Arithmetic Coordinates on Dynamical Moduli Spaces

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Special Session on Arithmetic Dynamics

Seattle Joint Mathematics Meetings

January 6, 2016

Background

Let K be a complete, algebraically closed, non-archimedean field, with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{m}$, and residue field $\widetilde{k}$. Let $|\cdot|$ be the absolute value on K and let $\mathrm{ord}(\cdot)$ be the associated additive valuation.

We write $\mathbf{P}^1_K$ for the Berkovich Projective Line over K, and ζ_G for the Gauss point.

Let $\varphi \in K(z)$ be a rational function of degree $d \geq 2$. If $\gamma \in \mathrm{GL}_2(K)$, write $\varphi^\gamma = \gamma^{-1} \circ \varphi \circ \gamma$.

A *normalized representation* for φ^γ is a pair (F_γ, G_γ) of homogeneous polynomials of degree d in $\mathcal{O}[X, Y]$, with at least one coefficient a unit of $\mathcal{O}$, and $\varphi^\gamma(z) = F_\gamma(z, 1)/G_\gamma(z, 1)$. It is unique up to scaling by a unit of $\mathcal{O}$.

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The Function $\mathrm{ordRes}_\varphi(\cdot)$

The number $\mathrm{ord}(\mathrm{Res}(\varphi^\gamma)) := \mathrm{ord}(\mathrm{Res}(F_\gamma, G_\gamma))$ is well-defined, and independent of the choice of normalized representation.

The function $\gamma \mapsto \mathrm{ord}(\mathrm{Res}(\varphi^\gamma))$ factors through a function $\mathrm{ordRes}_\varphi(\cdot) : \mathbf{P}^1_K \to [0, \infty]$, which is defined on type II points by

$$\mathrm{ordRes}_\varphi(\gamma(\zeta_G)) = \mathrm{ord}(\mathrm{Res}(\varphi^\gamma)) .$$

Theorem

The function $\mathrm{ordRes}_\varphi(\cdot)$ has the following properties:

- *It is continuous with respect to the strong topology on $\mathbf{P}^1_K$.*
- *It takes the value ∞ on $\mathbb{P}^1(K)$ and is finite on $\mathbf{P}^1_K \backslash \mathbb{P}^1(K)$.*
- *It is piecewise affine and convex up on each path, with respect to the logarithmic path distance.*
- *It achieves a minimum on $\mathbf{P}^1_K$.*

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The Minimal Resultant Locus

The Minimum Resultant Locus MinResLoc(φ) is the set of points in $\mathbf{P}^1_K$ where $\mathrm{ordRes}_\varphi(\cdot)$ takes on its minimum. It is either a single type *II* point, or a segment with type *II* endpoints. If d is even, it is a single point.

The Crucial Set

- There is a canonical way to assign non-negative integer weights $w_\varphi(P)$ to points in the interior of $\mathbf{P}^1_K$, such that $\sum_P w_\varphi(P) = d - 1$.
  The weights arise by taking the Laplacian of $\mathrm{ordRes}_\varphi(\cdot)$, restricted to the tree spanned by the classical fixed points and the Berkovich repelling fixed points of φ.
- The set of points which receive weight is called the **crucial set** of φ. It is a conjugation equivariant. There are only finitely many possible configurations of the crucial set, for a given d.

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The crucial set and MinResLoc(φ) form a bridge between analytic and moduli-theoretic properties of $\varphi(z)$

Theorem

The Minimal Resultant Locus MinResLoc(φ) *is the Barycenter of the crucial set (with weights $w_\varphi(P)$).*

Theorem

A conjugate φ^γ is semi-stable in the sense of Geometric Invariant Theory if and only if $\gamma(\zeta_G)$ belongs to MinResLoc(φ) *.*

The direction $\rightarrow$ in the second theorem is due to Szpiro, Tepper, and Williams.

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The theme of this talk is that the crucial set should determine where φ lies in the dynamical moduli space $\mathcal{M}_d(K)$:

Conjecture

There are a compactification $\overline{\mathcal{M}}_d$ of $\mathcal{M}_d/\operatorname{Spec}(\mathcal{O})$ and an algebraic stratification of the special fibre $\overline{\mathcal{M}}_d/\operatorname{Spec}(\widetilde{k})$ determined by the configurations of the crucial set, such that if $[\varphi] \in \mathcal{M}_d(K)$ is the point corresponding to φ, then $[\varphi] \pmod{\mathfrak{m}}$ lies in the stratum corresponding to the crucial set of φ.

We will call such a compactification *a good arithmetic compactification* of $\mathcal{M}_d$, and the functions which embed it, *good arithmetic coordinates*.

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We will show that the conjecture holds for quadratic rational functions and cubic polynomials, and discuss progress towards establishing it for cubic rational functions.

This represents joint work with students in two VIGRE groups and an REU:

John Doyle and Kenneth Jacobs (*Configuration of the Crucial set for a Quadratic Rational Map*, ArXiv 1507.03535)

Ebony Harvey, Allan Lacy, Marko Milosevich, Lori Watson, (*Configurations of the Crucial Set for a Cubic Polynomial*, in preparation)

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The Weights $w_\varphi(P)$

- There are two reasons a point can receive weight. Only points of type II, in $\mathbf{H}^1_K := \mathbf{P}^1_K \backslash \mathbb{P}^1(K)$, can have $w_\varphi(P) > 0$.

- If $\varphi(P) = P$, then $w_\varphi(P) = \deg_\varphi(P) - 1 + N_{\text{Shearing}}(P)$. Here $\deg_\varphi(P)$ is the degree of the reduction of φ at P. A tangent direction $\vec{v} \in T_P$ is called a *shearing direction* if it contains a classical fixed point, but is moved by φ. The number of shearing directions $N_{\text{Shearing}}(P)$ is a measure of the discrepancy between the local and global behavior of φ at P.

- If $\varphi(P) \neq P$, and P is a branch point of the tree Γ_{Fix} spanned by the classical fixed points of φ, then $w_\varphi(P) = \text{valence}(P) - 2$.

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Example: The weights $w_\varphi(P)$ for a quadratic function

If $\varphi(z) \in K(z)$ is quadratic, then $\deg(\varphi) - 1 = 1$ so there is a unique point in the crucial set. This point is turns out to be the unique point in the Minimal Resultant Locus, that is, $\mathrm{ordRes}_\varphi(P)$ is minimal.

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Configuration I: φ has *potential good reduction*; there is a point where $\deg_\varphi(P) = 2$.

An example with $P = \zeta_G$: $\varphi(z) = z^2/(z^2-1)$ specializes to $\widetilde{\varphi}(z) = z^2/(z^2-1) \pmod{\mathfrak{m}}$. $w_\varphi(P) = 2 - 1 + 0 = 1$.

Configuration II: φ has *potential multiplicative reduction*; there is a branch point P of Γ_{Fix} with $\varphi(P) = P$, such that after a change of coordinates, $\widetilde{\varphi}(P) = \widetilde{\lambda} z \pmod{\mathfrak{m}}$ for some $\widetilde{\lambda} \neq \widetilde{0}, \widetilde{1} \in \widetilde{k}$.

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Fixed points $0, 1, \infty$. $\vec{v}_1$ is sheared. $w_\varphi(P) = 1 - 1 + 1 = 1$.

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An example with $P = \zeta_G$: $\varphi(z) = \frac{z(z+1)}{z+p}$ specializes to $\widetilde{\varphi}(z) = z + 1 \pmod{\mathfrak{m}}$. Fixed points $0, 1/p, \infty$. $\vec{v}_0$ is sheared. $w_\varphi(P) = 1 - 1 + 1 = 1$.

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An example with $P = \zeta_G$: $\varphi(z) = \frac{z(z-1)}{p} + 1$ has fixed points $0, 1, \infty$, and $\varphi(\zeta_G) = \zeta_{0,p} \neq \zeta_G$. $w_\varphi(P) = \max(0, 3-2) = 1$.

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The Coarse Moduli Space for Quadratic Functions

For a quadratic rational function, let σ_1, σ_2 be the usual symmetric functions in the multipliers at the fixed points. It was shown by Silverman that $\mathcal{M}_2 / \operatorname{Spec}(\mathbb{Z})$ exists as a coarse moduli space, and that $\mathcal{M}_2 \cong \mathbb{A}^2 / \operatorname{Spec}(\mathbb{Z})$, with coordinate functions σ_1, σ_2.

Its natural compactification is $\mathbb{P}^2 / \operatorname{Spec}(\mathbb{Z})$. We will identify $(x, y) \in \mathbb{A}^2$ with $(x : y : 1) \in \mathbb{P}^2$ and base change to $\mathcal{O}$.

The following theorem, which strengthens a theorem of Diane Yap, says $\mathbb{P}^2$ is a good arithmetic compactification of $\mathcal{M}_2$, and that σ_1 and σ_2 are good arithmetic coordinates.

Theorem (J. Doyle, K. Jacobs, R. Rumely)

Let K be a complete, algebraically closed, non-archimedean valued field with ring of integers $\mathcal{O}$ and residue field $\widetilde{k}$. Let $\varphi(z) \in K(z)$ have degree 2.

Write $s([\varphi]) = (\sigma_1(\varphi) : \sigma_2(\varphi) : 1) \in \mathbb{P}^2(K)$, and let $\widetilde{s([\varphi])}$ be its specialization in $\mathbb{P}^2(\widetilde{k})$. Then

(A) φ has potential good reduction iff $\widetilde{s([\varphi])} \in \mathbb{A}^2(\widetilde{k})$.

(B) φ has potential multiplicative reduction iff $\widetilde{s([\varphi])} = (\widetilde{1} : \widetilde{x} : \widetilde{0})$ where $\widetilde{x} \neq \widetilde{2}$; if $\widetilde{\varphi}(z) \equiv \widetilde{\lambda} z$ then $\widetilde{x} = \widetilde{\lambda} + \widetilde{1}/\widetilde{\lambda}$.

(C) φ has potential additive reduction iff $\widetilde{s([\varphi])} = (\widetilde{1} : \widetilde{2} : \widetilde{0})$.

(D) φ has potential constant reduction iff $\widetilde{s([\varphi])} = (\widetilde{1} : \widetilde{0} : \widetilde{0})$.

Cubic Polynomials

Let $\varphi(z) = az^3 + bz^2 + cz + d \in K[z]$ be a cubic polynomial.

Under affine conjugacy, the *Monic Centered Normal Form* for $\varphi(z)$ is

$$\varphi^\gamma(z) \;=\; z^3 + Cz + D$$

where C is unique and D is determined up to a factor of ± 1.

The Monic Centered Normal Form exists when $\mathrm{char}(K) \neq 3$, but need not exist when $\mathrm{char}(K) = 3$.

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Fixed Point Normal Form

The VIGRE group found another normal form for a cubic polynomial which always exists.

$\varphi^\gamma(z)$ is in *Fixed Point Normal Form* if it is monic, and one of its fixed points is 0. If the fixed points are $0, F_1, F_2$ then

$$\begin{aligned} \varphi^\gamma(z) &= z(z-F_1)(z-F_2)+z \\ &= z^3-(F_1+F_2)z^2+(1+F_1F_2)z \end{aligned}$$

It determined up to six possibilities; if φ^γ is conjugated by $\gamma_1(z) \in \{\pm z, \pm z + F_1, \pm z + F_2\}$ the resulting polynomials $\varphi^{\gamma\circ\gamma_1}$ are also in fixed point normal form, with fixed point sets

$$\begin{aligned} &\{0, F_1, F_2\}, && \{0, -F_1, -F_2\}, \\ &\{0, -F_1, F_2 - F_1\}, && \{0, F_1, F_1 - F_2\}, \\ &\{0, -F_2, F_1 - F_2\}, && \{0, F_2, F_2 - F_1\}. \end{aligned}$$

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$$\begin{array}{ll}\{0,F_1,F_2\}\,, & \{0,-F_1,-F_2\}\,, \\ \{0,-F_1,F_2-F_1\}\,, & \{0,F_1,F_1-F_2\}\,, \\ \{0,-F_2,F_1-F_2\}\,, & \{0,F_2,F_2-F_1\}\,.\end{array}$$

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The Group of Symmetries G

If one orders the fixed points F_1, F_2, a generic cubic $\varphi(z) \in K[z]$ corresponds to 12 ordered pairs, acted on by a group of 2×2 matrices $G \cong \{\pm 1\} \times S_3$:

$$\pm\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \pm\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

$$\pm\begin{bmatrix} F_2 - F_1 \\ -F_1 \end{bmatrix} = \pm\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \pm\begin{bmatrix} -F_1 \\ F_2 - F_1 \end{bmatrix} = \pm\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix},$$

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Now let F_1 and F_2 be variables, so Spec $(K[F_1, F_2]) \cong \mathbb{A}^2$.

The moduli space $\mathcal{P}^3(K)$ of cubic polynomials is isomorphic to the quotient of $\mathbb{A}^2(K)$ by G, for the action above, and if $K[F_1, F_2]^G$ is the ring of invariants, then $\mathcal{P}^3 \cong \text{Spec}\,(K[F_1, F_2]^G)$.

Theorem

Let H be a field, and let F_1 and F_2 be independent variables. Let G act on $H[F_1, F_2]$ via the action above. Put $\mu_2 = F_1^2 - F_1F_2 + F_2^2$, $\mu_3 = F_1(F_1 - F_2)F_2$, $\mu_6 = \mu_3^2 = F_1^2(F_1 - F_2)^2F_2^2$. Then
(A) If $\text{char}(H) \neq 2$, one has $H[F_1, F_2]^G = H[\mu_2, \mu_6]$.
(B) If $\text{char}(H) = 2$, one has $H[F_1, F_2]^G = H[\mu_2, \mu_3]$.

Note that when $\text{char}(H) = 2$, $\mu_3^2 = \mu_6$ uniquely determines μ_3.

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The symmetric functions in the multipliers, σ_1, σ_3

For $\varphi(z) = az^3 + bz^2 + cz + d$ one has

$$\begin{aligned}
\sigma_1 &= (b^2 - 3ac + 6a)/a, \\
\sigma_2 &= (2b^2 - 6ac + 9a)/a, \\
\sigma_3 &= (27a^2d^2 - 28abcd + 28abd + 4ac^3 - 12ac^2 \\
&\qquad +9ac + 4b^3d - b^2c^2 + 2b^2c)/a
\end{aligned}$$

Theorem

Let H be a field, and let $\varphi(z) \in H[z]$ be a cubic polynomial. Then $\sigma_2 = 2\sigma_1 - 3$, and
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The Coarse Moduli Space of Cubic Polynomials

Theorem

Let H be an algebraically closed field. A coarse moduli space for cubic polynomials over H exists, and is isomorphic to $\mathbb{A}^2/\operatorname{Spec}(H)$, with coordinate functions σ_1, σ_3.

Let K be a complete, algebraically closed, nonarchimedean valued field.

Let $\varphi(z) = z^3 - (F_1 + F_2)z^2 + (1 + F_1 F_2)z$ be a cubic polynomial in fixed point normal form. Using the symmetry group G one can arrange that $0 \le |F_1| \le |F_2$ and that if $0 \ne |F_1| = |F_2|$, then F_1 and F_2 lie in distinct tangent directions at $\zeta_{0,|F_2|} \in \mathbf{P}^1_K$

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Configurations of the Crucial Set

Configuration I: If $|F_1|, |F_2| \leq 1$, then φ has *potential good reduction* at $P = \zeta_G$, and there are no shearing directions. $w_\varphi(P) = 3 - 1 + 0 = 2$.

Configuration II: If $|F_1| = |F_2| > 1$, there is a branch point $P = \zeta_{0,|F_2|}$ of Γ_{Fix} with valence 4 which moved by φ. $w_\varphi(P) \max(0, 4 - 2) = 2$.

Configuration III: If $|F_2| > 1$ and $1/|F_2| < |F_1| < |F_2$, then there are branch points of Γ_{Fix} with valence 3 at $P_1 = \zeta_{0,|F_1|}$ and $P_2 = \zeta_{0,|F_2|}$; both are moved. $w_\varphi(P_1) = w_\varphi(P_2) = \max(0, 3 - 2) = 1$.

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Configurations of the Crucial Set

Configuration IV: If $|F_2| > 1$ and $|F_1| = 1/|F_2| < 1$, then the branch point $P_2 = \zeta_{0,|F_2|}$ of Γ_{Fix} is moved, and the branch point $P_1 = \zeta_{0,1/|F_2|}$ of Γ_{Fix} is a fixed point of degree 2 with no shearing directions.
$w_\varphi(P_1) = w_\varphi(P_2) = 1$.

Configuration V: If $|F_2| > 1$ and $|F_1| < 1/|F_2| < 1$, then the branch point $P_2 = \zeta_{0,|F_2|}$ of Γ_{Fix} is moved, and the point $P_1 = \zeta_{0,1/|F_2|} \in \Gamma_{\mathrm{Fix}}$ is a fixed point of degree 1 with one shearing direction $\vec{v}_0$.
$w_\varphi(P_1) = w_\varphi(P_2) = 1$.

Configurations of the Crucial Set

Configuration IV: If $|F_2| > 1$ and $|F_1| = 1/|F_2| < 1$, then the branch point $P_2 = \zeta_{0,|F_2|}$ of Γ_{Fix} is moved, and the branch point $P_1 = \zeta_{0,1/|F_2|}$ of Γ_{Fix} is a fixed point of degree 2 with no shearing directions.
$w_\varphi(P_1) = w_\varphi(P_2) = 1$.

Configuration V: If $|F_2| > 1$ and $|F_1| < 1/|F_2| < 1$, then the branch point $P_2 = \zeta_{0,|F_2|}$ of Γ_{Fix} is moved, and the point $P_1 = \zeta_{0,1/|F_2|} \in \Gamma_{\text{Fix}}$ is a fixed point of degree 1 with one shearing direction $\vec{v}_0$.
$w_\varphi(P_1) = w_\varphi(P_2) = 1$.

Theorem (E. Harvey, L. Watson, M. Milosevic, R. Rumely)

Let K be a complete, algebraically closed, non-archimedean valued field with ring of integers $\mathcal{O}$ and residue field $\widetilde{k}$. Let $\varphi(z) \in K(z)$ be a cubic polynomial.

Write $s([\varphi]) = (\sigma_1(\varphi) : \sigma_3(\varphi) : 1) \in \mathbb{P}^2(K)$, and let $\widetilde{s([\varphi])}$ be its specialization in $\mathbb{P}^2(\widetilde{k})$. Then

(A) If φ has potential good reduction then $\widetilde{s([\varphi])} \in \mathbb{A}^2(\widetilde{k})$.

(B) If the crucial set has Configuration II, $\widetilde{s([\varphi])} = (\widetilde{0} : \widetilde{1} : \widetilde{0})$.

(C) If the crucial set has Configuration III, $\widetilde{s([\varphi])} = (\widetilde{0} : \widetilde{1} : \widetilde{0})$.

(D) If the crucial set has Configuration IV, $\widetilde{s([\varphi])} = (\widetilde{1} : \widetilde{x} : \widetilde{0})$

where $\widetilde{x} = (\widetilde{F_1 F_2})^2 + \widetilde{1} \neq \widetilde{1}$. (In this situation $|F_1 F_2| = 1$).

(E) If the crucial set has Configuration V, $\widetilde{s([\varphi])} = (\widetilde{1} : \widetilde{1} : \widetilde{0})$.

Although for both configurations II and III, $s([\varphi])$ specializes to the same point $(\widetilde{0} : \widetilde{1} : \widetilde{0})$ of $\mathbb{P}^1(\widetilde{k})$, they can be separated by blowing up that point with respect to an appropriate ideal, introducing an extra copy of $\mathbb{P}^1(\widetilde{k})$ in the special fibre.

This leads to a good arithmetic compactification for the moduli space of cubic polynomials.