LOW REGULARITY GLOBAL WELL-POSEDNESS FOR THE ZAKHAROV AND KLEIN-GORDON-SCHRÖDINGER SYSTEMS

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ABSTRACT. We prove low-regularity global well-posedness for the 1d Zakharov system and 3d Klein-Gordon-Schrödinger system, which are systems in two variables $u : \mathbb{R}_x^d \times \mathbb{R}_t \to \mathbb{C}$ and $n : \mathbb{R}_x^d \times \mathbb{R}_t \to \mathbb{R}$. The Zakharov system is known to be locally well-posed in $(u, n) \in L^2 \times H^{-1/2}$ and the Klein-Gordon-Schrödinger system is known to be locally well-posed in $(u, n) \in L^2 \times L^2$. Here, we show that the Zakharov and Klein-Gordon-Schrödinger systems are globally well-posed in these spaces, respectively, by using an available conservation law for the L^2 norm of u and controlling the growth of n via the estimates in the local theory.

1. INTRODUCTION

The initial-value problem for the one-dimensional Zakharov system is

$$(1.1) \qquad \begin{cases} i\partial_t u + \partial_x^2 u = nu \\ \partial_t^2 n - \partial_x^2 n = \partial_x^2 |u|^2 \\ u(x,0) = u_0(x), \ n(x,0) = n_0(x), \ \partial_t n(x,0) = n_1(x). \end{cases}$$

Here $u : [0, T^*) \times \mathbb{R} \longmapsto \mathbb{R}, n : [0, T^*) \times \mathbb{R} \longmapsto \mathbb{R}$. This problem arises in plasma physics. Sufficiently regular solutions of (1.1) satisfy conservation of mass

$$(1.2) \qquad M[u](t) = \int |u(t)|^2 \, dx = \int |u_0|^2 \, dx = M[u_0]$$

and conservation of the Hamiltonian

$$(1.3) \qquad \begin{aligned} H[u, n, \nu](t) &= \int \left(|\partial_x u(t)|^2 + \tfrac{1}{2} n(t)^2 + n(t)|u(t)|^2 + \tfrac{1}{2}\nu(t)^2 \right) \, dx \\ &= H[u_0, n_0, \nu_0] \end{aligned}$$

where $\nu(t)$ is such that $\partial_t n = \partial_x \nu$ and $\partial_t \nu = \partial_x(n + |u|^2)$.

The local-in-time theory in $X_{s,b}$ spaces has been established in [4], [11], the latter paper obtaining local well-posedness (LWP) for the one-dimensional equation (1.1) with $(u_0, n_0, n_1) \in L^2 \times H^{-1/2} \times H^{-3/2}$ and for some more regular spaces $H^k \times H^s \times H^{s-1}$

with various[1] k, s. As an immediate consequence of the local theory and (1.2),(1.3), one has global well-posedness (GWP) for $k = 1$, $s = 0$. Pecher [16], using the low-high frequency decomposition method of Bourgain [6], proved GWP for $\frac{9}{10} < k < 1$, $s = 0$. This result was improved in [18] using the I-method of [8] to obtain GWP for[2] $\frac{5}{6} < k < 1$, $s = k - 1$. The preceding GWP results are all based on the conservation of the Hamiltonian (1.3) or certain variants of the Hamiltonian. In this paper, we prove GWP for $k = 0$, $s = -\frac{1}{2}$, using a scheme based on mass conservation (1.2) and subcritical slack in certain multilinear estimates at this regularity threshold. In [12], it is shown that the one-dimensional LWP theory of [11] is effectively sharp by adapting techniques of [5] and [7]. Thus, we establish GWP in the largest space for which LWP holds.

Theorem 1.1. *The Zakharov system* (1.1) *is globally well-posed for* $(u_0, n_0, n_1) \in L^2 \times H^{-1/2} \times H^{-3/2}$ *and the solution* (u, n) *satisfies* (1.2) *and*

$$\|n(t)\|_{H_x^{-\frac{1}{2}}} + \|\partial_t n(t)\|_{H_x^{-\frac{3}{2}}} \leq \exp(c|t| \|u_0\|_{L^2}^2) \max(\|n_0\|_{H^{-\frac{1}{2}}} + \|n_1\|_{H^{-\frac{3}{2}}}, \|u_0\|_{L^2}^2)$$

Remark 1.2. *Since Theorem 1.1 is based on the mass conservation property* (1.2) *and the local theory, the same result applies to certain Hamiltonian generalizations of* (1.1) *for which global well-posedness was previously unknown. Indeed, if we write*

$$H[u, n, \nu](t) = \int |\partial_x u(t)|^2 + \frac{\alpha}{2}|n(t)|^2 + \frac{\beta}{2}|\nu(t)|^2 + \gamma n(t)|u(t)|^2 dx$$

and calculate

$$\partial_t \begin{bmatrix} u \\ n \\ \nu \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & \nabla\cdot \\ 0 & \nabla & 0 \end{bmatrix} \begin{bmatrix} H_{\overline{u}} \\ H_n \\ H_\nu \end{bmatrix}$$

we find the evolution system

$$\begin{cases} i\partial_t u + \partial_x^2 u = \gamma n u \\ \partial_t^2 n - \alpha\beta\partial_x^2 n = \beta\gamma\partial_x^2 |u|^2. \end{cases}$$

If we then choose $\alpha = \beta = -1$ *and* $\gamma = -1$, *we encounter a Hamiltonian evolution problem similar to* (1.1) *but with* $+nu$ *replaced by* $-nu$. *The local theory for these problems coincides but the appearance of* $\alpha = \beta = -1$ *in the Hamiltonian* H *precludes its use in obtaining a globalizing estimate.*

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[1]The paper [11] actually gives a systematic treatment of LWP for higher dimensional versions of (1.1) as well. Their result in dimension one uses the calculus techniques for obtaining $X_{s,b}$ bilinear estimates developed by Kenig, Ponce and Vega [14], [15]. The only LWP result in [11] when $k = 0$ is for dimension one, $s = -\frac{1}{2}$, and thus we have restricted exclusively to this case.

[2]Although the additional assumption $n_1 \in \dot{H}^{-1}$ appears in the papers [16] and [18], it can likely be removed by introducing suitable low frequency modifications to the energy identity; see [10] Lemma A.1 p. 358.

The initial-value problem for the d-dimensional Klein-Gordon-Schrödinger system with Yukawa coupling is

(1.4)
$$\begin{cases} i\partial_t u + \Delta u = -\gamma n u & x \in \mathbb{R}^d, t \in \mathbb{R} \\ \partial_t^2 n + \alpha\beta(1-\Delta)n = -\beta\gamma|u|^2 \\ u(x,0) = u_0(x), \ n(x,0) = n_0(x), \ \partial_t n(x,0) = n_1(x). \end{cases}$$

Here α, β, γ are real constants. The solution satisfies conservation of mass

(1.5)
$$M[u](t) = \int |u(t)|^2 \, dx = \int |u_0|^2 \, dx = M[u_0]$$

and conservation of the Hamiltonian

$$H[u,n,\nu](t) = \int \left(|\nabla u(t)|^2 + \frac{1}{2\beta}|\partial_t n(t)|^2 + \frac{\alpha}{2}|\sqrt{-\Delta+1}\, n(t)|^2 + \gamma n(t)|u(t)|^2 \right) dx$$

(1.6)
$$= H[u_0, n_0, \nu_0].$$

Pecher [17] proved that (1.4) is LWP for $d = 3$ in $L^2 \times L^2 \times H^{-1}$ and some more regular spaces $H^k \times H^s \times H^{s-1}$ for various k, s by following the scheme developed for the Zakharov system in [11]. Provided that $\alpha > 0$ and $\beta > 0$, energy conservation (1.6) yields GWP in the setting $1 \leq d \leq 3$ and $k = 1$, $s = 1$. In the case when $\alpha > 0$ and $\beta > 0$ where the energy gives control on the H^1 norm, the low-high frequency separation method of Bourgain [6] has been applied to (1.4) in [17] and the method of almost conservation laws of [8] has been applied to (1.4) in [22], to obtain GWP under the following assumptions: for $d = 1$, $k = s$, $s > \frac{1}{2}$; for $d = 2$, $k = s$, $s > \frac{\sqrt{17}-3}{2}$; for $d = 3$, $k = s$, $s > \frac{7}{10}$; for $d = 3$, $k, s > \frac{7}{10}$, $k + s > \frac{3}{2}$. Moreover, in each of these cases, a polynomial in time bound is obtained for the growth of the norms. The method of almost conservation laws has been also applied in [1], to obtain global solutions for a wave-Schrödinger system in dimensions 3 and 4 that in particular, for $d = 3$, implies GWP for (1.4) with $k = s$, $s > \frac{\sqrt{57}-5}{4}$. In this paper, we prove GWP for $d = 3$, $k = s = 0$, by a scheme involving (1.5) and direct application of the Strichartz estimates for the Schrödinger operator and Minkowski's integral inequality applied to the Klein-Gordon Duhamel term.[3]

Theorem 1.3. *If $\alpha\beta > 0$, the Klein-Gordon-Schrödinger system (1.4) in dimension $d = 3$ is globally well-posed for $(u_0, n_0, n_1) \in L^2 \times L^2 \times H^{-1}$. Moreover, the solution (u,n) satisfies (1.5) and*

(1.7) $\quad \|n(t)\|_{L^2} + \|\partial_t n(t)\|_{H^{-1}} \leq \exp(c|t|\|u_0\|_{L^2}^2) \max((\|n_0\|_{L^2} + \|n_1\|_{H^{-1}}), \|u_0\|_{L^2}^2).$

Remark 1.4. *In the case where $\alpha < 0$ and $\beta < 0$, global well-posedness of (1.4) for large smooth data was previously unknown. Since our proof of Theorem 1.3 is based*

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[3]Similar results hold for $d = 1$, $d = 2$, although for expositional convenience, we have restricted to the most delicate case $d = 3$.

on the conservation of $\|u(t)\|_{L^2}$, we do not require any Sobolev norm control obtained from the Hamiltonian and obtain global well-posedness for this case as well.

The proof of both Theorem 1.1 and 1.3 apply essentially the same scheme, although invoke a different space-time norm in the local theory estimates.

1.1. **Outline of method.** We describe the globalization scheme for the Zakharov system and the Klein-Gordon-Schrödinger system using the abstract initial value problem posed at some time $t = T_j$

$$(1.8) \qquad \begin{cases} Ku = F(u, n) \\ Ln = G(u) \\ (u, n)(T_j) = (u_j, n_j). \end{cases}$$

Here K and L are linear differential operators of evolution type, F is a nonlinear term coupling the two equations together and G is a nonlinear term depending only upon u. The fact that G does not depend upon n is used in our scheme. Let $W(t)n_0$ denote the linear group $W(t)$ applied to initial data n_0 solving the initial value problem $Ln = 0, n(0) = n_0$. Similarly, let $S(t)u_0$ denote the solution of $Ku = 0, u(0) = u_0$. We denote with $Wn_0 + L^{-1}g$ the solution of the linear initial value problem $Lu = g, n(0) = n_0$. Similarly, $Su_0 + K^{-1}g$ denotes the solution of $Ku = g, u(0) = u_0$.

We solve the second equation in our system to define n in terms of the initial data n_j and u

$$(1.9) \qquad n = Wn_j + L^{-1}G(u)$$

and insert the result into the solution formula for u to obtain an integrodifferential equation for u

$$(1.10) \qquad u = Su_j + K^{-1}F(u, Wn_j + L^{-1}G(u)).$$

Local well-posedness for problems of the form (1.8) often follows from a fixed point argument applied to (1.10). The fixed point analysis is carried out in a Banach space $X_{[T_j, T_{j-1}]}$ of functions defined on the spacetime slab $[T_j, T_{j+1}] \times \mathbb{R}^d$. The initial data are considered in function spaces having the unitarity property with respect to the linear solution maps

$$(1.11) \qquad \|W(t)n_0\|_{\mathcal{W}} = \|n_0\|_{\mathcal{W}}, \ \|S(t)u_0\|_{\mathcal{S}} = \|u_0\|_{\mathcal{S}}, \ \forall \, t,$$

and $X_{[T_j, T_{j+1}]} \subset C([T_j, T_{j+1}]; \mathcal{S})$. For the applications we have in mind, the length of the time interval $\Delta_j := |[T_j, T_{j+1}]|$ is chosen to be small enough to prove a contraction estimate and the smallness condition is of the form

$$\Delta_j \leq \min(\|u_j\|_{\mathcal{S}}^{-\gamma}, \|n_j\|_{\mathcal{W}}^{-\beta})$$

for certain $\gamma, \beta > 0$.

Suppose that $\|u(t)\|_{\mathcal{S}} = \|u_0\|_{\mathcal{S}}$ for all times t where solutions of (1.8) are well defined. If we iterate the local well-posedness argument, we will have successive time intervals $[T_j, T_{j+1}]$ with uniformly lower bounded lengths unless $\|n_j\|_{\mathcal{W}}$ grows without bound as we increase j. Suppose then at some time T_j we have $\|n_j\|_{\mathcal{W}} \gg \|u_j\|_{\mathcal{S}}^{\gamma/\beta}$ so that $\Delta_j = \|n_j\|_{\mathcal{W}}^{-\beta}$.

Since we have that (1.9) and (1.11) hold, any growth in $\|n(t)\|_{\mathcal{W}}$ as t moves through the time interval $[T_j, T_{j+1}]$ is due to the nonlinear influence of u upon n through the term $L^{-1}G(u)$. Therefore, an estimate of the form

(1.12) $$\|L^{-1}G(u)\|_{L^\infty_{[T_j,T_{j+1}]}\mathcal{W}} \leq \Delta_j^\delta \widetilde{G}(\|u\|_{L^\infty_{[T_j,T_{j+1}]}\mathcal{S}}) \ll \|n_j\|_{\mathcal{W}}$$

permits an iteration of the local theory. Observe that the appearance of the conserved $\mathcal{S}$ norm of u in this step suggests that we should retain this smallness property of the $\mathcal{W}$ increment of n over $[T_j, T_{j+1}]$ uniformly with respect to j. We then iterate the local well-posedness argument

$$m = O\left(\frac{\|n_j\|_{\mathcal{W}}}{\Delta^\delta \widetilde{G}(\|u_0\|_{\mathcal{S}})}\right)$$

times with time steps of uniform size $\Delta = (2\|n_j\|_{\mathcal{W}})^{-\beta}$. This extends the solution to the time interval $[T_j, T_j + m\Delta]$ with

$$m\Delta = C(\|u_0\|_{\mathcal{S}})\|n_j\|_{\mathcal{W}}^{1-\beta+\delta\beta}.$$

If $1 - \beta + \delta\beta \geq 0$, the scheme progresses to give global well-posedness for (1.8).

Implementing this abstract scheme for specific systems requires a quantification of the parameters β and δ using the local-in-time theory for the system. Notice that one way to force $1 - \beta + \delta\beta \geq 0$ is by demanding $\beta \leq 1$. But β is always bigger than 1. Still $1 - \beta + \delta\beta \geq 0$ can be greater or equal to zero because of the contribution of $\delta\beta$ term for certain $\beta > 1$ and $\delta > 0$. Calculations are required to obtain the parameters β, δ in any particular system. Unfortunately, we will often find that δ is very close to zero. Thus the condition that $1 - \beta - \delta\beta \geq 0$ fails to hold for many physical systems. Nevertheless for the Klein-Gordon-Schrödinger system the local well-posedness theory that we develop using the Strichartz's norms is sufficient for the above condition to hold. In particular we have that $\beta = 4$ and $\delta = \frac{3}{4}$ (see the proof of Theorem 1.3) and thus $1 - \beta + \delta\beta = 0$. This approach cannot be used for the Zakharov system. The main reason is that the nonlinearity $G(u)$ has two derivatives (see equation (1.1)) and the local estimates are not as generous. The idea now is to perform the contraction argument for (1.10) in a ball

$$B_{X_{[T'_j, T'_{j+1}]}} = \left\{u : \|u\|_{X_{[T'_j, T'_{j+1}]}} \leq (\Delta'_j)^\alpha \|u_0\|_{L^2}\right\}$$

where $\Delta'_j = |[T'_j, T'_{j+1}]|$. This idea is implemented here through the use of the $X^{s,b}$ spaces with $b < 1/2$. An easy consequence of this new iteration is that the local time

interval Δ'_j is larger or in other words (since $\Delta' < 1$) β is smaller. In addition when we calculate the growth of the $\|n(t)\|_{\mathcal{W}}$ norm, it takes more time for this norm to double in size. In other words the new δ' is bigger. Thus since $m' > m$ and $\Delta' > \Delta$ we have a better chance to meet the requirement of

$$m\Delta \gtrsim 1 \iff 1 - \beta + \delta\beta \geq 0.$$

The details are explained in the proof of Theorem 1.1.

Remark 1.5. *The abstract scheme described above can be applied to other evolution systems that have common features with systems (1.1) and (1.4). In particular, the scheme requires a satisfactory local well-posedness theory in a Banach space that embeds in $C([T_j, T_{j+1}]; \mathcal{S})$, with $\|u(t)\|_{\mathcal{S}} = \|u_0\|_{\mathcal{S}}$ holding true, and that the nonlinear term of the second equation is independent of n. As examples we mention the following systems.*

The initial-value problem for the coupled Schrödinger-Airy equation

(1.13)
$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha un + \beta|u|^2u & x \in \mathbb{R}, t \in \mathbb{R} \\ \partial_t n + \partial_x^3 n = \gamma\partial_x|u|^2 \\ u(x,0) = u_0(x), \ n(x,0) = n_0(x) \end{cases}$$

This system arises in the theory of capillary-gravity waves. The local well-posedness theory has been successively sharpened in [2], [3], [9], the last paper establishing local well-posedness for $(u_0, n_0) \in L^2 \times H^s$ for $-\frac{3}{4} < s \leq -\frac{1}{2}$, and in some more regular spaces. In [20], Pecher proved global well-posedness using I-method techniques for the harder Schrödinger-KdV system where the left hand side of the second equation of (1.13) includes $n\partial_x n$, with $(u_0, n_0) \in H^s \times H^s$ and $s > \frac{3}{5}$ when $\beta = 0$, and also for $s > \frac{2}{3}$ when $\beta \neq 0$ by dropping down from the $s = 1$ setting in which conservation of energy yields global well-posedness.

Our scheme also applies to the Schrödinger-Benjamin-Ono system

(1.14)
$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha un & x \in \mathbb{R}, t \in \mathbb{R} \\ \partial_t n + \nu\partial_x|\partial_x|n = \beta\partial_x|u|^2 \\ u(x,0) = u_0(x), \ n(x,0) = n_0(x) \end{cases}$$

with $\alpha, \beta, \nu \in \mathbb{R}$. This system has been studied in [3] where local well-posedness for $u_0 \in H^s$ and $n_0 \in H^{s-\frac{1}{2}}$, with $s \geq 0$ and $|\nu| \neq 1$ is established. In particular it is locally well-posed for $(u_0, n_0) \in L^2 \times H^{-1/2}$. Pecher proved [19] global well-posedness for $s > 1/3$ under the parameter constraints $\nu > 0$, $\frac{\alpha}{\beta} < 0$ and also proved local well-posedness without the restriction $|\nu| \neq 1$ but only for $s > 0$. In a forthcoming paper, we establish global well-posedness results for (1.13) and (1.14) with $|\nu| \neq 1$ for $(u_0, n_0) \in L^2 \times H^{-1/2}$.

2. Basic estimates for the group and Duhamel terms

Let $U(t) = e^{it\Delta}$ denote the free linear Schrödinger group. For the 1d wave equation, it is convenient to factor the wave operator $\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)$, and work with "reduced" components, as was done in [11]. Low frequencies in the time-derivative initial data create some minor difficulties, which we address in a manner slightly different than was done in [11]. Consider an initial data pair (n_0, n_1), and we look to solve $(\partial_t^2 - \partial_x^2)n = 0$ such that $n(0) = n_0$, $\partial_t n(0) = n_1$. Split $n_1 = n_{1L} + n_{1H}$ into low and high frequencies, and set $\hat{\nu}(\xi) = \frac{\hat{n}_{1H}(\xi)}{i\xi}$, so that $\partial_x \nu = n_{1H}$. Let

$$
\begin{aligned}
W_+(n_0, n_1)(t, x) &= \tfrac{1}{2}n_0(x - t) - \tfrac{1}{2}\nu(x - t) + \tfrac{1}{2}\int_{x-t}^{x} n_{1L}(y)\, dy \\
W_-(n_0, n_1)(t, x) &= \tfrac{1}{2}n_0(x + t) + \tfrac{1}{2}\nu(x + t) + \tfrac{1}{2}\int_{x}^{x+t} n_{1L}(y)\, dy
\end{aligned}
$$

(2.1)

so that

$$
(\partial_t \pm \partial_x)W_\pm(n_0, n_1)(t, x) = \tfrac{1}{2}n_{1L}(x)
$$
$$
W_\pm(n_0, n_1)(x, 0) = \tfrac{1}{2}n_0(x) \mp \tfrac{1}{2}\nu(x)
$$
$$
\partial_t W_\pm(n_0, n_1)(x, 0) = \mp\tfrac{1}{2}\partial_x n_0(x) + \tfrac{1}{2}\partial_x \nu(x) + \tfrac{1}{2}n_{1L}(x)
$$

and thus $n = W_+(n_0, n_1) + W_-(n_0, n_1)$ has the desired properties. We shall also use the notation $W(n_0, n_1) = W_+(n_0, n_1) + W_-(n_0, n_1)$. Let

$$
G(t)(n_0, n_1) = \cos[t(I - \Delta)^{1/2}]n_0 + \frac{\sin[t(I - \Delta)^{1/2}]}{(I - \Delta)^{1/2}}n_1
$$

be the free linear Klein-Gordon group, so that $(\partial_t^2 + (1 - \Delta))G(t)(n_0, n_1) = 0$, $G(0)(n_0, n_1) = n_0$, $\partial_t G(0)(n_0, n_1) = n_1$. Since our analysis involves tracking quantities whose size *increments*, rather than *doubles*, from one step to the next, it is imperative that we be precise about the definition of the following Sobolev norms. When we write the norm H^s, we shall mean exactly

$$
\|f\|_{H^s} = \left(\int_\xi (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\, d\xi \right)^{1/2}.
$$

Define the norm

$$
\|f\|_{A^s} = \left(\int_{|\xi| \leq 1} |\hat{f}(\xi)|^2\, d\xi + \int_{|\xi| \geq 1} |\xi|^{2s} |\hat{f}(\xi)|^2\, d\xi \right)^{1/2}
$$

Of course, $\|f\|_{A^s} \sim \|f\|_{H^s}$. Let

(2.2)
$$
\|(n_0, n_1)\|_{\mathcal{W}} = \left(\|n_0\|_{A_x^{-1/2}}^2 + \|n_1\|_{A_x^{-3/2}}^2 \right)^{1/2}.
$$

When working with a function of t, we use the shorthand $\|n(t)\|_{\mathcal{W}} = \|(n(t), \partial_t n(t))\|_{\mathcal{W}}$. In our treatment of the Zakharov system, we shall track the size of the wave component $n(t)$ in the above norm. Let

$$\|(n_0, n_1)\|_{\mathcal{G}} = \left(\|n_0\|_{L_x^2}^2 + \|n_1\|_{H_x^{-1}}^2 \right)^{1/2}. \tag{2.3}$$

Again, for functions of t, we use the shorthand $\|n(t)\|_{\mathcal{G}} = \|(n(t), \partial_t n(t))\|_{\mathcal{G}}$. In our treatment of the Klein-Gordon-Schrödinger system, we shall track the size of the wave component $n(t)$ in the above norm.

In our treatment of the Zakharov system, we shall need to work in the Bourgain spaces. We define the Schrödinger-Bourgain space $X_{0,\alpha}^S$, $\alpha \in \mathbb{R}$, by the norm

$$\|z\|_{X_{0,\alpha}^S} = \left(\iint_{\xi,\tau} \langle \tau + |\xi|^2 \rangle^{2\alpha} |\hat{z}(\xi, \tau)|^2 \, d\xi \, d\tau \right)^{1/2},$$

and the one-dimensional reduced-wave-Bourgain spaces $X_{-\frac{1}{2},\alpha}^{W\pm}$, for $\alpha \in \mathbb{R}$, as

$$\|z\|_{X_{-\frac{1}{2},\alpha}^{W\pm}} = \left(\iint_{\xi,\tau} \langle \xi \rangle^{-1} \langle \tau \pm \xi \rangle^{2\alpha} |\hat{z}(\xi, \tau)|^2 \, d\xi \, d\tau \right)^{1/2}.$$

Let $\psi \in C_0^\infty(\mathbb{R})$ satisfy $\psi(t) = 1$ on $[-1, 1]$ and $\psi(t) = 0$ outside of $[-2, 2]$. Let $\psi_T(t) = \psi(t/T)$, which will serve as a time cutoff for the Bourgain space estimates. For clarity, we write $\psi_1(t) = \psi(t)$. The following two lemmas are standard in the subject, although we are focusing attention particularly on the exponent of T in these estimates.

Lemma 2.1 (Group estimates). *Suppose $T \leq 1$.*
  (a) Schrödinger. $\|U(t)u_0\|_{C(\mathbb{R}_t; L_x^2)} = \|u_0\|_{L_x^2}$.
      If $0 \leq b_1 \leq \frac{1}{2}$, then $\|\psi_T(t)U(t)u_0\|_{X_{0,b_1}^S} \lesssim T^{\frac{1}{2}-b_1} \|u_0\|_{L^2}$.
      (Strichartz Estimates). *If $2 \leq q \leq \infty$, $2 \leq r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, excluding the case $d = 2$, $q = 2$, $r = \infty$, then $\|U(t)u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2}$*
  (b) 1-d Wave. $\|W(t)(n_0, n_1)\|_{C([0,T]; \mathcal{W}_x)} \leq (1 + T)\|(n_0, n_1)\|_{\mathcal{W}}$.
      If $0 \leq b \leq \frac{1}{2}$, $\|\psi_T(t)W_\pm(t)(n_0, n_1)\|_{X_{-\frac{1}{2},b}^{W\pm}} \lesssim T^{\frac{1}{2}-b}\|(n_0, n_1)\|_{\mathcal{W}}$.
  (c) Klein-Gordon. $\|G(t)(n_0, n_1)\|_{C(\mathbb{R}_t; \mathcal{G}_x)} = \|(n_0, n_1)\|_{\mathcal{G}}$.

Remark 2.2. *It is important that the first estimate in* (b) *and the identity in* (c) *do not have implicit constant multiples on the right-hand side, as these estimates will be used to deduce almost conservation laws. The $(1+T)$ prefactor in the first estimate of* (b) *arises from the low frequency terms. Had we made the assumption that $n_1 \in \dot{H}^{-1}$, this term could be removed and the norm $\mathcal{W}$ redefined so that equality is obtained. The $(1+T)$ prefactor will not cause trouble in our iteration since T will be selected so that $T\|(n_0, n_1)\|_{\mathcal{W}}$ functions as an increment whose size is on par with the increment arising from the Duhamel terms (see the proof of Theorem 1.1 for details).*

Proof. The Strichartz estimates quoted in (a) were established in [21] (for a more recent reference, see [13]). The first assertion in (a) is immediate by Plancherel's theorem. For the second assertion in (a), we note that $[\psi_T(t)U(t)u_0]\widehat{\ }(\xi, \tau) = (\psi_T)\widehat{\ }(\tau + \xi^2)\hat{u}_0(\xi)$, and consequently

$$\|\psi_T(t)U(t)u_0\|_{X_{0,b_1}^S} \leq c\|\psi_T\|_{H^{b_1}}\|u_0\|_{L^2}$$

To complete the proof of the estimate, we note that

$$\|\psi_T\|_{H^{b_1}} \leq \|\psi_T\|_{L^2} + \|\psi_T\|_{\dot{H}^{b_1}} = T^{\frac{1}{2}}\|\psi_1\|_{L^2} + T^{\frac{1}{2}-b_1}\|\psi_1\|_{\dot{H}^{b_1}}$$

by scaling.

For the first assertion in (b), let $f(x,t)$ solve the linear wave equation

$$\partial_t^2 f - \partial_x^2 f = 0 \tag{2.4}$$

with initial data $f(x,0) = n_0(x)$, $\partial_t f(x,0) = n_1(x)$. Let P_H be the projection onto frequencies $|\xi| \geq 1$, and P_L be the projection onto frequencies $|\xi| \leq 1$. Let $D^{-3/2}$ be the multiplier operator with symbol $|\xi|^{-3/2}$. By applying $D^{-3/2}P_H$ to (2.4), multiplying by $D^{-3/2}P_H\partial_t f$ and integrating in x, we obtain the conservation identity

$$\|P_H f(T)\|^2_{A_x^{-1/2}} + \|\partial_t P_H f(T)\|^2_{A_x^{-3/2}} = \|P_H n_0\|^2_{A^{-1/2}} + \|P_H n_1\|^2_{A^{-3/2}}. \tag{2.5}$$

To obtain low frequency estimates, we work directly from the explicit formula

$$f(x,t) = \tfrac{1}{2}n_0(x+t) + \tfrac{1}{2}n_0(x-t) + \tfrac{1}{2}\int_{x-t}^{x+t} n_1(y)\,dy. \tag{2.6}$$

By applying P_L and then directly estimating, we obtain

$$\|P_L f(T)\|_{L_x^2} \leq \|P_L n_0\|_{L^2} + T\|P_L n_1\|_{L^2}. \tag{2.7}$$

After applying ∂_t to (2.6), it can be rewritten as

$$\partial_t f(x,t) = \tfrac{1}{2}\int_{x-t}^{x+t} \partial^2 n_0(y)\,dy + \tfrac{1}{2}n_1(x+t) + \tfrac{1}{2}n_1(x-t).$$

Applying P_L and then directly estimating, we obtain

$$\|P_L \partial_t f(T)\|_{L_x^2} \leq T\|P_L n_0\|_{L^2} + \|P_L n_1\|_{L^2} \tag{2.8}$$

Combining (2.5), (2.7), and (2.8), we obtain the claim.

The second part of (b) is proved similarly to the second part of (a).

For (c), let $f(t,x)$ solve

$$\partial_t^2 f - \Delta f + f = 0 \tag{2.9}$$

with initial data $(f(0), \partial_t f(0)) = (n_0, n_1)$. Let E be the multiplier operator with symbol $(1+|\xi|^2)^{-1/2}$. Apply E to (2.9), then multiply by $\partial_t E f$, and finally integrate in x to obtain the asserted conservation law. $\square$

Let

$$U *_R z(t,x) = \int_0^t U(t-t')z(t',x)\,dt'$$

denote the Duhamel operator corresponding to the Schrödinger operator, so that $(i\partial_t + \Delta)U *_R z(t,x) = iz(t,x)$, $U *_R z(0,x) = 0$. Let

$$W_\pm *_R z(t,x) = \tfrac{1}{2}\int_0^t z(t-s, x\mp s)\,ds$$

so that

$$(\partial_t \pm \partial_x)W_\pm *_R z(t,x) = \tfrac{1}{2}z(t,x)$$

$$W_\pm *_R z(0,x) = 0, \quad \partial_t W_\pm *_R z(0,x) = \tfrac{1}{2}z(0,x).$$

It follows that if we set $n = W_+ *_R z - W_- *_R z$, then $(\partial_t^2 - \partial_x^2)n = \partial_x z$ and $n(0,x) = 0$, $\partial_t n(0,x) = 0$, so we define

(2.10) $$W *_R z = W_+ *_R z - W_- *_R z.$$

For the Klein-Gordon equation, let

$$G *_R z(t,x) = \int_0^t \frac{\sin[(t-t')(I-\Delta)^{1/2}]}{(I-\Delta)^{1/2}} z(t',x)\,dt'$$

so that $(\partial_t^2 + (I-\Delta))G *_R z = z$, $G *_R z(0,x) = 0$, $\partial_t G *_R z(0,x) = 0$.

Lemma 2.3 (Duhamel estimates). *Suppose $T \le 1$.*

- (a) Schrödinger. *If $0 \le c_1 < \tfrac{1}{2}$, then $\|U *_R z\|_{C([0,T];L_x^2)} \lesssim T^{\frac{1}{2}-c_1}\|z\|_{X_{0,-c_1}^S}$.*
  If $0 \le c_1 < \tfrac{1}{2}$, $0 \le b_1$, $b_1 + c_1 \le 1$, then $\|\psi_T U *_R z\|_{X_{0,b_1}^S} \lesssim T^{1-b_1-c_1}\|z\|_{X_{0,-c_1}^S}$.
  (Strichartz Estimates). If $2 \le q \le \infty$, $2 \le r \le \infty$, $\tfrac{2}{q} + \tfrac{d}{r} = \tfrac{d}{2}$ excluding the case $d = 2$, $q = 2$, $r = \infty$, and similarly for $\tilde{q}, \tilde{r}$, then $\|U *_R z\|_{C([0,T];L_x^2)} + \|U *_R z\|_{L_{[0,T]}^q L_x^r} \lesssim \|z\|_{L_{[0,T]}^{\tilde{q}'} L_x^{\tilde{r}'}}$, where $'$ indicates the Hölder dual exponent ($\tfrac{1}{p} + \tfrac{1}{p'} = 1$).
- (b) 1-d Wave. *If $0 \le c < \tfrac{1}{2}$, then $\|W *_R z\|_{C([0,T];\mathcal{W}_x)} \lesssim T^{\frac{1}{2}-c}\|z\|_{X_{-\frac{1}{2},-c}^{W\pm}}$.*
  If $0 \le c < \tfrac{1}{2}$, $0 \le b$, $b+c \le 1$, then $\|\psi_T W_\pm *_R z\|_{X_{-\frac{1}{2},b}^{W\pm}} \lesssim T^{1-b-c}\|z\|_{X_{-\frac{1}{2},-c}^{W\pm}}$.
- (c) Klein-Gordon. *$\|G *_R z\|_{C([0,T];\mathcal{G}_x)} \lesssim \|z\|_{L_{[0,T]}^1 H_x^{-1}}$.*

Proof. The second assertion in each of (a) and (b) is [11] Lemma 2.1(ii). For the Stricharz estimates quoted in (a), see [21][13]. We next establish the first part of (a). We begin by establishing the bound

(2.11) $$\|\psi_T(t)U *_R z(x,t)\|_{L_t^\infty L_x^2} \le cT^{\frac{1}{2}-c_1}\|z\|_{X_{0,-c_1}^S}.$$

Let $f_\xi(t) = e^{it\xi^2}\hat{z}(\xi, t)$, where $\hat{\ }$ denotes the Fourier transform in the x-variable only. We have

$$\|\psi_T(t)U *_R z(x, t)\|_{L_t^\infty L_x^2}$$

$$= \left\|\psi_T(t)\int_0^t f_\xi(t')dt'\right\|_{L_t^\infty L_\xi^2}$$

(2.12)
$$\leq \left\|\psi_T(t)\int_0^t f_\xi(t')dt'\right\|_{L_\xi^2 L_t^\infty}.$$

Below we shall show that for a function $f(t)$ of the t-variable alone, we have the estimate

(2.13)
$$\left\|\psi_T(t)\int_0^t f(t')dt'\right\|_{L_t^\infty} \leq cT^{\frac{1}{2}-c_1}\|f\|_{H^{-c_1}}.$$

Assuming this, then it follows from (2.12) that

$$\|\psi_T(t)U *_R z(x, t)\|_{L_t^\infty L_x^2} \leq cT^{\frac{1}{2}-c_1}\|\,\|f_\xi\|_{H_t^{-c_1}}\|_{L_\xi^2} = cT^{\frac{1}{2}-c_1}\|z\|_{X_{0,-c_1}^S}$$

completing the proof of (2.11). Now we show (2.13). Break $f(t) = f_+(t) + f_-(t)$ where $\hat{f}_-(\tau) = \chi_{|\tau|<\frac{1}{T}}\hat{f}(\tau)$ and $\hat{f}_+(\tau) = \chi_{|\tau|>\frac{1}{T}}\hat{f}(\tau)$. Then for f_-, we have

$$\left\|\psi_T(t)\int_0^t f_-(s)\,ds\right\|_{L_t^\infty} \leq T^{1/2}\|f_-\|_{L^2} \leq T^{\frac{1}{2}-c_1}\|f_-\|_{H^{-c_1}}.$$

We compute

$$\int_0^t f_+(s)\,ds = (\chi_{[-t,0]} * f_+)(0)$$

$$= \frac{1}{2\pi}\int (\chi_{[-t,0]} * f_+)\widehat{\ }(\sigma)\,d\sigma$$

$$= \frac{1}{2\pi}\int_\sigma \frac{1 - e^{-it\sigma}}{i\sigma}\hat{f}_+(\sigma)\,d\sigma$$

and hence

$$\left[\psi_T(t)\int_0^t f_+(s)\,ds\right]\widehat{\ }(\tau) = \int_\sigma \frac{T\hat{\psi}(T(\tau - \sigma)) - T\hat{\psi}(T\tau)}{\sigma}\hat{f}_+(\sigma)\,d\sigma.$$

Thus,

$$\left\| \psi_T(t) \int_0^t f_+(s)\,ds \right\|_{L_t^\infty}$$

$$\leq \|T\hat{\psi}(T\tau)\|_{L_\tau^1} \left\| \frac{\hat{f}_+(\tau)}{\tau} \right\|_{L_\tau^1}$$

$$\leq \left(\int_{|\tau| > \frac{1}{T}} \frac{d\tau}{|\tau|^{2-2c_1}} \right)^{1/2} \|f\|_{H^{-c_1}}$$

$$\leq T^{\frac{1}{2}-c_1} \|f\|_{H^{-c_1}}$$

establishing (2.13). It remains only to show continuity, i.e. that for a fixed $z \in X_{0,-c_1}^S$ and each $\epsilon > 0$, there is $\delta = \delta(\epsilon, z) > 0$ such that if $|t_2 - t_1| < \delta$, then

$$\|\psi_T U *_R z(x, t_2) - \psi_T U *_R z(x, t_1)\|_{L_x^2} < \epsilon.$$

By an $\epsilon/3$ argument appealing to (2.11), it suffices to establish this statement for z belonging to the dense class $\mathcal{S}(\mathbb{R}^2) \subset X_{0,-c_1}^S$. However, if $z \in \mathcal{S}(\mathbb{R}^2)$, we have $\partial_t(U *_R z) = z + i\Delta(U *_R z)$ and the fundamental theorem of calculus and (2.11) imply that

$$\|U *_R z(\cdot, t_2) - U *_R z(\cdot, t_1)\|_{L_x^2} \leq c(t_2 - t_1)(\|z\|_{L_t^\infty L_x^2} + \|\Delta z\|_{X_{0,-c_1}^S}).$$

The proof of the first assertion of (b) proceeds in analogy to the above proof, first establishing the bound

$$\|W_\pm *_R z\|_{L_t^\infty H_x^{-1/2}} \lesssim T^{\frac{1}{2}-c} \|z\|_{X_{-\frac{1}{2},-c}^{W_\pm}}.$$

The continuity statement is deduced by a density argument as in the previous paragraph, and finally the bound as stated on $W *_R z$ follows by the identity

$$\partial_t W *_R z = \partial_x W_+ *_R z + \partial_x W_- *_R z.$$

The proof of (c) follows from an application of Minkowskii's integral inequality, with the continuity statement deduced by a density argument as in the previous paragraph. $\qquad\square$

3. 1-D ZAKHAROV SYSTEM

In this section, we prove Theorem 1.1. We shall make use of the conservation law (1.2) to control the growth of $u(t)$ from one local time step to the next. We track the growth of $n(t)$ in the norm $\mathcal{W}$ defined in (2.2) using the estimates from the local theory. We now state the needed estimates from the local theory of [11].

Lemma 3.1 (Multilinear estimates).

(a) *If* $\frac{1}{4} < b_1, c_1, b < \frac{1}{2}$ *and* $b + b_1 + c_1 \geq 1$, *then* $\|n_\pm u\|_{X_{0,-c_1}^S} \lesssim \|n_\pm\|_{X_{-\frac{1}{2},b}^{W_\pm}} \|u\|_{X_{0,b_1}^S}$.

(b) If $\frac{1}{4} < b_1, c < \frac{1}{2}$ and $2b_1 + c \geq 1$, then $\|\partial_x(u_1 \bar{u}_2)\|_{X^{W_\pm}_{-\frac{1}{2},-c}} \lesssim \|u_1\|_{X^S_{0,b_1}} \|u_2\|_{X^S_{0,b_1}}$.

We remark that we can simultaneously achieve both optimal conditions $b + b_1 + c_1 = 1$ and $2b_1 + c = 1$, for example by taking all four indices $b = b_1 = c = c_1 = \frac{1}{3}$.

Proof. (a) is the case $k = 0$, $\ell = -\frac{1}{2}$ in [11] Lemma 4.3, and (b) is the case $k = 0$, $l = -\frac{1}{2}$ in [11] Lemma 4.4. The assumptions $b + c_1 > \frac{3}{4}$, $b + b_1 > \frac{3}{4}$ for Lemma 4.3 and $b_1 + c > \frac{3}{4}$ for Lemma 4.4 that appear in [11] are not needed and we only have the requirements $b + b_1 + c_1 \geq 1$ for Lemma 4.3 and $2b_1 + c \geq 1$ for Lemma 4.4. The reason is that equation (4.30) in [11] on p. 424 is finite even if $\alpha_1 < \frac{1}{2}$ since the range of integration is finite[4] (from 0 to $\xi_1^2/4$). Because relaxing this condition is essential to our method, we have included these proofs in the appendix so that they can be examined by the reader. $\qquad\square$

Proof of Theorem 1.1. As discussed above, we can reduce the wave component $n = n_+ + n_-$ and recast (1.1) as

(3.1)
$$\begin{cases} i\partial_t u + \partial_x^2 u = (n_+ + n_-)u & x \in \mathbb{R}, t \in \mathbb{R} \\ (\partial_t \pm \partial_x)n_\pm = \pm\frac{1}{2}\partial_x|u|^2 + \frac{1}{2}n_{1L} \end{cases}$$

which has the integral equation formulation

$$u(t) = U(t)u_0 - iU *_R [(n_+ + n_-)u](t)$$
$$n_\pm(t) = W_\pm(t)(n_0, n_1) \pm W_\pm *_R (\partial_x|u|^2)(t).$$

Fix $0 < T < 1$, and consider the maps Λ_S, $\Lambda_{W\pm}$

(3.2)
$$\Lambda_S(u, n_\pm) = \psi_T U u_0 + \psi_T U *_R [(n_+ + n_-)u]$$

(3.3)
$$\Lambda_{W\pm}(u) = \psi_T W_\pm(n_0, n_1) \pm \psi_T W_\pm *_R (\partial_x|u|^2).$$

We seek a fixed point $(u(t), n_\pm(t)) = (\Lambda_S(u, n_\pm), \Lambda_{W\pm}(u))$. Estimating (3.2) in X^S_{0,b_1}, applying the first estimates in Lemma 2.1(a), 2.3(a) and following through with Lemma 3.1(a); and estimating (3.3) in $X^{W\pm}_{-\frac{1}{2},b}$, applying the first estimates in Lemma 2.1(b), 2.3(b) and following through with Lemma 3.1(b), we obtain

$$\|\Lambda_S(u, n_\pm)\|_{X^S_{0,b_1}} \lesssim T^{\frac{1}{2}-b_1}\|u_0\|_{L^2} + T^{1-b_1-c_1}\|n_\pm\|_{X^{W\pm}_{-\frac{1}{2},b}}\|u\|_{X^S_{0,b_1}}$$

$$\|\Lambda_{W\pm}(u)\|_{X^{W\pm}_{-\frac{1}{2},b}} \lesssim T^{\frac{1}{2}-b}\|(n_0, n_1)\|_{\mathcal{W}} + T^{1-b-c}\|u\|^2_{X^S_{0,b_1}}$$

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[4]This comment applies in the $k = 0$, $\ell = -\frac{1}{2}$ setting but perhaps not in the general setting in which Lemmas 4.3,4.4 are stated.

and also

$$\|\Lambda_S(u_1, n_{1\pm}) - \Lambda_S(u_2, n_{2\pm})\|_{X_{0,b_1}^S}$$
$$\lesssim T^{1-b_1-c_1}(\|n_{1\pm}\|_{X_{-\frac{1}{2},b}^{W\pm}}\|u_1 - u_2\|_{X_{0,b_1}^S} + \|n_{1\pm} - n_{2\pm}\|_{X_{-\frac{1}{2},b}^{W\pm}}\|u_2\|_{X_{0,b_1}^S})$$
$$\|\Lambda_{W\pm}(u_1) - \Lambda_{W\pm}(u_2)\|_{X_{-\frac{1}{2},b}^{W\pm}} \lesssim T^{1-b-c}(\|u_1\|_{X_{0,b_1}^S} + \|u_2\|_{X_{0,b_1}^S})\|u_1 - u_2\|_{X_{0,b_1}^S}.$$

By taking T such that

$$T^{\frac{3}{2}-2b_1-c_1}\|u_0\|_{L^2} \lesssim 1, \qquad T^{\frac{3}{2}-b-b_1-c}\|u_0\|_{L^2} \lesssim 1$$

(3.4)
$$T^{\frac{3}{2}-b-b_1-c_1}\|(n_0, n_1)\|_{\mathcal{W}} \lesssim 1$$

(3.5)
$$T^{\frac{3}{2}-2b_1-c}\|u_0\|_{L^2}^2 \lesssim \|(n_0, n_1)\|_{\mathcal{W}}$$

one obtains sufficient conditions for a contraction argument yielding the existence of a fixed point $u \in X_{0,b_1}^S$, $n_\pm \in X_{-\frac{1}{2},b}^{W\pm}$ of (3.2)-(3.3) such that

(3.6)
$$\|u\|_{X_{0,b_1}^S} \lesssim T^{\frac{1}{2}-b_1}\|u_0\|_{L^2}, \qquad \|n_\pm\|_{X_{-\frac{1}{2},b}^{W\pm}} \lesssim T^{\frac{1}{2}-b}\|(n_0, n_1)\|_{\mathcal{W}}.$$

Similarly estimating (3.2) in $C([0,T]; L_x^2)$ by applying Lemmas 2.1(a),2.3(a) and (3.6) shows that in fact $u \in C([0,T]; L_x^2)$. We may therefore invoke the conservation law (1.2) to conclude $\|u(T)\|_{L_x^2} = \|u_0\|_{L^2}$ and thus are concerned only with the possibility of growth in $\|n(t)\|_{\mathcal{W}}$ from one time step to the next. Suppose that after some number of iterations we reach a time where $\|n(t)\|_{\mathcal{W}} \gg \|u(t)\|_{L_x^2}^2 = \|u_0\|_{L_x^2}^2$. Take this time position as the initial time $t = 0$ so that $\|u_0\|_{L^2}^2 \ll \|(n_0, n_1)\|_{\mathcal{W}}$. Then (3.5) is automatically satisfied and by (3.4), we may select a time increment of size

(3.7)
$$T \sim \|(n_0, n_1)\|_{\mathcal{W}}^{-1/(\frac{3}{2}-b-b_1-c_1)} = \|(n_0, n_1)\|_{\mathcal{W}}^{-2}$$

where the right-hand side follows by selecting the optimal condition $b + b_1 + c_1 = 1$ in Lemma 3.1(a). Since

$$n = W(n_0, n_1) + W *_R (\partial_x|u|^2)$$

we can apply Lemma 2.1(b), 2.3(b) and follow through with (3.6) to obtain

$$\|n(T)\|_{\mathcal{W}} \le (1+T)\|(n_0, n_1)\|_{\mathcal{W}} + CT^{\frac{3}{2}-(2b_1+c)}\|u_0\|_{L^2}^2$$
$$\le \|(n_0, n_1)\|_{\mathcal{W}} + CT^{\frac{1}{2}}(\|u_0\|_{L^2}^2 + 1)$$

where C is some fixed constant. The second line above follows by selecting the optimal condition $2b_1+c = 1$ in Lemma 3.1(b), and using (3.7) to obtain $T\|(n_0, n_1)\|_{\mathcal{W}} \le CT^{\frac{1}{2}}$. From this we see that we can carry out m iterations on time intervals, each of length (3.7), where

(3.8)
$$m \sim \frac{\|(n_0, n_1)\|_{\mathcal{W}}}{T^{\frac{1}{2}}(\|u_0\|_{L^2}^2 + 1)}$$

before the quantity $\|n(t)\|_{\mathcal{W}}$ doubles. The total time we advance after these m iterations, by (3.7) and (3.8), is

$$mT \sim \frac{1}{\|u_0\|_{L^2}^2 + 1}$$

which is independent of $\|n(t)\|_{\mathcal{W}}$.

We can now repeat this entire procedure, each time advancing a time of length $\sim (\|u_0\|_{L^2}^2 + 1)^{-1}$ (independent of the size of $\|n(t)\|_{\mathcal{W}}$). Upon each repetition, the size of $\|n(t)\|_{\mathcal{W}}$ will at most double, giving the exponential-in-time upper bound stated in Theorem 1.1.

$\square$

4. 3-D Klein-Gordon Schrödinger system

The goal of this section is to prove Theorem 1.3. For the Klein-Gordon-Schrödinger system (1.4), no special multilinear estimates are needed. Instead, we will work in standard space-time norms and use Sobolev imbedding and the Hölder inequality.

We shall use the conservation law (1.5) to control the growth of $u(t)$ from one time step to the next and track the growth of $\|n(t)\|_{\mathcal{G}}$, where the $\mathcal{G}$ norm was defined in (2.3), by direct estimation. For expositional convenience, we restrict to dimension $d = 3$ although similar results do hold for $d = 1$ and $d = 2$.

Proof of Theorem 1.3. (1.4) has the integral equation formulation

$$u(t) = U(t)u_0 + iU *_R [nu](t)$$
$$n(t) = G(t)(n_0, n_1) + G *_R (|u|^2)(t).$$

Define the maps Λ_S, Λ_G as

(4.1) $$\Lambda_S(u, n) = Uu_0 + iU *_R [nu]$$

(4.2) $$\Lambda_G(u) = G(n_0, n_1) + G *_R (|u|^2).$$

Let $\text{Str} = L_{[0,T]}^{10/3} L_x^{10/3} \cap L_{[0,T]}^8 L_x^{12/5}$. We seek a fixed point $(u(t), n(t)) = (\Lambda_S(u, n), \Lambda_G(u))$ in the space $[C([0,T]; L_x^2) \cap \text{Str}] \times C([0,T]; \mathcal{G}_x)$. Apply Lemma 2.3(a) with $(\tilde{q}, \tilde{r}) = (\frac{20}{9}, 5)$ for $d = 3$ to obtain

$$\|\Lambda_S(u, n)\|_{C([0,T]; L_x^2) \cap \text{Str}} \lesssim \|u_0\|_{L^2} + T^{1/4} \|n\|_{L_{[0,T]}^\infty L_x^2} \|u\|_{L_{[0,T]}^{\frac{10}{3}} L_x^{\frac{10}{3}}}.$$

Estimate (4.2) in $C([0,T]; \mathcal{G}_x)$ and apply Lemma 2.3(c) followed by Sobolev imbedding to obtain

(4.3) $$\|\Lambda_G(u)\|_{C([0,T]; \mathcal{G}_x)} \leq \|(n_0, n_1)\|_{\mathcal{G}} + cT^{3/4} \|u\|_{L_{[0,T]}^8 L_x^{12/5}}^2$$

where we estimated as: $\||u|^2\|_{L_{[0,T]}^1 H_x^{-1}} \lesssim \||u|^2\|_{L_{[0,T]}^1 L_x^{6/5}} \leq T^{3/4} \|u\|_{L_{[0,T]}^8 L_x^{12/5}}^2$. There are similar estimates for the differences $\Lambda_S(u_1, n_1) - \Lambda_S(u_2, n_2)$ and $\Lambda_G(u_1) - \Lambda_G(u_2)$.

If T is such that

$$T^{1/4}\|u_0\|_{L^2} \lesssim 1$$

(4.4)
$$T^{1/4}\|(n_0, n_1)\|_{\mathcal{G}} \lesssim 1$$

(4.5)
$$T^{3/4}\|u_0\|_{L^2}^2 \lesssim \|(n_0, n_1)\|_{\mathcal{G}}$$

then a contraction argument implies there is a solution (u, n) to (1.4) on $[0, T]$ such that

(4.6)
$$\|u\|_{C([0,T];L_x^2) \cap \mathrm{Str}} \lesssim \|u_0\|_{L^2}$$

(4.7)
$$\|n\|_{C([0,T];\mathcal{G}_x)} \leq \|(n_0, n_1)\|_{\mathcal{G}} + cT^{3/4}\|u_0\|_{L^2}^2.$$

By the conservation of mass, we have $\|u(t)\|_{L_x^2} = \|u_0\|_{L^2}$, and are thus concerned only with the possibility that $\|n(t)\|_{\mathcal{G}}$ grows excessively from one local increment to the next. Suppose that after some number of iterations $\|n(t)\|_{\mathcal{G}} \gg \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2$. Consider this time as the initial time so that $\|(n_0, n_1)\|_{\mathcal{G}} \gg \|u_0\|_{L^2}^2$. Then (4.5) is automatically satisfied, and by (4.4), we may thus take

(4.8)
$$T \sim \|(n_0, n_1)\|_{\mathcal{G}}^{-4}.$$

We see from (4.7) that, after m iterations, each of size (4.8), where

$$m \sim \frac{\|(n_0, n_1)\|_{\mathcal{G}}}{T^{3/4}\|u_0\|_{L^2}^2}$$

the quantity $\|n(t)\|_{\mathcal{G}}$ at most doubles. The total time advanced after these m iterations is

$$mT \sim \frac{1}{\|u_0\|_{L^2}^2}.$$

We can now repeat this entire procedure, each time advancing a time of length $\sim \|u_0\|_{L^2}^{-2}$ (independent of the size of $\|n(t)\|_{\mathcal{G}}$). Upon each repetition, the size of $\|n(t)\|_{\mathcal{G}}$ will at most double, giving the exponential-in-time upper bound stated in Theorem 1.3.

$$\square$$

Appendix A. Proof of the multilinear estimates (expository)

In this section, we prove Lemma 3.1. The material here is taken from [11] Lemma 4.3, 4.4 with only a slight modification at one stage. This modification was described in a note under the heading "proof" following the statement of Lemma 3.1. Given its importance in our scheme, the full proof is included here in detail.

We need the calculus lemmas:

Lemma A.1 ([11] Lemma 4.1). *Let $f \in L^q(\mathbb{R})$, $g \in L^{q'}(\mathbb{R})$ for $1 \leq q, q' \leq \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Assume that f, g are nonnegative, even, and nonincreasing for positive argument. Then $f * g$ enjoys the same properties.*

Define

$$[\lambda]_+ = \begin{cases} \lambda & \text{if } \lambda > 0 \\ \epsilon & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda < 0 \end{cases}$$

Lemma A.2 ([11] Lemma 4.2). *Let* $0 \leq a_- \leq a_+$ *and* $a_+ + a_- > \frac{1}{2}$. *Then* $\forall\, s \in \mathbb{R}$,

$$\int_y \langle y - s \rangle^{-2a_+} \langle y + s \rangle^{-2a_-}\, dy \leq c\langle s \rangle^{-\alpha}$$

where $\alpha = 2a_- - [1 - 2a_+]_+$.

Proof of Lemma 3.1(a). We shall only do the $+$ case. The estimate is equivalent to

$$|S| \leq c\|v\|_2 \|v_1\|_2 \|v_2\|_2$$

where

(A.1) $$S = \int_* \frac{\hat{v}\hat{v}_1\hat{v}_2\langle \xi \rangle^{1/2}}{\langle \sigma \rangle^b \langle \sigma_1 \rangle^{c_1} \langle \sigma_2 \rangle^{b_1}}$$

with $\hat{v} = \hat{v}(\xi, \tau)$, $\hat{v}_1 = \hat{v}_1(\xi_1, \tau_1)$, $\hat{v}_2 = \hat{v}_2(\xi_2, \tau_2)$, $\sigma_1 = \tau_1 + \xi_1^2$, $\sigma_2 = \tau_2 + \xi_2^2$, $\sigma = \tau + \xi$, and $*$ indicates the restriction $\xi_1 = \xi + \xi_2$, $\tau_1 = \tau + \tau_2$. Indeed, for $\hat{v}_1 \in L^2$,

$$\int_{\xi_1, \tau_1} \widehat{n_+ u}(\xi_1, \tau_1)\langle \sigma_1 \rangle^{-c_1} \hat{v}_1(\xi_1, \tau_1)\, d\xi_1 d\tau_1$$

$$= \int_{\xi_1, \tau_1} \left[\int_{\substack{\xi_1 = \xi + \xi_2 \\ \tau_1 = \tau + \tau_2}} \hat{n}_+(\xi, \tau)\hat{u}(\xi_2, \tau_2) \right] \langle \sigma_1 \rangle^{-c_1} \hat{v}_1(\xi_1, \tau_1) d\xi_1 d\tau_1.$$

Let $\hat{v}(\xi, \tau) = \hat{n}_+(\xi, \tau)\langle \xi \rangle^{-1/2}\langle \sigma \rangle^b$ and $\hat{v}_2(\xi_2, \tau_2) = \hat{u}(\xi_2, \tau_2)\langle \sigma_2 \rangle^{b_1}$ to obtain (A.1).
  We note here that

$$\sigma_1 - \sigma - \sigma_2 = \tau_1 + \xi_1^2 - (\tau + \xi) - (\tau_2 + \xi_2^2)$$
$$= \xi_1^2 - \xi - \xi_2^2$$
$$= (\xi_1 - \tfrac{1}{2})^2 - (\xi_2 - \tfrac{1}{2})^2$$

In the case when $|\xi| \leq 1$, it suffices to estimate

$$\int_{\xi_1,\tau_1}\int_{\substack{\xi_1=\xi+\xi_2\\\tau_1=\tau+\tau_2}}\frac{\hat{v}\hat{v}_1\hat{v}_2}{\langle\sigma\rangle^{1/4}\langle\sigma_1\rangle^{1/4+}\langle\sigma_2\rangle^{1/4+}}$$

$$=\int_{\xi_1,\tau_1}\left(\frac{\hat{v}_1}{\langle\sigma_1\rangle^{1/4+}}\right)\left(\int_{\substack{\xi_1=\xi+\xi_2\\\tau_1=\tau+\tau_2}}\frac{\hat{v}\hat{v}_2}{\langle\sigma\rangle^{1/4}\langle\sigma_2\rangle^{1/4+}}\,d\xi_2d\tau_2\right)d\xi_1d\tau_1$$

$$=\int_{x_1,t_1}\left[\frac{\hat{v}_1}{\langle\sigma_1\rangle^{1/4+}}\right]^{\vee}\left[\int_{\substack{\xi=\xi_1+\xi_2\\\tau=\tau_1+\tau_2}}\frac{\hat{v}\hat{v}_2}{\langle\sigma\rangle^{1/4}\langle\sigma_2\rangle^{1/4+}}\,d\xi_2d\tau_2\right]^{\vee}$$

$$=\int_{x,t}\left[\frac{\hat{v}_1}{\langle\sigma_1\rangle^{1/4+}}\right]^{\vee}\left[\frac{\hat{v}}{\langle\sigma\rangle^{1/4}}\right]^{\vee}\left[\frac{\hat{v}_2}{\langle\sigma_2\rangle^{1/4+}}\right]^{\vee}$$

$$\leq\left\|\left[\frac{\hat{v}_1}{\langle\sigma_1\rangle^{1/4+}}\right]^{\vee}\right\|_{L_t^{8/3}L_x^4}\left\|\left[\frac{\hat{v}}{\langle\sigma\rangle^{1/4}}\right]^{\vee}\right\|_{L_t^4L_x^2}\left\|\left[\frac{\hat{v}_2}{\langle\sigma_2\rangle^{1/4+}}\right]^{\vee}\right\|_{L_t^{8/3}L_x^4}.$$

Now

$$\left\|\left[\frac{\hat{v}}{\langle\sigma\rangle^{1/4}}\right]^{\vee}\right\|_{L_t^4L_x^2}=\left\|\int_\xi e^{ix\xi}\int_\tau e^{it\tau}\frac{\hat{v}(\xi,\tau)}{\langle\tau+\xi\rangle^{1/4}}\,d\tau\,d\xi\right\|_{L_t^4L_x^2}$$

$$=\left\|\int_\tau e^{it\tau}\frac{\hat{v}(\xi,\tau)}{\langle\tau+\xi\rangle^{1/4}}\,d\tau\right\|_{L_t^4L_\xi^2}$$

$$\leq\left\|\int_\tau e^{it\tau}\frac{\hat{v}(\xi,\tau)}{\langle\tau+\xi\rangle^{1/4}}\,d\tau\right\|_{L_\xi^2L_t^4}\quad\text{by Minkowskii}$$

$$\leq\left\|\int_\tau e^{it\tau}\frac{\hat{v}(\xi,\tau-\xi)}{\langle\tau\rangle^{1/4}}\,d\tau\right\|_{L_\xi^2L_t^4}$$

$$\leq\left\|\int_\tau e^{it\tau}\hat{v}(\xi,\tau-\xi)\,d\tau\right\|_{L_\xi^2L_t^2}\quad\text{by Sobolev}$$

$$=\|\hat{v}\|_{L_\xi^2L_\tau^2}=\|v\|_{L^2}.$$

For $j=1,2$, the estimate on

$$\left\|\left[\frac{\hat{v}_j}{\langle\sigma_j\rangle^{1/4+}}\right]^{\vee}\right\|_{L_t^{8/3}L_x^4}$$

is obtained by interpolating halfway between $\|[\langle\sigma_j\rangle^{-d}\hat{v}_j]^{\vee}\|_{L_t^4L_x^\infty}\leq\|v_j\|_{L_t^2L_x^2}$ for $d>\frac{1}{2}$ (Strichartz) and $\|[\hat{v}_j]^{\vee}\|_{L_t^2L_x^2}=\|v_j\|_{L_t^2L_x^2}$. This leaves us to estimate

(A.2)
$$S'=\int_*\frac{\hat{v}\hat{v}_1\hat{v}_2|\xi|^{1/2}}{\langle\sigma\rangle^b\langle\sigma_1\rangle^{c_1}\langle\sigma_2\rangle^{b_1}}$$

with $\hat{v} = \hat{v}(\xi, \tau)$, $\hat{v}_1 = \hat{v}_1(\xi_1, \tau_1)$, $\hat{v}_2 = \hat{v}_2(\xi_2, \tau_2)$, $\sigma_1 = \tau_1 + \xi_1^2$, $\sigma_2 = \tau_2 + \xi_2^2$, $\sigma = \tau + \xi$, and $*$ indicates the restriction $\xi_1 = \xi + \xi_2$, $\tau_1 = \tau + \tau_2$.

Region σ dominant, $|\sigma| \geq \max(|\sigma_1|, |\sigma_2|)$.

$$S' \leq \left(\int_{\xi,\sigma} |\hat{v}|^2 \right)^{1/2} \left(\int_{\xi,\sigma} \langle\sigma\rangle^{-2b} \left| \int_{\xi_2,\sigma_2} \frac{|\hat{v}_1 \hat{v}_2||\xi|^{1/2}}{\langle\sigma_1\rangle^{c_1}\langle\sigma_2\rangle^{b_1}} \right|^2 \right)^{1/2}$$

$$\leq \left(\int_{\xi,\sigma} |\hat{v}|^2 \right)^{1/2} \left(\int_{\xi,\sigma} \langle\sigma\rangle^{-2b} \left(\int_{\xi_2,\sigma_2} |\hat{v}_1\hat{v}_2|^2 \right) \left(\int_{\xi_2,\sigma_2} \frac{|\xi|}{\langle\sigma\rangle^{2c_1}\langle\sigma_2\rangle^{2b_1}} \right) \right)^{1/2}$$

$$(A.3) \qquad \leq \left(\sup_{\sigma,\xi} \langle\sigma\rangle^{-2b} \int_{\sigma_2} \int_{\xi_2} \frac{|\xi|}{\langle\sigma_1\rangle^{2c_1}\langle\sigma_2\rangle^{2b_1}} d\xi_2 d\sigma_2 \right)^{1/2} \|v\|_2 \|v_1\|_2 \|v_2\|_2.$$

The inner integral is taken over fixed σ, ξ, σ_2. Since $\xi_1 = \xi + \xi_2$, we have $d\xi_1 = d\xi_2$, and since $\sigma_1 - \sigma - \sigma_2 = (\xi_1 - \frac{1}{2})^2 - (\xi_2 - \frac{1}{2})^2$, we have

$$d\sigma_1 = [+2(\xi_1 - \tfrac{1}{2}) - 2(\xi_2 - \tfrac{1}{2})]d\xi_1 = 2\xi d\xi_2.$$

Thus, the quantity in parentheses in (A.3) is bounded by

$$\sup_\sigma \langle\sigma\rangle^{-2b} \int_{\sigma_2} \int_{\sigma_1} \frac{d\sigma_1 d\sigma_2}{\langle\sigma_1\rangle^{2c_1}\langle\sigma_2\rangle^{2b_1}}$$

$$\leq \sup_\sigma \langle\sigma\rangle^{-2b} \langle\sigma\rangle^{[1-2c_1]_+} \langle\sigma\rangle^{[1-2b_1]_+}$$

$$\leq \sup_\sigma \langle\sigma\rangle^{-2b+[1-2c_1]_+ +[1-2b_1]_+}$$

since σ is dominant. If $b, c_1, b_1 < \frac{1}{2}$, then the exponent is $2 - 2b - 2c_1 - 2b_1$, and it suffices to have $b + c_1 + b_1 \geq 1$.

Region σ_1 dominant, $|\sigma_1| \geq \max(|\sigma|, |\sigma_2|)$.
   By the Cauchy-Schwarz method, it suffices to show

$$(A.4) \qquad \sup_{\xi_1,\sigma_1} \langle\sigma_1\rangle^{-2c_1} \int_{\sigma_2} \int_{\xi_2} |\xi|\langle\sigma\rangle^{-2b}\langle\sigma_2\rangle^{-2b_1} d\xi_2 d\sigma_2$$

is finite.
Subregion $|\xi_1 - \frac{1}{2}| \leq 2|\xi_2 - \frac{1}{2}|$. Then $|\xi| \leq 3|\xi_2 - \frac{1}{2}|$. The inner integral over ξ_2 is taken with σ_1, ξ_1, σ_2 fixed. Since $\sigma_1 - \sigma - \sigma_2 = (\xi_1 - \frac{1}{2})^2 - (\xi_2 - \frac{1}{2})^2$, we have $d\sigma = 2(\xi_2 - \frac{1}{2})d\xi_2$ and thus (A.4) is bounded by

$$\sup_{\sigma_1} \langle\sigma_1\rangle^{-2c_1} \int_{\sigma_2=0}^{|\sigma_1|} \langle\sigma_2\rangle^{-2b_1} d\sigma_2 \int_{\sigma=0}^{|\sigma_1|} \langle\sigma\rangle^{-2b} d\sigma$$

$$\leq \sup_{\sigma_1} \langle\sigma_1\rangle^{-2c_1+[1-2b_1]_+ +[1-2b]_+}.$$

If $b, b_1 < \frac{1}{2}$, then the exponent here is $2 - 2b_1 - 2b - 2c_1$, and thus we need $b_1 + b + c_1 \geq 1$.
Subregion $|\xi_1 - \frac{1}{2}| \geq 2|\xi_2 - \frac{1}{2}|$. Since $\xi = \xi_1 - \xi_2$, we have $|\xi| \leq \frac{3}{2}|\xi_1 - \frac{1}{2}|$. Also
$\frac{3}{4}(\xi_1 - \frac{1}{2})^2 \leq (\xi_1 - \frac{1}{2})^2 - (\xi_2 - \frac{1}{2})^2 = \sigma_1 - \sigma_2 - \sigma \leq 3|\sigma_1|$, and thus $(\xi_1 - \frac{1}{2})^2 \leq 4|\sigma_1|$.
Thus (A.4) is bounded by

$$\sup_{\xi_1, \sigma_1} \langle \xi_1 \rangle^{1-4c_1} \int_{\sigma_2} \int_{\xi_2} \langle \sigma \rangle^{-2b} \langle \sigma_2 \rangle^{-2b_1} \, d\xi_2 \, d\sigma_2.$$

We change variables $y = (\xi_2 - \frac{1}{2})^2$ to obtain

$$(A.5) \qquad \sup_{\xi_1, \sigma_1} \langle \xi_1 \rangle^{1-4c_1} \int_{y=-\langle \xi_1 \rangle^2}^{y=\langle \xi_1 \rangle^2} |y|^{-1/2} \int_{\sigma_2} \langle \sigma \rangle^{-2b} \langle \sigma_2 \rangle^{-2b_1} \, d\sigma_2 \, dy.$$

The inner integral over σ_2 is taken with fixed $y = (\xi_2 - \frac{1}{2})^2$, ξ_1, σ_1, and thus $\sigma_2 + \sigma = \sigma_1 - (\xi_1 - \frac{1}{2})^2 - (\xi_2 - \frac{1}{2})^2 = \sigma_1 - (\xi_1 - \frac{1}{2})^2 - y$ is fixed. By Lemma A.2,

$$\int_{\sigma_2} \langle \sigma \rangle^{-2b} \langle \sigma_2 \rangle^{-2b_1} \, d\sigma_2 \leq c \langle \sigma_1 - (\xi_1 - \tfrac{1}{2})^2 - y \rangle^{-\alpha} \qquad \text{if } b_1 + b > \tfrac{1}{2}$$

where $\alpha = 2b - [1 - 2b_1]_+$ if $b_1 \geq b$ or $= 2b_1 - [1 - 2b]_+$ if $b \geq b_1$. By Lemma A.1 with
$f(y) = \chi_{-\langle \xi_1 \rangle^2 \leq y \leq \langle \xi_1 \rangle^2}(y)|y|^{-1/2}$ and $g(y) = \langle y \rangle^{-\alpha}$,

$$\sup_{\sigma_1} \int_{y=-\langle \xi_1 \rangle^2}^{+\langle \xi_1 \rangle^2} |y|^{-1/2} \langle \sigma_1 - (\xi_1 - \tfrac{1}{2})^2 - y \rangle^{-\alpha} \, dy$$

$$\leq \int_{y=-\langle \xi_1 \rangle^2}^{+\langle \xi_1 \rangle^2} |y|^{-1/2} \langle y \rangle^{-\alpha} \, dy$$

$$\leq \langle \xi_1 \rangle^{[1-2\alpha]_+}$$

and hence (A.5) is controlled by

$$\sup_{\xi_1} \langle \xi_1 \rangle^{1-4c_1+[1-2\alpha]_+}.$$

We now consider the exponent. Suppose $b, b_1 < \frac{1}{2}$ but $b_1 + b > \frac{1}{2}$. Then $\alpha = -1 + 2b + 2b_1$.
Case 1. $\alpha > \frac{1}{2} \iff b + b_1 > \frac{3}{4}$. Then we need $c_1 \geq \frac{1}{4}$.
Case 2. $\alpha = \frac{1}{2} \iff b + b_1 = \frac{3}{4}$. Then we need $c_1 > \frac{1}{4}$.
Case 3. $\alpha < \frac{1}{2} \iff b + b_1 < \frac{3}{4}$. Then the exponent is $4 - 4c_1 - 4b - 4b_1$, and we need $b + b_1 + c_1 \geq 1$.

Region σ_2 dominant, $|\sigma_2| \geq \max(|\sigma|, |\sigma_1|)$. This is analogous to the σ_1 dominant case, but we carry it out anyway. By the Cauchy-Schwarz method, need to show

$$(A.6) \qquad \sup_{\xi_2, \sigma_2} \langle \sigma_2 \rangle^{-2b_1} \iint_{\sigma_1, \xi_1} |\xi| \langle \sigma \rangle^{-2b} \langle \sigma_1 \rangle^{-2c_1} \, d\sigma_1 \, d\xi_1$$

is finite.
Subregion $|\xi_2 - \frac{1}{2}| \leq 2|\xi_1 - \frac{1}{2}|$. Then $|\xi| \leq 3|\xi_1 - \frac{1}{2}|$. The inner integral over ξ_1

is taken with σ_1, ξ_2, σ_2 fixed. Since $\sigma_1 - \sigma - \sigma_2 = (\xi_1 - \frac{1}{2})^2 - (\xi_2 - \frac{1}{2})^2$, we have $-d\sigma = 2(\xi_1 - \frac{1}{2})d\xi_1$. Thus (A.6) is bounded by

$$\sup_{\sigma_2}\langle\sigma_2\rangle^{-2b_1} \int_{\sigma_1}\langle\sigma_1\rangle^{-2c_1}d\sigma_1 \int_\sigma \langle\sigma\rangle^{-2b}\,d\sigma$$
$$\leq \sup_{\sigma_2}\langle\sigma_2\rangle^{-2b_1+[1-2c_1]_++[1-2b]_+}.$$

If $c_1, b < \frac{1}{2}$, then need $b_1 + c_1 + b \geq 1$.

Subregion $|\xi_2 - \frac{1}{2}| \geq 2|\xi_1 - \frac{1}{2}|$. Then $|\xi| \leq \frac{3}{2}|\xi_2 - \frac{1}{2}|$. Also, $\frac{3}{4}(\xi_2 - \frac{1}{2})^2 \leq (\xi_2 - \frac{1}{2})^2 - (\xi_1 - \frac{1}{2})^2 = -\sigma_1 + \sigma + \sigma_2 \leq 3|\sigma_2|$ and hence $(\xi_2 - \frac{1}{2})^2 \leq 4|\sigma_2|$. We change variable $y = (\xi_1 - \frac{1}{2})^2$ to obtain that (A.6) is bounded by

(A.7) $$\sup_{\sigma_2,\xi_2}\langle\xi_2\rangle^{1-4b_1} \int_{y=-\langle\xi_2\rangle^2}^{\langle\xi_2\rangle^2} |y|^{-1/2} \int_{\sigma_1}\langle\sigma\rangle^{-2b}\langle\sigma_1\rangle^{-2c_1}\,d\sigma_1\,dy.$$

Since $-\sigma + \sigma_1 = \sigma_2 + (\xi_1 - \frac{1}{2})^2 - (\xi_2 - \frac{1}{2})^2 = \sigma_2 - (\xi_2 - \frac{1}{2})^2 + y$ is fixed, by Lemma A.2,

$$\int_{\sigma_1}\langle\sigma\rangle^{-2b}\langle\sigma_1\rangle^{-2c_1}d\sigma_1 \leq \langle\sigma_2 - (\xi_2 - \tfrac{1}{2})^2 + y\rangle^{-\alpha} \qquad \text{if } b + c_1 > \tfrac{1}{2}$$

with $\alpha = 2b - [1 - 2c_1]_+$ if $c_1 \geq b$ and $\alpha = 2c - [1 - 2b]_+$ if $b \geq c_1$. By Lemma A.1 with $f(y) = \chi_{-\langle\xi_2\rangle^2 \leq y \leq \langle\xi_2\rangle^2}(y)|y|^{-1/2}$ and $g(y) = \langle y\rangle^{-\alpha}$,

$$\sup_{\sigma_2} \int_{y=-\langle\xi_2\rangle^2}^{\langle\xi_2\rangle^2} |y|^{-1/2}\langle\sigma_2 - (\xi_2 - \tfrac{1}{2})^2 + y\rangle^{-\alpha}dy$$
$$\leq \int_{y=-\langle\xi_2\rangle^2}^{\langle\xi_2\rangle^2} |y|^{-1/2}\langle y\rangle^{-\alpha}\,dy$$
$$\leq \langle\xi_2\rangle^{[1-2\alpha]_+}.$$

Hence (A.7) is bounded by

$$\sup_{\xi_2}\langle\xi_2\rangle^{1-4b_1+[1-2\alpha]_+}.$$

We now consider the exponent. If $b, c_1 < \frac{1}{2}$ and $b + c_1 > \frac{1}{2}$, then $\alpha = -1 + 2b + 2c_1$.

Case 1. $\alpha < \frac{1}{2} \iff b + c_1 < \frac{3}{4}$. Then the exponent is $4 - 4b_1 - 4b - 4c_1$ so we need $b + b_1 + c_1 \geq 1$.

Case 2. $\alpha = \frac{1}{2} \iff b + c_1 = \frac{3}{4}$. Here, we need $b_1 > \frac{1}{4}$.

Case 3. $\alpha > \frac{1}{2} \iff b + c_1 > \frac{3}{4}$. Here, we need $b_1 \geq \frac{1}{4}$. □

Proof of Lemma 3.1 (b). We show that the proof of Lemma 3.1 (b) is actually identical to that of Lemma 3.1 (a). We discuss only the $+$ case. The estimate is equivalent to showing

$$|W| \leq c\|v\|_2\|v_1\|_2\|v_2\|_2$$

where

$$(A.8) \qquad W = \int_* \frac{\hat{v}\hat{v}_1\hat{v}_2|\xi|\langle\xi\rangle^{-1/2}}{\langle\sigma\rangle^c\langle\sigma_1\rangle^{b_1}\langle\sigma_2\rangle^{b_1}}$$

with $\hat{v} = \hat{v}(\xi,\tau)$, $\hat{v}_1 = \hat{v}_1(\xi_1,\tau_1)$, $\hat{v}_2(\xi_2,\tau_2)$, $\sigma = \tau + \xi$, $\sigma_1 = \tau_1 + \xi_1^2$, $\sigma_2 = \tau_2 + \xi_2^2$, and $*$ indicates the restriction $\xi = \xi_1 - \xi_2$, $\tau = \tau_1 - \tau_2$. Indeed, for $\hat{v} \in L^2$,

$$\int_{\xi,\tau} [\partial_x(u_1\bar{u}_2)]\widehat{\ }(\xi,\tau)\langle\xi\rangle^{-1/2}\langle\sigma\rangle^{-c}\hat{v}(\xi,\tau)d\xi d\tau$$

$$= \int_{\xi,\tau} |\xi| \left[\int_{\substack{\xi=\xi_1-\xi_2\\\tau=\tau_1-\tau_2}} \hat{u}_1(\xi_1,\tau_1)\hat{u}_2(\xi_2,\tau_2)\right] \langle\xi\rangle^{-1/2}\langle\sigma\rangle^{-c}\hat{v}(\xi,\tau)d\xi d\tau.$$

Set $\hat{v}_1(\xi_1,\tau_1) = \hat{u}_1(\xi_1,\tau_1)\langle\sigma_1\rangle^{b_1}$, $\hat{v}_2(\xi_2,\tau_2) = \hat{u}_2(\xi_2,\tau_2)\langle\sigma_2\rangle^{b_1}$ to obtain (A.8).

We note that (A.8) is the same as (A.1) considered in the proof of Lemma 3.1 (a) with b, c_1 in (A.1) replaced by c, b_1, respectively. Thus, the condition $b + b_1 + c_1 \geq 1$ in Lemma 3.1 (a) becomes $c + 2b_1 \geq 1$ in Lemma 3.1 (b) □

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