(\mu,\nu)|^2 \left( \int_\sigma d\sigma \right) d\mu \, d\nu \right)^{1/2} .$$

The measure of the support of $\sigma$, for fixed $\mu = \xi_1 + \xi_2$, in Case 1 is $LN_1^{-1}$ and in Case 2 is $LAN_1^{-1}$. Thus, we obtain (A.1).

*Case B.* Now suppose that $L \le N$. Let $\{E_j\}$ be a partition of $[0, +\infty)$ into intervals of length $L$. Then the left side of (A.1) becomes

$$\sum_j \int g_1(\xi_1) g_2(\xi_2) f(\xi_1+\xi_2, \cdot) \chi_{E_j}(|\xi_1+\xi_2|) \, d\xi_1 \, d\xi_2.$$

For a fixed $j$, we have that $|\xi|$ is localized to an interval of length $L$, and since $|\tau - |\xi|| \le L$, we obtain that $|\xi_2|^2 - |\xi_1|^2$ is localized to an interval of size $L$, from which it follows that $|\xi_2| - |\xi_1|$ is localized to an interval of length $L/N_1$. We can now follow the argument of Case A to obtain the bound

$$|T(g_{1,c_1}, g_{2,c_2}, f)| \lesssim \frac{A^{1/2} L^{1/2}}{N_1} \sum_j \|g_1(\xi_1) g_2(\xi_2) \chi_{E_j}(|\xi_1+\xi_2|)\|_{L^2_{\xi_1 \xi_2}} \|f(\xi,\tau) \chi_{E_j}(|\xi|)\|_{L^2_{\xi\tau}} .$$

Applying Cauchy-Schwarz with respect to $j$ we complete the proof of (A.1). $\square$

**Lemma Appendix A.1.** *Suppose $N_1 \gtrsim 1$, $1 \lesssim A \ll N_1$, $k \ll N_1^2$ and that $x, y \ge 0$ satisfy*

$$k \le x^2 - y^2 \le k + N_1 A^{-1} , \qquad \frac{1}{4} N_1 \le x, y \le 4 N_1 .$$

*Decompose $[\frac{1}{4}N_1, 4N_1]$ into a sequence of intervals $\{I_j\}$ each of length $A^{-1}$. Then there is a mapping $j \mapsto k(j)$ such that*

$$y \in I_j \ \Rightarrow \ x \in I_{k(j)-100} \cup \cdots \cup I_{k(j)+100} .$$

*Moreover, as $j$ ranges over the full set of intervals, $k(j)$ hits a particular element no more than 100 times.*

*Proof.* We take $I_j = [A^{-1}(j - \frac{1}{2}), A^{-1}(j + \frac{1}{2}))]$ (so $j$ ranges from $AN_1/4$ to $4AN_1$). Suppose that $y \in I_j$. Then $|y - A^{-1}j| \leq A^{-1}$, and therefore

$$k - 4N_1A^{-1} \leq x^2 - A^{-2}j^2 \leq k + 4N_1A^{-1},$$

which implies that

$$(A^{-2}j^2 + k - 4N_1A^{-1})^{-1/2} \leq x \leq (A^{-2}j^2 + k + 4N_1A^{-1})^{-1/2}.$$

The length of this interval is

$$\frac{8N_1A^{-1}}{(A^{-2}j^2 + k - 4N_1A^{-1})^{-1/2} + (A^{-2}j^2 + k + 4N_1A^{-1})^{-1/2}} \lesssim A^{-1}.$$

Also, as we increment from $j$ to $j+1$, the left endpoint of the interval advances by an amount

$$\frac{2A^{-2}j}{(A^{-2}j^2 + k - 4N_1A^{-1})^{-1/2} + (A^{-2}j^2 + k + 4N_1A^{-1})^{-1/2}} \gtrsim A^{-1},$$

and the claim follows. □

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