On the distribution of orbits in affine varieties

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A motivating question

**Question:** Let $X$ be a variety, let $\phi : X \rightarrow X$ be a morphism, let $\alpha \in X$ be a point, and let $V \subset X$ be a subvariety. What can you say about the set

$$Z_{\phi,\alpha,V} = \{ n \in \mathbb{N} | \phi^n(\alpha) \in V \}?$$

Could $Z_{\phi,\alpha,V}$ be arbitrary? Or must it have some structure?
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**First observation:** If it happens that

- $\phi^b(\alpha) \in V$ for some $b \geq 0$, and
- $\phi^a(V) \subseteq V$ for some $a \geq 1$,

then $Z_{\phi, \alpha, V}$ contains the infinite arithmetic progression $a\mathbb{N} + b$. 
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**Second observation:** Conversely, if $Z_{\phi, \alpha, V}$ contains an infinite arithmetic progression, then a Zariski-closure argument can be used to produce a periodic subvariety of $V$ meeting the orbit of $\alpha$. 
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**Question (cont.):** Apart from the possible existence of infinite arithmetic progressions in $Z_{\phi, \alpha, V}$, what else can you say about its structure? If it contains no infinite arithmetic progressions, how large can it be?
Theorem: Let $X$ be a variety, let $\phi : X \to X$ be a morphism, let $\alpha \in X$, and let $V$ be a Zariski-closed subset of $X$. Then

$$\{ n \in \mathbb{N} | \phi^n(\alpha) \in V \} = A \cup B$$

where $A$ is a (possibly empty) finite union of infinite arithmetic progressions, and $B$ is a set of (Banach) density zero.
Statement of a Theorem

Theorem: Let $X$ be a variety, let $\phi : X \rightarrow X$ be a morphism, let $\alpha \in X$, and let $V$ be a Zariski-closed subset of $X$. Then

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Remarks:

- The dynamical Mordell-Lang conjecture states that, in characteristic zero, $B$ is actually finite.
- $S \subseteq \mathbb{N}$ has Banach density zero if $\frac{|S \cap I|}{|I|} \rightarrow 0$ as $|I| \rightarrow +\infty$ over all intervals $I \subset \mathbb{N}$.
- Actually holds for arbitrary Noetherian spaces.
Statement of a Theorem

This theorem is due to:

- L. Denis: special case of automorphisms of $\mathbb{P}^n$ in characteristic $p$.
- W. Gignac: ergodic theory and measure theory on Zariski spaces (should also mention C. Favre)
- Bell-Ghioca-Tucker: elementary arguments (they also get some quantitative results)
- Petsche: special case of affine varieties
A proof for affine varieties

**Step 1:** Reduce the Theorem to the following formally weaker statement.

**Theorem:** Let $X$ be an affine variety, let $\phi : X \to X$ be a morphism, let $\alpha \in X$ be a point, and let $V$ be a Zariski-closed subset of $X$. If the set $\{ n \in \mathbb{N} \mid \phi^n(\alpha) \in V \}$ contains no infinite arithmetic progressions, then it has (Banach) density zero.

**Step 2:** Introduce a Berkovich space. We have $X = \text{Spec} A$ for some finitely generated $k$-algebra $A$. Let $X = \mathcal{M}(A)$, the Berkovich spectrum of $A$ as a trivially-normed Banach ring.
A proof for affine varieties

**Step 1:** Reduce the Theorem to the following formally weaker statement.

**Theorem:** Let $X$ be an affine variety, let $\phi : X \to X$ be a morphism, let $\alpha \in X$ be a point, and let $V$ be a Zariski-closed subset of $X$. If the set $\{n \in \mathbb{N} \mid \phi^n(\alpha) \in V\}$ contains no infinite arithmetic progressions, then it has (Banach) density zero.

**Step 2:** Introduce a Berkovich space.

We have $X = \text{Spec} \, \mathcal{A}$ for some finitely generated $k$-algebra $\mathcal{A}$. Let $X = M(\mathcal{A})$, the Berkovich spectrum of $\mathcal{A}$ as a trivially-normed Banach ring.
Step 2 (cont.): More precisely, $X$ is the set of all functions $[\cdot] : \mathcal{A} \to \mathbb{R}$ satisfying

(I) $0 \leq [f] \leq 1$ for all $f \in \mathcal{A}$;

(II) $[0] = 0$ and $[r] = 1$ for all nonzero $r \in k$;

(III) $[f + g] \leq \max\{[f], [g]\}$ for all $f, g \in \mathcal{A}$;

(IV) $[fg] = [f][g]$ for all $f, g \in \mathcal{A}$.

- In words, each $[\cdot]$ is “a multiplicative seminorm on $\mathcal{A}$ restricting to the trivial absolute value on $k$.”

- Give $X$ the coarsest topology under which each function $X \to \mathbb{R}$, given by $[\cdot] \mapsto [f]$ for some $f \in \mathcal{A}$, is continuous.
A proof for affine varieties

Facts:

- $X$ is a compact Hausdorff space.
- There exists a natural reduction map
  \[ \pi : X \rightarrow X \quad \pi([\cdot]) = \{ f \in A \mid [f] < 1 \}. \]
  
  (Recall $X = \text{Spec}(A)$ is the prime ideal spectrum of $A$.)
- The reduction map $\pi$ is surjective, and it is anti-continuous with respect to the Hausdorff topology on $X$ and the Zariski topology on $X$. 
**Ergodic theory interlude**

**Definition:** Let $M$ be a metric space, and let $T : M \to M$ be a function. A point $\alpha \in M$ is a **recurrent point** for $T$ if some subsequence of the forward orbit $\{ T^k(\alpha) \}_{k=0}^{\infty}$ converges to $\alpha$.

**Poincare recurrence theorem (a la Furstenberg):** Let $M$ be a compact metric space, let $T : M \to M$ be a continuous function, and let $\mu$ be a $T$-invariant unit Borel measure on $M$. Then $\mu$-almost all points of $M$ are recurrent for $T$.

**Idea of the proof:** The forward orbit of a non-recurrent point takes up too much room for there to be very many of them.
A proof for affine varieties

Sketch of an argument:

- Consider $\phi : X \to X$, $\alpha \in X$, and a Zariski-closed subset $V$ of $X$ such that $Z_{\phi, \alpha, V} = \{ n \in \mathbb{N} \mid \phi^n(\alpha) \in V \}$ has positive upper-Banach density.

- **Going up:** Lift the dynamical system to a continuous map $T : X \to X$. Via Prokhorov’s theorem, there exists a $T$-invariant probability measure $\mu$ on $X$ charging $\pi^{-1}(V)$. Via the Poincaré recurrence theorem, there exists a $T$-recurrent point $\zeta \in \pi^{-1}(V) \cap \text{supp}(\mu)$.

- **Going back down:** $\pi(\zeta)$ is a periodic subvariety of $V$ meeting the forward orbit of $\alpha$. This leads to an infinite arithmetic progression in $Z_{\phi, \alpha, V}$.
An example in characteristic $p$

Example:

- $K = \mathbb{F}_p(t)$, field of rational functions in one variable $t$.
- $\phi : K^2 \to K^2$ defined by $\phi(x, y) = (tx, (1 - t)y)$.
- $\alpha = (1, 1)$ and $V = \{x + y = 1\}$ in $K^2$.
- Then $\phi^n(\alpha) = (t^n, (1 - t)^n)$ and

$$\{n \in \mathbb{N} \mid \phi^n(\alpha) \in V\} = \{p^\ell \mid \ell \geq 0\}.$$