Higher-Dimensional Nonarchimedean Dynamics

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Warning. We will routinely pass from φ to an iterate. So everything here that is stated for a fixed point is also valid for periodic points.

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One benefit of linearization: there exist nicely intersecting analytic hypersurfaces: $x_i = 0$ for each i (if $\varphi_* T_\mathbf{0}$ is diagonal).

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Main Lemma (1). Suppose $\lambda_1, \ldots, \lambda_r \notin \langle \lambda_{r+1}, \ldots, \lambda_n \rangle$. Then $\exists! f_i \in K[[x_{r+1}, \ldots, x_n]], i = 1, \ldots, r$ such that the system of formal equations $x_i = f_i$ is φ-invariant.

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Main Lemma (2). If $\lambda_1, \ldots, \lambda_r \notin \overline{\langle \lambda_{r+1}, \ldots, \lambda_n \rangle}$ in the analytic topology on a valued field K, then the f_is have positive radius of convergence, and define a φ-invariant analytic subvariety V, tangent to $x_1 = \ldots = x_r = 0$.

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Proof idea. We construct f_i explicitly, and get denominators

$$\lambda_i - \prod_{\alpha_j \geq 0, \sum \alpha_j > 0} \lambda_{r+1}^{\alpha_{r+1}} \ldots \lambda_n^{\alpha_n}$$

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Example. Suppose $|\lambda_i| > 1$ when $i \leq r$ and $|\lambda_i| < 1$ when $i > r$. Then V is repelling, i.e. points near $\mathbf{0}$ that are not on V get farther away from V, but on V itself, points near $\mathbf{0}$ are attracted to $\mathbf{0}$.

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This generalizes hyperbolic dynamics, which pulls apart attracting and repelling directions over $\mathbb{C}$; see Yoccoz 1995.

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It is natural to ask whether periodic points are isolated in several variables.

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If $n = 1$, this proof also works if char $K = p$ and λ is not a root of unity, because the constant coefficient valuation grows as $O(k)$. But if $n > 1$ then $k \notin O(\sqrt[n]{k})$ and the argument fails.

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Conjecture. (Zhang) Let $\psi : \mathbb{P}^n_{\overline{\mathbb{Q}}} \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$, $\deg \psi > 1$. Then $\exists x \in \mathbb{P}^n(\overline{\mathbb{Q}})$ such that the forward orbit of x is Zariski-dense.

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The main lemmas help us in extending ABR to additional cases.

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In the second case, assume $\lambda_2 = 1$, and choose a completion K such that $|\lambda_1| < 1$. Apply the main lemma with $r = 1$. If K is p-adic, the proof of the first case works, even without ABR.

Application 2: Zhang's Conjecture—Main Results

Theorem 2. (L.) Zhang's conjecture is true if there exists a fixed (or periodic) point whose eigenvalues satisfy *at least one* of the following two conditions:

1. One eigenvalue is zero and the rest are multiplicatively independent.
2. $n = 2$, one eigenvalue is a root of unity and the other is not.

Proof idea. In the first case, assume $\lambda_1 = 0$, and choose a completion K such that the other multipliers are indifferent. Apply the main lemma with $r = 1$ and Amerik-Bogomolov-Rovinsky.

In the second case, assume $\lambda_2 = 1$, and choose a completion K such that $|\lambda_1| < 1$. Apply the main lemma with $r = 1$. If K is p-adic, the proof of the first case works, even without ABR.

If $K = \mathbb{C}$, the situation is more difficult, because 0 is in the Julia set of $\varphi|_V$. We choose x such that it is attracted to an attracting petal near 0, and then argue the orbit cannot possibly be contained in an analytic subvariety.