Eventually stable rational functions

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Introduction

Joint work with Alon Levy

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Call f *stable over* K if f^n is irreducible over K for all $n \geq 1$.

Sample Theorem (Fein-Danielson 2001)

If $d \geq 2$ and $f(x) = x^d + c \in \mathbb{Z}[x]$ is irreducible, then f is stable over $\mathbb{Q}$.

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- $f(x) = x^2 - x - 1$ has f and f^2 irreducible over $\mathbb{Q}$, but $f^3(x) = (x^4 - 3x^3 + 4x - 1)(x^4 - x^3 - 3x^2 + x + 1)$.

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- $f(x) = x^2 - 4/3$ is irreducible over $\mathbb{Q}$, but $f^2(x) = (x^2 - 2x + 2/3)(x^2 - 2x - 2/3)$.

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Even worse: Let $f(x) = x^2 + 1$. It follows from work of Odoni and Stoll that given $m, r \geq 1$ there exists a number field K such that f^m is irreducible over K but a sufficiently large iterate of f has r distinct irreducible factors over K.

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- There exists $n \geq 1$ such that (ϕ^n, α) is eventually stable.
- For every sequence $\{\beta_n\}$ in $\mathbb{P}^1(\overline{K})$ satisfying $\phi(\beta_1) = \alpha$ and $\phi(\beta_n) = \beta_{n-1}$ for $n \geq 2$, we have $[K(\beta_n) : K(\beta_{n-1})] = d$ for all n sufficiently large.

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Let K^{sep} be a separable closure of K, $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$, $\phi^{-n}(\alpha) = \{\beta \in \overline{K} : \phi^n(\beta) = \alpha\}$, and suppose $\phi^{-n}(\alpha) \subset K^{\mathrm{sep}}$ for every $n \geq 1$.

- The number of G_K-orbits on $\phi^{-n}(\alpha)$ is bounded as n grows.
- There exists $m \geq 0$ such that for all $\beta \in \phi^{-m}(\alpha)$, G_K acts transitively on $\phi^{-n}(\beta)$ for all $n \geq 1$.

Main conjecture

Conjecture (Everywhere eventual stability conjecture)

Let $\phi \in K(z)$, and suppose that $\alpha \in \mathbb{P}^1(K)$ is not periodic for ϕ.

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Example 1: $K = \mathbb{Q}, f(x) = x^2 - 1$. Then $x \mid f^2(x)$, so $f^n(x) \mid f^{n+2}(x)$ for all $n \geq 1$, so $x \mid f^2(x) \mid f^4(x) \mid \cdots$. Thus $(f, 0)$ not eventually stable.

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Example 2: $K = \mathbb{F}_5, f(x) = x^2 + 2$. Using a theorem of Stickelberger, can show that $(f, 0)$ is not eventually stable even though 0 is not periodic for f.

Some past results

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(Hamblen-J-Madhu) K a field of characteristic $\nmid d$, $\phi(x) = x^d + c \in K[x]$, $d \geq 2$. If there is a discrete valuation v on K with $v(c) > 0$, then $(\phi, 0)$ is eventually stable over K.

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Open Question: Is $f(x) = x^2 + (1/c)$ eventually stable for all $c \in \mathbb{Z} \setminus \{0, -1\}$?

An observation

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Observation: In some sense, the proof of Eisenstein's criterion gives too much away.

v a discrete valuation on a field K, normalized so that $v(K^*) = \mathbb{Z}$.
$R = \{z \in K : v(z) \geq 0\}$
$\mathfrak{p} = \{z \in K : v(z) > 0\}$ (unique maximal ideal of R)
$k = R/\mathfrak{p}$ (residue field)
$\tilde{f}$ = polynomial obtained from $f \in R[x]$ by reducing each coefficient modulo $\mathfrak{p}$.

Proposition (Eisentein's criterion)

Let v be a discrete valuation on a field K, let $f(x) = a_d x^d + \cdots + a_0 \in R[x]$ for $d \geq 1$, and suppose $v(a_d) = 0$, $v(a_i) > 0$ for all $i = 1, \ldots, d-1$, and $v(a_0) = 1$. Then $f(x)$ has at most 1 irreducible factor over K.

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Proof: Write $f(x) = a_d f_0(x)$ with $f_0(x) \in R[x]$ monic, and note $v(f_0(0)) = v(a_0) = 1$. Suppose that $f_0(x) = g_1(x) g_2(x)$ is a factorization of f_0 into monics in $K[x]$ with $\deg g_i = e_i \geq 1$ and $\sum e_i = d$. By Gauss' Lemma, we may assume $g_i \in R[x]$ for all i. Then $\tilde{g}_i$ is monic of degree e_i, and we have $x^d = \tilde{g}_1(x)\tilde{g}_2(x)$ in $k[x]$. Because $k[x]$ is a UFD, we must have $\tilde{g}_i(x) = x^{e_i}$ for all i, and hence $v(g_i(0)) > 0$ for all i. This implies $v(f_0(0)) = \sum v(g_i(0)) > 1$, a contradiction.

Proposition (Generalized Eisenstein's criterion)

Let v be a discrete valuation on a field K, let $f(x) = a_d x^d + \cdots + a_0 \in R[x]$ for $d \geq 1$, and suppose $v(a_d) = 0$, $v(a_i) > 0$ for all $i = 1, \ldots, d-1$, and $v(a_0) = m \geq 1$. Then $f(x)$ has at most m irreducible factors over K.

Proof: Write $f(x) = a_d f_0(x)$ with $f_0(x) \in R[x]$ monic, and note $v(f_0(0)) = v(a_0) = m$. Suppose that $f_0(x) = g_1(x) \cdots g_{m+1}(x)$ is a factorization of f_0 into monics in $K[x]$ with $\deg g_i = e_i \geq 1$ and $\sum e_i = d$. By Gauss' Lemma, we may assume $g_i \in R[x]$ for all i. Then $\tilde{g}_i$ is monic of degree e_i, and we have $x^d = \tilde{g}_1(x) \cdots \tilde{g}_{m+1}(x)$ in $k[x]$. Because $k[x]$ is a UFD, we must have $\tilde{g}_i(x) = x^{e_i}$ for all i, and hence $v(g_i(0)) > 0$ for all i. This implies $v(f_0(0)) = \sum v(g_i(0)) > m$, a contradiction.

Proposition

Let v be a discrete valuation on a field K, let $\alpha \in K$, and let $f, g \in R[x]$ satisfy $v(f(0) = v(g(0) = r > 0$ $(r \neq \infty)$ and $v(f'(0)) > 0$. Then $v(f(g(0))) = r$.

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Proof: Let $f(x) = a_d x^d + \cdots + a_0$, $g(x) = b_e x^e + \cdots + b_0$. By assumption $v(a_0) = v(b_0) = r$, and $v(a_1) > 0$.

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But $f(g(0)) = f(b_0) = a_d b_0^d + \cdots + a_2 b_0^2 + a_1 b_0 + a_0$. The strong triangle inequality then gives $v(f(g(0))) = v(a_0)$.

Corollary

Let v be a discrete valuation on a field K, let $\alpha \in K$, and let $f \in R[x]$ satisfy $v(f(0)) = r > 0$ ($r \neq \infty$) and $v(f'(0)) > 0$. Then $v(f^n(0)) = r$ for all $n \geq 1$.

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Proposition (Generalized Eisenstein's criterion)

Let v be a discrete valuation on a field K, let $f(x) \in R[x]$ have degree $d \geq 2$, and suppose $\tilde{f}(x) = cx^d$ for $c \in k^*$. Then $f(x)$ has at most $v(f(0))$ irreducible factors over K.

Theorem

Let v be a discrete valuation on a field K, let $f \in R[x]$ have degree $d \geq 2$, and let $\alpha \in R$ with $f(\alpha) \neq \alpha$. Suppose that $\tilde{f}(x) - \tilde{\alpha} = c(x - \tilde{\alpha})^d$ for $c \in k^*$. Then for all $n \geq 1$, $f^n(x) - \alpha$ has at most $v(f(\alpha) - \alpha)$ irreducible factors over K. In particular, (f, α) is eventually stable over K.

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Remark: With minor modifications, the theorem can be made to work for $\alpha \in K \setminus R$. That case is already covered by Ingram's result.

Theorem

Let v be a discrete valuation on a field K, let k be a finite field of characteristic p, and let $f \in R[x]$ have degree $d = p^k$ for $k \geq 1$. Suppose that $\tilde{f}(x) = cx^{p^k} + b$ for $c \in k^*$ and $b \in k$. Then (f, α) is eventually stable over K for all $\alpha \in R$ not periodic under f.

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Remark: the number of irreducible factors of $f^n(x) - \alpha$ is bounded by $v(f^m(\alpha) - \alpha)$, where $m = \min\{i \geq 1 : v(f^i(\alpha)) - \alpha) > 0\}$.

Proof: Let $\alpha \in K$ be non-periodic for f. Observe that $\tilde{f}$ acts on $\overline{k}$ as a permutation, so there exists $m \geq 1$ with $\tilde{f}^m(\tilde{\alpha}) = \tilde{\alpha}$, and $(\widetilde{f^m})^{-1}(\tilde{\alpha}) = \{\tilde{\alpha}\}$ as a map of $\overline{k}$. But $f^m(\alpha) \neq \alpha$, so we may apply the previous result.

An application

Let $f(x) = x^2 + 1$, K be a number field, and $v_{\mathfrak{p}}$ the $\mathfrak{p}$-adic valuation associated to any prime of $\mathfrak{p}$ lying above (2). Then (f, α) is eventually stable over K for any $\alpha \in R$ not periodic under f.

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Let $K = \mathbb{Q}$ and $v = v_2$, so that $k = \mathbb{F}_2$. Then $\tilde{f}$ acts on k as 2-cycle.

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