MA 153 Midterm March 7, 2012

PLEASE try to write neatly!!

1. Answer the following questions with *true* or *false*, followed by a one sentence explanation, which should make it clear that you understand why the answer you give is correct. Your comment should not be longer than one or two lines. Each question is worth 6 points.

a) The map $\phi : D_8 \to D_8$ given by $\phi(x) = x^{-1}$ is a homomorphism. – False. For the map from x to x^{-1} to be a homomorphism, it must be the cast that $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in D_8$. But this is not the case as D_8 is not abelian.

b) The permutation (123)(345) has order 3. – False. Writing it as a product of disjoint cycles we see that it equals the single cycle (12345), which has order 5.

c) A cyclic group of order 18 must contain an element of order 9. – True. Every subgroup of a cyclic group is cyclic, and there exists a subgroup of order d for every d dividing 18. As $9|18$ there is a cyclic subgroup of order 9 and it's generator has order 9.

d) The group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$ is isomorphic to $\mathbf{Z}/6\mathbf{Z}$. – True. The generator is $(\bar{1}, \bar{1})$.

e) The map ϕ from $\mathbf{Z}/3\mathbf{Z} \to (\mathbf{Z}/7\mathbf{Z})^*$ determined by setting $\phi(\bar{1}) = \bar{2}$ is well defined. – True. The determined map is $\phi(n) = 2^n$. To be well defined it must be the case that for any integer k, $\phi(\overline{1+3k}) \equiv \bar{2} \pmod 7$, that is, that $2^{1+3k} \equiv 2 \pmod 7$. But this is the case as

$$2^{1+3k} \equiv 2(2^3)k \equiv 2 \cdot 1^k \equiv 2 \pmod 7.$$

f) The normalizer of a subgroup must contain the centralizer of that subgroup. –True. If H is the centralizer and $h \in H$, then for any g in the group, $ghg^{-1} = h$.

2. Give simple one or two line answers to the following questions. Each is worth 6 points.

a) Find the kernel of the homomorphism ϕ from $\mathbf{Z}/15\mathbf{Z} \to \mathbf{Z}/15\mathbf{Z}$ given by setting $\phi(x) = 3x$. $\bar{n}$ is mapped to $\bar{0}$ when $3n \equiv 0 \pmod{15}$. This is the case for $\bar{0}, \bar{5}, \overline{10}$, so the kernel equals $\{\bar{0}, \bar{5}, \overline{10}\}$.

b) Let Z_{12} be a cyclic group of order 12 generated by x. Let K be the subgroup $K = \{1, x^3, x^6, x^9\}$. Write down a surjective homomorphism from Z_{12} to Z_3, a cyclic group of order 3 generated by y, with kernel K. Define ϕ by $\phi(x^\ell) = y^\ell$. That's all you need to do.

c) Consider the action $\mathbf{Z}/6\mathbf{Z}$ on the set S of cosets $\{\{\bar{0}, \bar{3}\}, \{\bar{1}, \bar{4}\}, \{\bar{2}, \bar{5}\}\}$ by addition, i.e. $\bar{n} \cdot \{\bar{0}, \bar{3}\} = \{\bar{n} + \bar{0}, \bar{n} + \bar{3}\}$. What is the stabilizer of $\{\bar{1}, \bar{4}\}$? The stabilizer is the subgroup equal to $\{\bar{0}, \bar{3}\}$, as

$$0 + \{\bar{1}, \bar{4}\} = 3 + \{\bar{1}, \bar{4}\} = \{\bar{1}, \bar{4}\},$$

and similarly for $\{\bar{2}, \bar{5}\}\}$.

d) Why is the kernel of a homomorphism normal? To be normal it must be true for every g in the group and every k in the kernel, that $\phi(gkg^{-1}) = e$. This is true as

$$\phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g)^{-1}) = \phi(g)e\phi(g)^{-1}) = e$$

e) If A, B are subgroups of G, with A normal in G, and if $|A| = p$, $|B| = q$, with p, q distinct primes, why is the order of $AB = \{ab|a \in A, b \in B\}$ equal to pq? By a proposition in the text, $|AB| = |A||B|/|A \cap B|$, so the order of AB can be smaller than pq only if $A \cap B \neq e$. But $A \cap B$ is a subgroup of both A and B and hence its order must divide p and q. But the only integer that does this is 1.

3. To be turned in Thursday morning: Problem 34 of Section 3.1. D_{2n} is, as usual, the dihedral group generated by r, s satisfying $r^n = s^2 = 1$ and $rs = sr^{-1}$. Suppose k divides n. Prove that H, the cyclic subgroup generated by r^k, is a normal subgroup of D_{2n}, and also that D_{2n}/H is isomorphic to D_{2k}.

First, The different groups D_{2n} and D_{2k} need different names for their generators. Let R, S, with $R^k = S^2 = 1$ and $RS = SR^{-1}$ be a presentation for D_{2k}. Let $m = n/k$. Then, there are several ways to proceed.

One way: Define a mapping $\phi : D_{2n} \to D_{2k}$ by $\phi(s^a r^b) = S^a R^b$. Another way to say this is: Define ϕ on the generators by $\phi(s) = S$, $\phi(r) = R$, and extend it to a mapping from D_{2n} to D_{2k} by $\phi(s^a r^b) = \phi(s)^a \phi(r)^b$. This will be well defined if for any ℓ_1, ℓ_2, it is also true that $\phi(s^{a+2\ell_1} r^{b+n\ell_2}) = S^a R^b$. This is, in fact, the case as

$$\phi(s^{a+2\ell_1} r^{b+n\ell_2}) = S^{a+2\ell_1} R^{b+n\ell_2} = S^a S^{2\ell_1} R^b R^{km\ell_2} = S^a \cdot 1 \cdot R^b \cdot 1 = S^a R^b.$$

It will be a homomorphism if for any a, b, c, d, $\phi((s^a r^b)(s^c r^d)) = \phi(s^a r^b)\phi(s^c r^d)$. If $c \equiv 0 \pmod 2$ this is the statement that $\phi(s^a r^{b+d}) = \phi(s^a r^b)\phi(r^d)$, which is true as $S^a R^{b+d} = S^a R^b R^d$. If $c \equiv 1 \pmod 2$ this is the statement that $\phi(s^a r^b s r^d) = \phi(s^a r^b)\phi(s r^d)$, which is true as

$$\phi(s^a r^b s r^d) = \phi(s^{a+1} r^{-b+d}) = S^{a+1} R^{d-b}$$

and

$$\phi(s^a r^b)\phi(s r^d) = S^a R^b S R^d = S^{a+1} R^{d-b}.$$

A more efficient way to prove well defined and a homomorphism at once, is to verify that $\phi(s), \phi(r)$ satisfy the same relations as s, r. To verify this, we check that $\phi(s)^2 = S^2 = 1$, $\phi(r)^n = R^n = (R^k)^{n/k} = 1^m = 1$, and

$$\phi(r)\phi(s) = RS = SR^{-1} = \phi(s)\phi(r)^{-1}.$$

Continuing, the map is surjective as R, S generate D_{2k}. Its kernel is the set of

$$\{s^a r^b | a = 0, 1; b \equiv 0 \pmod k\}$$

such that $\phi(s^a r^b) = S^a R^b = 1$. But $S^a R^b = 1$ if and only if $a = 0$ and $b \equiv 0 \pmod{k}$, so the kernel of ϕ equals H. As H is the kernel of a homomorphism it is normal, and the map is surjective, it follows from the first isomorphism theorem D_{2n}/H is isomorphic to D_{2k}.

Another way: Verify directly that H is normal by checking that for any $s^a r^b$, and any r^{kl}, $(s^a r^b) r^{kl} (s^a r^b)^{-1} \in H$. To see this, note that there are two cases: $a = 0$ and $a = 1$. If $a = 0$ then $(s^a r^b) r^{kl} (s^a r^b)^{-1} = r^k$, while if $a = 1$, $(s^a r^b) r^{kl} (s^a r^b)^{-1} = r^{-k}$. It is not, unfortunately, true that $(s^a r^b)^{-1} = r^{-b} s^{-a}$ unless $a = 1$. Once you know that H is normal, define $\phi : D_{2k} \to D_{2n}/H$ by $\phi(S^a R^b) = s^a r^b H$. This is well defined because for any ℓ_1, ℓ_2,

$$\phi(S^{a+2\ell_1} R^{b+k\ell_2}) = s^{a+2\ell_1} r^{b+k\ell_2} H = s^a r^b H,$$

as $s^{2\ell_1}, r^{k\ell_2} \in H$. It's homomorphism because H is normal, making coset multiplication well defined.