Higher-Dimensional Nonarchimedean Dynamics

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Notation and Definitions

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Let $\phi(0) = 0$, with $\phi^* T_0$ in Jordan canonical form, with eigenvalues $\lambda_1, \ldots, \lambda_n$. We call them the multipliers at $0$.

When $K$ is a complete valued field, we'd like to study $\phi$ analytically. The best tool for this is the linearization.

Definition. We say $\phi$ (or $\psi$) is formally linearizable at the fixed point $0$ if there exists $L = (L_1, \ldots, L_n)$, $L_i \in K[[x_1, \ldots, x_n]]$ such that $\phi \circ L = L \circ \phi^* T_0$.

If $K$ is valued and $L$ has positive radius of convergence, we say $\phi$ is analytically linearizable.

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Background on Linearization

There is considerable past work on whether $\varphi$ is linearizable in 1 dimension. This depends on the value of the multiplier $\lambda$. 

$\varphi$ is formally linearizable iff $\lambda \neq 0$ or a root of unity, since the coefficients of $L$ have $\lambda^k - \lambda$ in the denominators.

Over any valued field, $\varphi$ is analytically linearizable if $|\lambda| \neq 0$, 1.

If $|\lambda| = 1$ but $\lambda$ is not a root of unity, it depends on $K$. If $K = \mathbb{C}_p$ and $\lambda$ is algebraic, $\varphi$ is analytically linearizable. If $K = \mathbb{C}$, $\lambda = e^{\pi i \theta}$, $\varphi$ is linearizable iff $\theta$ is not too irrational (Brjuno 1971-72, Yoccoz 1995).

In several variables, if $K = \mathbb{C}_p$ and the $\lambda_i$ are algebraic and multiplicatively independent, then $\varphi$ is analytically linearizable (Hermann-Yoccoz 1983).

This is not a necessary condition: at $(1, 1, \ldots, 1)$, the power map is linearizable via $L_i = e^{x_i}$ but $\lambda_1 = \ldots = \lambda_n$.

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**Main Lemma (1).** Suppose \( \lambda_1, \ldots, \lambda_r \notin \langle \lambda_{r+1}, \ldots, \lambda_n \rangle \). Then \( \exists ! f_i \in K[[x_{r+1}, \ldots, x_n]], i = 1, \ldots, r \) such that the system of formal equations \( x_i = f_i \) is \( \varphi \)-invariant.
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**Main Lemma (2).** If $\lambda_1, \ldots, \lambda_r \notin \langle \lambda_{r+1}, \ldots, \lambda_n \rangle$ in the analytic topology on a valued field $K$, then the $f_i$s have positive radius of convergence, and define a $\varphi$-invariant analytic subvariety $V$, tangent to $x_1 = \ldots = x_r = 0$. 
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**Proof idea.** We construct $f_i$ explicitly, and get denominators

$$
\lambda_j = \prod_{\alpha_j \geq 0, \sum \alpha_j > 0} \lambda_{r+1}^{\alpha_{r+1}} \cdots \lambda_n^{\alpha_n}
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The conditions of the main lemmas are satisfied when
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**Proposition.** The behavior of \( V \) (attracting, repelling, etc.) near \( 0 \) under \( \varphi \) depends on \( \lambda_i \) for \( i \leq r \). The behavior of \( 0 \) under \( \varphi|_V \) depends on \( \lambda_i \) for \( i > r \).
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**Example.** Suppose $|\lambda_i| > 1$ when $i \leq r$ and $|\lambda_i| < 1$ when $i > r$. Then $V$ is repelling, i.e. points near $0$ that are not on $V$ get farther away from $V$, but on $V$ itself, points near $0$ are attracted to $0$. 

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This generalizes hyperbolic dynamics, which pulls apart attracting and repelling directions over $\mathbb{C}$; see Yoccoz 1995.
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- If $|\lambda| < 1$, then $x$ is isolated (it has an attracting basin).
- If $\text{char} K = 0$ and $|\lambda| > 1$, then $x$ is not isolated (Julia 1918/C, B´ezivin 2001/Cp).
- If $|\lambda| = 1$ and $K = \mathbb{C}$, then isolated $\iff$ linearizable.
- If $|\lambda| = 1$ and $K$ is nonarchimedean, then conjecturally isolated, proven in all cases except if $\lambda$ is a root of unity and $\text{char} K = p$ (Benedetto 2000, Rivera-Letelier 2001-3, Lindahl-Rivera-Letelier 2014).

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*Proof idea.* There are three main ingredients to the proof:

1. Use the main lemmas with $|\lambda_i| < 1$ for $i \leq r$ and $|\lambda_i| = 1$ for $i > r$. Near $0$, periodic points only occur on $V$. This reduces the problem to when $|\lambda_i| = 1$ for all $i$.

2. Compute explicitly the lowest-degree nonzero terms of the equations $\varphi_k(x) = x$. These can be shown to have coefficients with valuations growing as $O(\log p_k)$.

3. Apply the theory of tropical intersection and the Newton polytope (Rabinoff 2012), argue that a $k$-cycle near $0$ is impossible for large $k$ if valuations grow as $O(n^{1/2}k^{1/2})$. If $n = 1$, this proof also works if $\text{char } K = p$ and $\lambda$ is not a root of unity, because the constant coefficient valuation grows as $O(k)$. But if $n > 1$ then $k/\in O(n^{1/2}k^{1/2})$ and the argument fails.
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1. Use the main lemmas with $|\lambda_i| < 1$ for $i \leq r$ and $|\lambda_i| = 1$ for $i > r$. Near 0, periodic points only occur on $V$. This reduces the problem to when $|\lambda_i| = 1$ for all $i$.

2. Compute explicitly the lowest-degree nonzero terms of the equations $\varphi^k(x) = x$. These can be shown to have coefficients with valuations growing as $O(\log_p k)$.

3. Apply the theory of tropical intersection and the Newton polytope (Rabinoff 2012), argue that a $k$-cycle near 0 is impossible for large $k$ if valuations grow as $O(\sqrt{n}k)$.
Application 1: Isolated Periodic Points—Main Result

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3. Apply the theory of tropical intersection and the Newton polytope (Rabinoff 2012), argue that a $k$-cycle near $0$ is impossible for large $k$ if valuations grow as $O(\sqrt[4]{k})$.

If $n = 1$, this proof also works if char $K = p$ and $\lambda$ is not a root of unity, because the constant coefficient valuation grows as $O(k)$. But if $n > 1$ then $k \notin O(\sqrt[4]{k})$ and the argument fails.
Conjecture. (Zhang) Let $\psi : \mathbb{P}^n_{\mathbb{Q}} \to \mathbb{P}^n_{\mathbb{Q}}$, deg $\psi > 1$. Then
$\exists x \in \mathbb{P}^n(\overline{\mathbb{Q}})$ such that the forward orbit of $x$ is Zariski-dense.
**Application 2: Zhang’s Conjecture—Background**

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The main lemmas help us in extending ABR to additional cases.
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Application 2: Zhang’s Conjecture—Main Results

**Theorem 2.** (L.) Zhang’s conjecture is true if there exists a fixed (or periodic) point whose eigenvalues satisfy *at least one* of the following two conditions:

1. One eigenvalue is zero and the rest are multiplicatively independent.
2. $n = 2$, one eigenvalue is a root of unity and the other is not.

**Proof idea.**

In the first case, assume $\lambda_1 = 0$, and choose a completion $K$ such that the other multipliers are indifferent. Apply the main lemma with $r = 1$ and Amerik-Bogomolov-Rovinsky.

In the second case, assume $\lambda_2 = 1$, and choose a completion $K$ such that $|\lambda_1| < 1$. Apply the main lemma with $r = 1$. If $K$ is $p$-adic, the proof of the first case works, even without ABR.

If $K = \mathbb{C}$, the situation is more difficult, because 0 is in the Julia set of $\phi|_V$. We choose $x$ such that it is attracted to an attracting petal near 0, and then argue the orbit cannot possibly be contained in an analytic subvariety.
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