On the distribution of orbits in affine varieties

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A motivating question

Question: Let X be a variety, let $\phi : X \to X$ be a morphism, let $\alpha \in X$ be a point, and let $V \subset X$ be a subvariety. What can you say about the set

$$Z_{\phi,\alpha,V} = \{n \in \mathbb{N} \mid \phi^n(\alpha) \in V\}?$$

Could $Z_{\phi,\alpha,V}$ be arbitrary? Or must it have some structure?

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First observation: If it happens that

- $\phi^b(\alpha) \in V$ for some $b \geq 0$, and
- $\phi^a(V) \subseteq V$ for some $a \geq 1$,

then $Z_{\phi,\alpha,V}$ contains the infinite arithmetic progression $a\mathbb{N} + b$.

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Second observation: Conversely, if $Z_{\phi,\alpha,V}$ contains an infinite arithmetic progression, then a Zariski-closure argument can be used to produce a periodic subvariety of V meeting the orbit of α.

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Question (cont.): Apart from the possible existence of infinite arithmetic progressions in $Z_{\phi,\alpha,V}$, what else can you say about its structure? If it contains no infinite arithmetic progressions, how large can it be?

Statement of a Theorem

Theorem: Let X be a variety, let $\phi : X \to X$ be a morphism, let $\alpha \in X$, and let V be a Zariski-closed subset of X. Then

$$\{n \in \mathbb{N} \mid \phi^n(\alpha) \in V\} = A \cup B$$

where A is a (possibly empty) finite union of infinite arithmetic progressions, and B is a set of (Banach) density zero.

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Remarks:

- The dynamical Mordell-Lang conjecture states that, in characteristic zero, B is actually finite.
- $S \subseteq \mathbb{N}$ has *Banach density zero* if $\frac{|S \cap I|}{|I|} \to 0$ as $|I| \to +\infty$ over all intervals $I \subset \mathbb{N}$.
- Actually holds for arbitrary Noetherian spaces.

Statement of a Theorem

This theorem is due to:

- L. Denis: special case of automorphisms of $\mathbb{P}^n$ in characteristic p.
- W. Gignac: ergodic theory and measure theory on Zariski spaces (should also mention C. Favre)
- Bell-Ghioca-Tucker: elementary arguments (they also get some quantitative results)
- Petsche: special case of affine varieties

A proof for affine varieties

STEP 1: Reduce the Theorem to the following formally weaker statement.

THEOREM: Let X be an affine variety, let $\phi : X \to X$ be a morphism, let $\alpha \in X$ be a point, and let V be a Zariski-closed subset of X. If the set $\{n \in \mathbb{N} \mid \phi^n(\alpha) \in V\}$ contains no infinite arithmetic progressions, then it has (Banach) density zero.

A proof for affine varieties

STEP 1: Reduce the Theorem to the following formally weaker statement.

THEOREM: Let X be an affine variety, let $\phi : X \to X$ be a morphism, let $\alpha \in X$ be a point, and let V be a Zariski-closed subset of X. If the set $\{n \in \mathbb{N} \mid \phi^n(\alpha) \in V\}$ contains no infinite arithmetic progressions, then it has (Banach) density zero.

STEP 2: Introduce a Berkovich space.

We have $X = \operatorname{Spec} \mathcal{A}$ for some finitely generated k-algebra $\mathcal{A}$. Let $\mathbf{X} = M(\mathcal{A})$, the Berkovich spectrum of A as a trivially-normed Banach ring.

A proof for affine varieties

Step 2 (cont.): More precisely, $\mathbf{X}$ is the set of all functions $[\cdot] : \mathcal{A} \to \mathbb{R}$ satisfying

(i) $0 \leq [f] \leq 1$ for all $f \in \mathcal{A}$;

(ii) $[0] = 0$ and $[r] = 1$ for all nonzero $r \in k$;

(iii) $[f + g] \leq \max\{[f], [g]\}$ for all $f, g \in \mathcal{A}$;

(iv) $[fg] = [f][g]$ for all $f, g \in \mathcal{A}$.

- In words, each $[\cdot]$ is "a multiplicative seminorm on $\mathcal{A}$ restricting to the trivial absolute value on k."
- Give $\mathbf{X}$ the coarsest topology under which each function $\mathbf{X} \to \mathbb{R}$, given by $[\cdot] \mapsto [f]$ for some $f \in \mathcal{A}$, is continuous.

A proof for affine varieties

Facts:

- $\mathbf{X}$ is a compact Hausdorff space.
- There exists a natural reduction map

$$\pi : \mathbf{X} \to X \qquad \pi([\cdot]) = \{f \in \mathcal{A} \mid [f] < 1\}.$$

(Recall $X = \mathrm{Spec}(\mathcal{A})$ is the prime ideal spectrum of $\mathcal{A}$.)

- The reduction map π is surjective, and it is anti-continuous with respect to the Hausdorff topology on $\mathbf{X}$ and the Zariski topology on X.

Ergodic theory interlude

Definition: Let M be a metric space, and let $T : M \to M$ be a function. A point $\alpha \in M$ is a **recurrent point** for T if some subsequence of the forward orbit $\{T^k(\alpha)\}_{k=0}^{\infty}$ converges to α.

Poincare recurrence theorem (a la Furstenberg): Let M be a compact metric space, let $T : M \to M$ be a continuous function, and let μ be a T-invariant unit Borel measure on M. Then μ-almost all points of M are recurrent for T.

Idea of the proof: The forward orbit of a non-recurrent point takes up too much room for there to be very many of them.

A proof for affine varieties

Sketch of an argument:

- Consider $\phi : X \to X$, $\alpha \in X$, and a Zariski-closed subset V of X such that $Z_{\phi,\alpha,V} = \{n \in \mathbb{N} \mid \phi^n(\alpha) \in V\}$ has positive upper-Banach density.
- **Going up:** Lift the dynamical system to a continuous map $T : \mathbf{X} \to \mathbf{X}$. Via Prokhorov's theorem, there exists a T-invariant probability measure μ on $\mathbf{X}$ charging $\pi^{-1}(V)$. Via the Poincaré recurrence theorem, there exists a T-recurrent point $\zeta \in \pi^{-1}(V) \cap \mathrm{supp}(\mu)$.
- **Going back down:** $\overline{\pi(\zeta)}$ is a periodic subvariety of V meeting the forward orbit of α. This leads to an infinite arithmetic progression in $Z_{\phi,\alpha,V}$.

An example in characteristic p

Example:

- $K = \mathbb{F}_p(t)$, field of rational functions in one variable t.
- $\phi : K^2 \to K^2$ defined by $\phi(x,y) = (tx, (1-t)y)$.
- $\alpha = (1,1)$ and $V = \{x + y = 1\}$ in K^2.
- Then $\phi^n(\alpha) = (t^n, (1-t)^n)$ and

$$\{n \in \mathbb{N} \mid \phi^n(\alpha) \in V\} = \{p^\ell \mid \ell \geq 0\}.$$