MATH1530, HOMEWORK 7

Section 4.1

1. Let G act on A. prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$. Deduce that if G acts transitively on A then the kernel of the action is $\cap_{g\in G} gG_ag^{-1}$.

Assume $b = g \cdot a$. Let $x \in G_a$. Then we need to show $gxg^{-1} \in G_b$. It follows from the computation $gxg^{-1} \cdot b = gx \cdot a = g \cdot a = b$. The other inclusion follows in the same way.

Now assume that G acts transitively on A. If h is in the kernel of the action, it means by definition $h \cdot a = a$ for all $a \in A$. Then $h \in G_b$ for all b, but we showed that $gG_ag^{-1} = G_{g\cdot a}$. That is, (for a fixed a) $h \in gG_ag^{-1}$ for all $g \in G$. On the other hand, any $b \in A$ is obtained from a fixed element $a \in A$ by some group element g by $b = g \cdot a$. Therefore for a fixed choice of $a \in A$, $G_{g\cdot a}$ go over all of G_b for $b \in A$. It shows that if $h \in \cap_{g\in G} gG_ag^{-1}$, h is in the kernel.

Section 4.2

2. Exhibit the image of each element of S_3 under the left regular representation of S_3 into S_6.

Let us do it for the case (12). This element, by left multiplication, sends $1 \mapsto (12)$, $(12) \mapsto 1$, $(13) \mapsto (132)$, $(23) \mapsto (123)$, $(123) \mapsto (23)$, $(132) \mapsto (13)$. We have assigned numbers for each element of S_3, and according to this numbering, (12) corresponds to the permutation $1 \mapsto 2 \mapsto 1$, $3 \mapsto 5 \mapsto 3$, $4 \mapsto 6 \mapsto 4$, or (12)(35)(46).

Likewise, (13) corresponds to (14)(25)(36) and (23) corresponds to (13)(26)(45). The element (123) corresponds to (156)(243) and (132) corresponds to (165)(234).

4. Use the left regular representation of Q_8 to produce two elements of S_8 which generate a subgroup of S_8 isomorphic to the quaternion group Q_8.

The answer will depend on your numbering of elements of Q_8.

For example, let us call $\{1, i, j, k, -1, -i, -j, -k\}$ by $\{1, 2, 3, 4, 5, 6, 7, 8\}$ respectively. Since the two elements i and j generate Q_8, their permutation representations will generate a subgroup in S_8 which is isomorphic to Q_8. Now it is a simple computation;

$i : 1 \mapsto 2(= i) \mapsto 5(= -1)6(= -i) \mapsto 1$, $3 \mapsto 4 \mapsto 7 \mapsto 8 \mapsto 3$. Hence $i = (1256)(3478)$ under this isomorphism. Likewise, $j = (1357)(2864)$.

7. Let Q_8 be the quaternion group of order 8. Prove that Q_8 is not isomorphic to a subgroup of S_n for $n \leq 7$.

We have essentially proved part (a) in problem 4.

Assume that Q_8 acts on a set A of element ≤ 7. For each $a \in A$, we consider Stab(a) which is a subgroup of Q_8 and has index equal to the cardinality of the orbit of a. But the orbit of a is clearly ≤ 7, which means this Stab(a) cannot be trivial (it is trivial if and only if the index is 8). However, any nontrivial subgroup of Q_8 contanis -1. (it is enough to see that any subgroup generated by one element not equal to the identity contains -1, which is trivial) In particular, we conclude that $< -1 > \leq \text{Stab}(a)$ for all $a \in A$. That is, $-1 \in \text{Ker}(\pi)$. Equivalently, given any

homomorphism $Q_8 \to S_n$ for $n \leq 7$, the image of -1 is the identity in S_n which means it can never be injective.

8. Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$.

We let K be the kernel of the homomorphism $\pi : G \mapsto S_n$ coming from the action of G to H-cosets. Being a kernel, K is normal in G, $K \leq H$ (proposition 3) and $|G : K| \leq n!$ as S_n has $n!$ elements and $|G/K| \leq |S_n|$ since G/K is isomorphic to a subgroup of S_n.

10. Prove that any nonabelian group of order 6 has a nonnormal subgroup of order 2. Use this to classify groups of order 6.

One can use Cauchy's theorem to obtain a simple proof. But we can instead do it directly:

The assumption that G is nonabelian gives us two elements $a, b \in G$ such that $ab \neq ba$. Clearly, they are not equal to the identity element, so we have found 5 distinct element in our group, namely $\{1, a, b, ab, ba\}$. (if we assume $a = ab$, it gives $b = 1$, a contradiction, and so on) Now we know that a has order either 2 or 3 (if 6, it means G is cyclic and abelian).

(i) Assume a has order 2.

Further assume that b also has order 2. Then we claim that aba is different from all five elements listed above. For example $1 = aba$ would imply that $a = ba$ or $b = 1$. $b = aba$ would imply $ab = ba$, and so on. (since $a^2 = b^2 = 1$). By symmetry, the same holds for bab, and we are forced to conclude $aba = bab$. From these relations, we can indeed complete the group multiplication table of G.

If b has order 3, we can easily see that b^2 is different from any five in the above list, so it is the sixth element of the group. Then we are forced to conclude that $a = bab$ and $aba = b^2$. From these relations, again we have all the information about G.

(ii) Assume a has order 3. If b has order 2, we are back in the same situation. If b has also order 3, it is a contradiction. (because we are forced to conclude $a^2 = b^2$, or $a^{-1} = b^{-1}$, or $a = b$)

To summarize, there are exactly two possibilites: In the first case, you can easily show that $a \mapsto (12)$, $b \mapsto (23)$ gives an isomorphism between G and S_3. In the second case, you can show that $a \mapsto (12)$, $b \mapsto (123)$ gives an isomorphism between G and S_3.

In particular, it implies that there is only one nonabelian group of order 6, namely S_3. And it is trivial to see that there exists two abelian groups of order 6 (up to isomorphism), Z_6 and $Z_2 \times Z_3$.

Section 4.3

2. Find conjugacy classes.

(a) D_8.

Recall that we have shown sometime before that $Z(D_8) = \{1, r^2\}$. That is, $\{1\}$ and $\{r^2\}$ are their own conjugacy classes. Now, $< r > \leq C_{D_8}(r) \leq D_8$, while $rs \neq sr$ so we are forced to conclude that $|C_{D_8}(r)| = 4$. This means there are $8/4 = 2$ conjugates of r, including itself. Since $srs = r^3$, $\{r, r^3\}$ form a conjugacy class.

The same argument applies to other elements; since r^2 is in the center, we know that $< s, r^2 > \leq C_{D_8}(s) \leq D_8$, and $rs \neq sr$ so $|C_{D_8}(s)| = 4$. Hence we conclude that $\{s, sr^2\}$ form another conjugacy class. Finally we see that$\{sr, sr^3\}$ should be another conjugacy class because they are not in the center.

(b) Q_8.

We know that the center is $\{1, -1\}$. Hence $\{1\}$, $\{-1\}$ are two conjugacy classes. We again use the same method: $< i > \leq C_{Q_8}(i) \leq Q_8$, but since $ij \neq ji$, $|C_{Q_8}(i)| = 4$, and since $jij = -i$, we

know that $\{i, -i\}$ is a conjugacy class. By symmetry, $\{j, -j\}$ and $\{k, -k\}$ are other two conjugacy classes.

5. If the center of G is of index n, prove that every conjugacy class has at most n elements.

Let $g \in G$. Then $Z(G) \leq C_G(g) \leq G$. The number of elements in the conjugacy class of g is precisely the index of $C_G(g)$ in G, but this is clearly at most n, from the equation $|G : Z(G)| = |G : C_G(g)| \cdot |C_G(g) : Z(G)| = n$.

6. Assume G is nonabelian of order 15. Prove that $Z(G) = 1$. Show that there is at most one possible class equation for G.

First statement follows from the following statement: If $G/Z(G)$ is cyclic, G is abelian. If $Z(G)$ has order 5 (or 3), $G/Z(G)$ has three elements (five elements, respectively), which means it is cyclic. So G will be abelian, which is a contradiction. Clearly $Z(G) \neq G$, so $Z(G) = 1$.

Now we consider the class equation: $15 = 1 + ...$ Our claim is that easy summand in ... is either 3 or 5. [Proof: Given any element $g \neq 1$, g has order either 5 or 3, simply because it cannot have order 15. And we know that $1 << g > \leq C_G(g) < G$, (g not in the center), so we conclude $< g >= C_G(g)$.] How many ways are there to represent 14 as a sum of threes and fives? Only one, $14 = 3 + 3 + 3 + 5$. We are done.