MATH1530, HOMEWORK 8

Section 4.3

6. Prove that characteristic subgroups are normal. Give an example of a normal subgroup that is not characteristic.

Let H be a characteristic subgroup of G. For any $g \in G$, $x \mapsto gxg^{-1}$ is an automorphism of G. By definition, H is invariant under this automorphism; that is, $gHg^{-1} = g$ for all $g \in G$.

If we want a group containing a normal subgroup that is not characteristic, it must be the case where $\mathrm{Aut}(G)$ contains elements not in $\mathrm{Inn}(G)$. Exercises 3 and 5 in this section prove that $\mathrm{Aut}(D_8) \cong D_8$ and since we know that $r^2 \in Z(D_8)$ for example, $|\mathrm{Inn}(D_8)| \leq 4$. Therefore, it gives us some hope that D_8 might have a normal subgroup that is not characteristic.

Consider the subgroup $\{1, r^2, sr, sr^3\}$ in D_8. One can check that this is normal, and from Exercises 3 and 5, we can conclude that a map $r \mapsto r$, $s \mapsto sr$ extends to an automorphism of D_8. If we apply this automorphism to $\{1, r^2, sr, sr^3\}$, we have $\{1, r^2, sr^2, s\}$, which is another normal subgroup of D_8. Hence this is not characteristic.

For another example, consider a normal subgroup $\{1, i, -1, -i\}$ in Q_8. If you believe that there is an automorphism of Q_8 sending $i \mapsto j$, it shows that this normal subgroup is not characteristic.

13. Let G be a group of order 203. Prove that if H is a normal subgroup of order 7 in G then $H \leq Z(G)$. Deduce that G is abelian in this case.

Let H be a normal subgroup of order 7. Apply Corollary 15 to the pair (H, G) to obtain the fact that $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$, which has 6 elements. By our assumption, $N_G(H) = G$ and G has order 203 that we are forced to conclude $C_G(H) = G$.

That is, $H \leq Z(G)$, and therefore $Z(G)$ has order 7 or 203. If we assume $Z(G)$ has order 7, $G/Z(G)$ is cyclic so G is abelian; i.e., a contradiction. Therefore, $Z(G)$ has order 203 and G is abelian.

Section 4.5

12. Let $2n = 2^a k$ where k is odd. Prove that the number of Sylow 2-subgroups of D_{2n} is k.

Let m be the number of Sylow 2-subgroups in D_{2n}. By Sylow's theorem, m divides k. Consider subsets $A_j = \{r^{ik}, sr^{ik+j}\}_{0 \leq i < 2^{a-1}}$ of D_{2n} for $j = 0, 1, ..., k-1$. It is very easy to check that they are all subgroups of order 2^a. That is, we have found k distinct Sylow 2-subgroups. Hence $m = k$.

16. Let $|G| = pqr$, where p, q and r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q, or r.

This is similar in spirit to the example in which they study groups of order 30. Let n_p, n_q, and n_r be the number of Sylow p, q, and r-subgroups, respectively. To begin with, $n_r | pq$ and r is larger than p and q, we have $n_r = 1$ or pq. If $n_r = 1$, it means the Sylow r-subgroup is normal so let us assume $n_r = pq$. Then all of these Sylow r-subgroups intersect trivially [being cyclic groups], so we have $pq(r-1)$ elements of order r.

Likewise, $n_q | pr$ and assuming $n_q \neq 1$, we have either $n_q = r$ or pr. Therefore, we can conclude that G has at least $r(q-1)$ elements of order q. In the same way, we conclude G has at least $q(p-1)$ elements of order p. Finally, $pqr - pq + rq - r + pq - q = pqr + rq - r - q > pqr$, which is a contradiction.

17. Prove that if $|G| = 105$ then G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.

To begin with, $n_5 | 21$ and $n_5 \equiv 1(5)$, so either G has a normal subgroup of order 5 or $n_5 = 21$. Likewise, either G has a normal subgroup of order 7 or $n_7 = 15$. If none of Sylow 5-subgroups and 7-subgroups are normal, we are forced to conclude that G has at least 84 elements of order 5 and 90 elements of order 7. It is ridiculous since G has only 105 elements.

Therefore, if we let P be a Sylow 5-subgroup and Q be a Sylow 7-subgroup, then we know one of them is normal, and therefore PQ is a subgroup of order 35 as we all know. Then PQ has index 3 in G and therefore PQ is normal since 3 is the smallest prime dividing the order of G. We also know that PQ has order 35 and 5 does not divide $7-1$, so PQ is cyclic as shown in page 143. In particular, P and Q are characteristic in PQ, which means they are both normal subgroups of G. We are done.