A nonarchimedean approach to local holomorphic dynamics in dimension two

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Superattracting fixed points

Setup: Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a dominant holomorphic germ for which 0 is a superattracting fixed point, i.e., $f'(0)$ is nilpotent.

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Method: We'll study an associated dynamical system $f_\bullet : \mathcal{V} \to \mathcal{V}$ on a certain space of valuations $\mathcal{V}$, and conclude something about f.

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The dynamical system $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ induces a dynamical system $f : \mathcal{U} \to \mathcal{U}$ in the usual way:

$$f(\nu)(\phi) := \nu(\phi \circ f).$$

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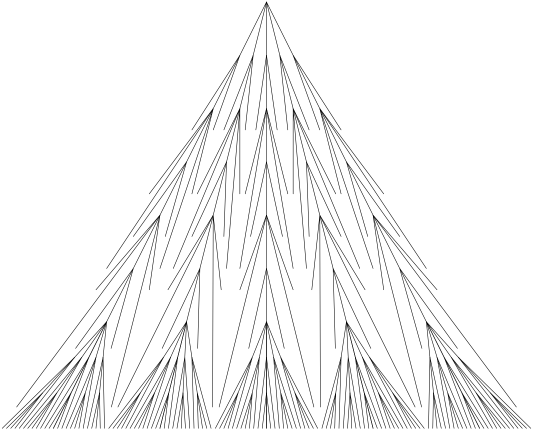
$$f_\bullet(\nu) := f(\nu)/c \quad \longleftarrow \text{ Normalize!}$$

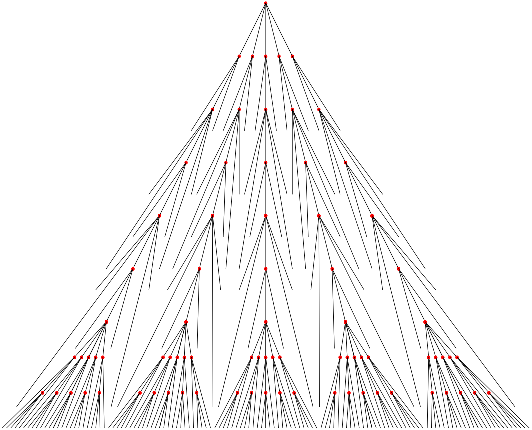
Structure of $\mathcal{V}$

The structure of $\mathcal{V}$ resembles that of a 1-d Berkovich disk. It is a compact Hausdorff, path connected, and has a tree structure.



Divisorial points

The branch points of $\mathcal{V}$ are the **divisorial** valuations.



Divisorial points

Construction:
If $\pi: X \to (\mathbb{C}^2, 0)$ is a modification of $(\mathbb{C}^2, 0)$ and $E \subseteq \pi^{-1}(0)$ is a prime divisor of X, the divisorial valuation $\nu_E \in \mathcal{V}$ associated to E is:

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Note: Understanding the orbits of exceptional prime divisors E is needed when studying *algebraic stability*, one of the central topics in complex dynamics in dimension ≥ 2.

Global setting

This method for studying stability using dynamics on normalized valuations spaces was initiated in the works of Favre-Jonsson.

Dynamical compactifications of $\mathbb{C}^2$ (2008)

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2. *the set of all such $\nu_\star$ is an interval in $\mathcal{V}$.*

Application to stability

Application (G.-Ruggiero)

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *be a dominant holomorphic germ for which the origin is a superattracting fixed point. Then one can find arbitrarily high modifications* $\pi : X \to (\mathbb{C}^2, 0)$ *over the origin for which the lift* $f : X \dashrightarrow X$ *is "eventually stable."*

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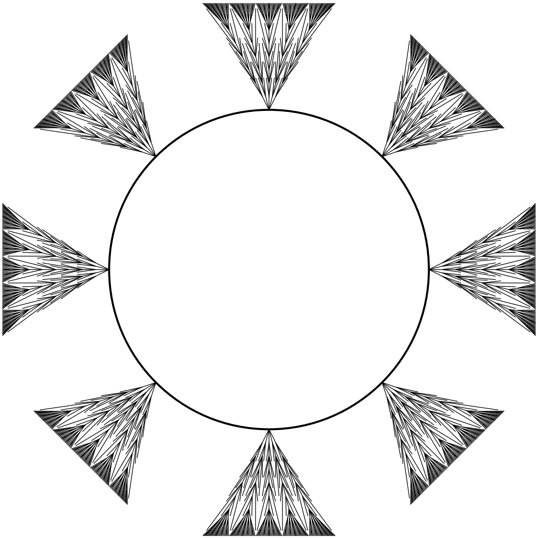
Eventually Stable: For each prime divisor $E \subseteq \pi^{-1}(0)$, one has $f^n(E) \not\subseteq \mathrm{Indet}(f : X \dashrightarrow X)$ for sufficiently large n.

The singular setting

Replace $(\mathbb{C}^2, 0)$ with (X, x_0), where $X =$ normal complex surface, and $x_0 \in X$ is a singularity of X.

One can associate to (X, x_0) an analogous valuation space $\mathcal{V}$ on which to study dynamics, with structure resembling Berkovich curves.

The space $\mathcal{V}$ for a cusp singularity



Dynamics and stability for singularities

Theorem (G.-Ruggiero)

Suppose $f : (X, x_0) \to (X, x_0)$ is a non-invertible holomorphic germ. Then, assuming (X, x_0) is not a cusp singularity, one can replace f by a suitable iterate so that

1. *every $\nu \in \mathcal{V}$ which is not an end of $\mathcal{V}$ lies in the basin of attraction of some fixed point $\nu_\star$ of $f_\bullet$, and*
2. *the set of all such $\nu_\star$ forms an interval or a circle in $\mathcal{V}$.*

Dynamics and stability for singularities

Application (G.-Ruggiero)

Assuming (X, x_0) is not a cusp, one can find arbitrarily high modifications $\pi : X' \to (X, x_0)$ over x_0 for which the lift $f : X' \dashrightarrow X'$ is "eventually stable."

Analogy to dynamics on hyperbolic Riemann surfaces

A classical theorem of Fatou classifies all the possible dynamics of holomorphic maps $f : \Sigma \to \Sigma$, where Σ is a hyperbolic Riemann surface.

Key tool: f is non-expansive with respect to the hyperbolic metric.

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In proving our main dynamical theorems, we introduce a metric with respect to which $f_\bullet$ is non-expansive.