Lower Bounds for non-Archimedean Lyapunov Exponents

Kenneth Jacobs, University of Georgia

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Generalities

K - complete, algebraically closed valued field
$\phi \in K(z)$, with $\deg(\phi) = d \geq 2$
$\mathbf{P}^1_{\mathrm{K}}$ - Berkovich projective line over K

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- Does this limit exist?
- How does it vary with z?

Lyapunov Exponent

Theorem (application of Birkhoff Ergodic Theorem)

For μ_ϕ-almost every $z \in \mathbf{P}^1_K$:

$$\int \log_v [\phi']_\zeta \, d\mu_\phi(\zeta) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log_v [\phi']_{\phi^k(z)} \ .$$

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We define the *Lyapunov Exponent* of ϕ to be

$$L_v(\phi) := \int_{\mathbf{P}^1_{\mathrm{K}}} \log_v [\phi']_\zeta \, d\mu_\phi(\zeta) \ .$$

Bounds on $L_v(\phi)$ over $\mathbb{C}$

Theorem (Lyubich, Friere-Lopes-Mañe, Ruelle)

The Lyapunov Exponent is bounded below

$$L_v(\phi) \geq \frac{1}{2} \log d$$

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Proof Ingredients:

- Entropy
- Equidistribution
- Ruelle's Inequality

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Example: $K = \mathbb{C}_p$
Let $\phi(z) = z^p$, so that $\phi'(z) = pz^{p-1}$. Then

$$L_v(\phi) = \int_{\mathbf{P}^1_{\mathrm{K}}} \log_v[\phi'] d\mu_\phi$$

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A Special Case

Theorem (J., 2015)

If ϕ has good reduction and also has separable reduction, then $L_v(\phi) = 0$.

Main Results

Theorem (J., 2015)

Let K be a complete, algebraically closed non-Archimedean field with $char(K) = 0$. Let $\phi \in K(z)$ be a non-trivial rational map of degree d.

Let $\kappa = \min(\log_v |m| \ : \ 1 \leq m \leq d)$, noting that $\kappa \leq 0$. Let $\mathcal{L}_\phi$ be the Lipschitz constant for the action of $\mathbb{P}^1(K)$ in the spherical metric. Then

$$L(\phi) \geq \kappa - (d+1) \log_v \mathcal{L}_\phi \ .$$

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Let K be a complete, algebraically closed non-Archimedean field, and suppose $char(K) = 0$ Let $\phi \in K(T)$ be a rational map of degree $d \geq 2$ such that $\log_v \mathrm{diam}_\infty(\cdot) \in L^1(\mu_\phi)$ and $\mathcal{J}(\phi) \subseteq \mathbf{H}^1_K$. Then

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Applies to:

- Maps of (potential) good reduction.
- Lattès Maps

Proof Idea

For all $\zeta \in \mathbf{H}^1_{\mathrm{K}}$:

$$\log_v[\phi']_\zeta \geq \kappa + \log_v \mathrm{diam}_\infty(\phi(\zeta)) - \log_v \mathrm{diam}_\infty(\zeta) \ .$$

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Let $\nu_n = \frac{1}{d^n}\phi^{n*}\delta_{\zeta_{\mathrm{Gauss}}}$. Then

$$\int \log_v[\phi']_\zeta d\nu_n \geq \qquad \kappa + \int \log_v \operatorname{diam}_\infty(\zeta) d(\nu_{n-1} - \nu_n)$$

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$$\begin{array}{rcl} \int \log_v[\phi']_\zeta d\nu_n \geq & & \kappa + \int \log_v \operatorname{diam}_\infty(\zeta) d(\nu_{n-1} - \nu_n) \\ \downarrow & & \downarrow \qquad\qquad \downarrow \\ L_v(\phi) \geq & & \kappa - (d+1)\log_v \mathcal{L}_\phi \ . \end{array}$$

Proof Idea - Logarithmic Equidistribution

(Uses machinery of Favre and Rivera-Letelier)

Theorem (J., 2015)

Let K be a complete, algebraically closed non-Archimedean valued field. Let $\phi \in K(z)$ have degree $d \geq 2$, and let ν be a probability measure on $\mathbf{P}^1_K$ with bounded potentials. Then for any rational function $g \in K(z)$, we have

$$\int \log_v [g]_z \ d\left(\frac{1}{d^n}\phi^{n*}\nu\right)(z) \to \int \log_v [g]_z \ d\mu_\phi(z) \ .$$

Proof Idea - The Error Term

(Uses machinery of Favre and Rivera-Letelier)

Γ_n: Tree spanned by $\phi^{-n}(\zeta_{\text{Gauss}})$ and $\phi^{-(n-1)}(\zeta_{\text{Gauss}})$
$\mu_{\mathrm{Br},\Gamma}$: branching measure for Γ
$g_1(\cdot, \zeta_{\text{Gauss}})$: potential function for $\frac{1}{d}\phi^*\delta_{\zeta_{\text{Gauss}}} - \delta_{\zeta_{\text{Gauss}}}$

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$$\geq -C \cdot \frac{\#\ \text{endpoints in}\ \Gamma_n}{d^{n-1}}$$